SINGULARITIES IN THE NILPOTENT SCHEME
OF A CLASSICAL GROUP

BY

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ABSTRACT. If $(X, x)$ is a pointed scheme over a ring $k$, we introduce a (generalized) partition $\text{ord}(x, X/k)$. If $G$ is a reductive group scheme over $k$, the existence of a nilpotent subscheme $N(G)$ of $\text{Lie}(G)$ is discussed. We prove that $\text{ord}(x, N(G)/k)$ characterizes the orbits in $N(G)$, their codimension and their adjacency structure, provided that $G$ is $\text{Gl}_n$, or $\text{Sp}_n$ and $1/2 \in k$. For $\text{So}_n$, only partial results are obtained. We give presentations of some singularities of $N(G)$. Tables for its orbit structure are added.

Introduction. Let $G$ be a reductive algebraic group over a field of characteristic $p$. Let $\mathfrak{g}$ be its Lie-algebra and $N(G)$ the closed subset of the nilpotent elements of $\mathfrak{g}$, cf. [19]. The $G$-orbits in $N(G)$ are characterized by weighted Dynkin diagrams, cf. [20, III]. Consider the following question. *Is it possible to classify the orbits in $N(G)$ using only the local structure of the variety $N(G)$?* We prove in (4.3) that the answer is positive if $G$ is $\text{Gl}_n$ or if $G$ is $\text{Sp}_n$ and $p \neq 2$.

To this end we introduce a local invariant "ord" for any pointed scheme in §1. We develop the theory of $N(G)$ over an arbitrary ground ring $k$ in §2. In §3 we restrict our attention to the classical group schemes. Using a cross section we obtain information about the orbit structure of $N(G)$. Our main theorem (4.2) relates $\text{ord}(x, N(G)/k)$ to the Jordan normal form of the nilpotent endomorphism induced by $x$ in the classical representation.

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Conventions and notations. The cardinality of a set $V$ is denoted by $\# V$. Any infinite cardinal is represented by $\infty$. If $x$ is a real number then $[x]$ is the greatest integer in $x$. All rings are commutative with 1. Let $M$ be a module over a ring $A$. If $M$ is free the rank of $M$ is denoted by $\text{rg}_AM$. An element $c \in A$ is called $M$-regular if $a: M \rightarrow M$ is injective. Let $a = (a_1, \ldots, a_r)$ be a
sequence in \( A \). The ideal generated by \( a \) is denoted by \((a)\). The sequence is called \( M \)-regular if \( a_i \) is \((M/(a_i))\)-regular for all \( i \), cf. [12, 0\text{IV} 15.1].

Unless stated otherwise \( k \) is an arbitrary ground ring. General references for schemes and group schemes are [11], [12] and [8]. If we consider a \( k \)-scheme as a functor from \( k \)-algebras to sets, cf. [11, p. 17], then the letter \( R \) is used to denote an arbitrary \( k \)-algebra. If \( X \) is a \( k \)-scheme and \( R \) is a \( k \)-algebra then \( X(\mathbb{C}) \) is the \( R \)-scheme \( X \otimes_k R \). If \( X \) is an affine scheme then its coordinate ring is denoted by \( A(X) \). If \( A \) is a local ring its maximal ideal is denoted by \( m_A \) and its residue field by \( k(A) \). If \( X \) is a scheme and \( x \in X \) then we write \( m_x := m_A \) and \( k(x) := k(A) \) where \( A := \mathcal{O}_{X,x} \).

1. A near-partition for a local \( k \)-algebra.

\textbf{(1.1) A near-partition} \( \lambda \) is a subset of \( \mathbb{N}^2 \) such that if \((m, n) \in \lambda \) and \( i < m \) and \( j < n \) then \((i, j) \in \lambda \). The set of near-partitions is denoted by \( \mathcal{N}P \). The duality mapping \( D: \mathcal{N}P \to \mathcal{N}P \) is induced by \((i, j) \mapsto (j, i) \). The set \( \mathcal{N}P \) is ordered by \( \lambda \leq \mu \) if and only if \( \lambda \subseteq \mu \). We write \(|\lambda| := \# \lambda \). A near-partition \( \lambda \) is called a partition if \(|\lambda| < \infty \). The set of partitions is denoted by \( \mathcal{P} \).

If \( \lambda \in \mathcal{N}P \), the nonincreasing sequences \( \lambda^* \) and \( \lambda_* \) in \( \{0\} \cup \mathbb{N} \cup \{\infty\} \) are defined by

\[
\lambda^* \geq i \leftrightarrow (n, i) \in \lambda \leftrightarrow \lambda_i \geq n.
\]

Clearly \( \lambda_i = (D\lambda)^i = \sup \{n \in \mathbb{N}|\lambda^n \geq i\} \), and dually. A near-partition \( \lambda \) is completely determined by its sequence \( \lambda^* \) (or \( \lambda_* \)). We write \( \lambda_* = (\lambda_1, \ldots, \lambda_r) \) if \( \lambda_i = 0 \) for \( i > r \). If \( \lambda, \mu \in \mathcal{N}P \), we define \( \lambda + \mu \in \mathcal{N}P \) by \((\lambda + \mu)^n := \lambda^n + \mu^n \), where \( x + \infty := \infty + x := \infty \) for all \( x \). If \( \lambda_* = (\lambda_1, \ldots, \lambda_r) \) and \( \mu_* = (\mu_1, \ldots, \mu_s) \) then \((\lambda + \mu)_* \) is the sequence obtained by ordering \((\lambda_1, \ldots, \lambda_r, \mu_1, \ldots, \mu_s) \), see [9, Proposition 6].

\textbf{(1.2) Definition.} A linear extension over a ring \( k \) is a surjective morphism \( \varepsilon: E \to A \) of local \( k \)-algebras such that \( m_E \ker(\varepsilon) = 0 \). Its near-partition \( \text{ord}(\varepsilon) \) is defined by

\[
\text{ord}^n(\varepsilon) := r_{E}(\mathcal{O}(E) \cap m_E^{n+1}).
\]

A linear extension \( \varepsilon: E \to A \) is called \textit{versal} over \( k \) if for any linear extension \( \xi: F \to B \) over \( k \) and any local \( k \)-morphism \( \phi: A \to B \) there exists a (clearly local) \( k \)-morphism \( \gamma: E \to F \) with \( \xi \circ \gamma = \phi \circ \varepsilon \), see diagram (i).

\textbf{(1.3) Proposition.} Let diagram (i) be a commutative diagram of \( k \)-algebras such that \( \varepsilon \) and \( \xi \) are linear extensions, that \( \phi \) is a flat local morphism and that \( m_A B = m_B \). Then we have \( \text{ord}(\varepsilon) \geq \text{ord}(\xi) \).
The Nilpotent Scheme of a Classical Group

(i)

\[
\begin{array}{ccc}
F & \xrightarrow{\xi} & B \\
\gamma & \uparrow & \phi \\
E & \xrightarrow{e} & A
\end{array}
\]

Proof. Let \( n \in \mathbb{N} \). We prove that \( \text{ord}^n(e) \geq \text{ord}^n(\xi) \). It suffices to prove that the ideal \( \ker(\xi) \cap m_E^{n+1} \) is generated by the image of \( \ker(e) \cap m_E^{n+1} \). We may assume that \( \ker(e) \cap m_E^{n+1} = 0 \). Now the mapping \( m_E^{n+1} \to A \) induced by \( e \) is an injection of \( A \)-modules. Since \( B \) is flat over \( A \), it follows that \( m_E^{n+1} \otimes_A B \to B \) is injective and hence that \( \text{Tor}^E(E/m_E^{n+1}, B) = 0 \). This implies injectivity of

\[
\ker(\xi) \otimes_E (E/m_E^{n+1}) \to F \otimes_E (E/m_E^{n+1})
\]

so that \( \ker(\xi) \cap m_E^{n+1}F = m_E^{n+1} \ker(\xi) = 0 \). On the other hand \( m_AF = m_B \) implies that \( m_EF + \ker(\xi) = m_F \), so that \( m_E^{n+1}F = m_F^{n+1} \). This proves \( \ker(\xi) \cap m_F^{n+1} = 0 \).

(1.4) Let \( A \) be a local \( k \)-algebra. If \( e: E \to A \) is a versal linear extension over \( k \) then (1.3) implies that \( \text{ord}(e) \geq \text{ord}(\xi) \) for any linear extension \( \xi: F \to A \) over \( k \). On the other hand there exists a versal linear extension \( e: E \to A \) over \( k \). In fact, write \( A = R/A \) where \( R \) is some polynomial \( k \)-algebra. Let \( M \) be the ideal in \( R \) such that \( m_A = M \). Then \( R/MJ \to A \) is a versal linear extension over \( k \), compare [15, p. 37]. Now we can give the following:

Definition. \( \text{ord}(A/k) := \text{ord}(e) \) where \( e: E \to A \) is some (or any) versal linear extension over \( k \).

Example. Let \( k \) be a field. Put \( H = k[T_1, \ldots, T_m] \). Let \( a = (a_1, \ldots, a_r) \) be a sequence in \( H \). Let \( a_i \) be homogeneous of degree \( 1 + \lambda_i \) where \( \lambda \) is a partition with \( \lambda_{r+1} = 0 \). Assume that the ideal \( (a) \) is not generated by a strict subsequence of \( a \). Consider the local ring \( A := (H/(a))_{p} \) where \( p = (T_1, \ldots, T_m) \). Then \( \text{ord}(A/k) = \lambda \).

In fact \( e: (H/p(a))_p \to A \) is a versal linear extension over \( k \) and \( \text{ord}^n(e) = \#\{i \mid \lambda_i \geq n\} = \lambda^n \).

(1.5) Proposition. Let \( A \) be a local \( k \)-algebra and \( R \) a \( k \)-algebra. Assume that \( A \) or \( R \) is flat over \( k \). Let \( p \in \text{Spec}(A \otimes_k R) \) contract to \( m_A \). Then \( \text{ord}(A/k) \leq \text{ord}((A \otimes_k R)_p/R) \).

Proof. Let \( e: E \to A \) be a versal linear extension over \( k \). Put \( I := \ker(e) \). Let \( q \in \text{Spec}(E \otimes R) \) be the inverse image of \( p \). Since \( A \) or \( R \) is flat over \( k \), \( (I \otimes R)_q \) is an ideal in \( (E \otimes R)_q \). Put \( F := (E \otimes R)_q/q(I \otimes R)_q \), so that \( \xi: F \to (A \otimes R)_p \) is a linear extension over \( R \). One verifies that \( I \otimes_{k(E)} k(F) \to \ker(\xi) \) is injective and hence that \( \text{ord}(e) \leq \text{ord}(\xi) \). This suffices.
(1.6) **Proposition.** Let $A$ be a local $k$-algebra, $x = (x_1, \ldots, x_m)$ an $A$-regular sequence in $m_A$ and $f$ a nonzero element of $\langle x \rangle$. Put $B = A/(f)$ and $C = A/(x)$. Let $r \in \mathbb{N}$ and let $\rho$ be a partition with $\rho \ast = (r - 1)$.

(a) If $f \in m_A^r$ then $\rho + \text{ord}(A/k) \leq \text{ord}(B/k)$.
(b) If $f \notin m_A^{r+1}$ then $\text{ord}(B/k) \leq \rho + \text{ord}(C/k)$.

**Proof.** Let $\varepsilon: E \to A$ be a versal linear extension over $k$. Put $I = \ker(\varepsilon)$. Choose $y_i \in E$ with $\varepsilon(y_i) = x_i$ and $g \in E$ with $\varepsilon(g) = f$. Put $F := E/m_E$ and $G := E/m_E(y)$. The linear extensions $\xi: F \to B$ and $\eta: G \to C$ are versal over $k$. Since $x$ is a regular sequence, we have $I \cap \langle y \rangle = 0$. So the induced mappings $I \to \ker(\xi)$ and $I \to \ker(\eta)$ are injective. This implies that $\text{ord}(\varepsilon) \leq \text{ord}(\xi)$ and $\text{ord}(\varepsilon) \leq \text{ord}(\eta)$.

(a) Now it suffices to prove:

(\*) If $n < r$ then $1 + \text{ord}^n(\varepsilon) = \text{ord}^n(\xi)$.

We may assume that $g \in m_E^{n+1}$. The cokernel of the injection $I \cap m_E^{n+1} \to \ker(\xi) \cap m_F^{n+1}$ is isomorphic to $\langle g \rangle/m_E$; this proves (\*).

(b) By (\*) it suffices to prove: If $f \notin m_A^{n+1}$ then $\text{ord}^n(\xi) \leq \text{ord}^n(\eta)$. We may assume $g \in \langle y \rangle$. Since $f \notin m_A^{n+1}$ we have $g \notin m_E^{n+1}$. Using that $I \cap \langle y \rangle = 0$, one shows that the mapping $\ker(\xi) \cap m_E^{n+1} \to \ker(\eta) \cap m_F^{n+1}$ is injective.

**Remark.** Usually (1.6) (a) is applied in the situation where $f$ itself is $A$-regular, $m = 1$ and $x_1 = f$.

(1.7) If $X$ is a $k$-scheme and $x \in X$ then $(X, x)$ is called a **pointed $k$-scheme**. We define $\text{ord}(x, X/k) := \text{ord}(O_{X, x}/k)$. Pointed $k$-schemes $(X, x)$ and $(Y, y)$ are called **smoothly equivalent** if there are smooth $k$-morphisms $f: Z \to X$, $g: Z \to Y$ and a point $z \in Z$ with $f(z) = x$, $g(z) = y$. This is an equivalence relation on the class of pointed $k$-schemes, to be denoted by $(X, x) \sim (Y, y)$. See [12, IV 17] for the definition and the basic properties of smooth morphisms.

**Theorem.** If $(X, x) \sim (Y, y)$ then $\text{ord}(x, X/k) = \text{ord}(y, Y/k)$.

**Proof.** We may assume that there is a smooth $k$-morphism $f: X \to Y$ with $f(x) = y$.

Using the regularity of the noetherian local ring $O_{X, x}/m_Y O_{X, x}$ and the arguments of the proof of [12, IV 19.2.9], we construct a subscheme $Z$ of $X$ containing $x$ such that $O_{Z, x} = O_{X, x}/(a)$ where $a$ is an $O_{X, x}$-regular sequence, that $O_{Y, y} \to O_{Z, x}$ is flat and that $m_Y O_{Z, x}$ is the maximal ideal of $O_{Z, x}$. By (1.6) (a) we have $\text{ord}(x, X/k) \leq \text{ord}(x, Z/k)$. Using (1.3) one proves that $\text{ord}(x, Z/k) \leq \text{ord}(y, Y/k)$.

We may assume that $Y = \text{Spec } A$ and $y = m_A$ where $A$ is a local $k$-algebra. Choose a versal linear extension $\varepsilon: E \to A$ over $k$. By [12, IV 18.1.1] there is a smooth $E$-algebra $R$ such that $\text{Spec}(A \otimes_E R)$ is isomorphic to an open neigh-
bourhood of $x$ in $X$. So $O_{x,x} \cong (A \otimes_k R)_p$ for some $p \in \text{Spec}(A \otimes_k R)$ contracting to $m_A$. By (1.5) we have $\text{ord}(A/E) \leq \text{ord}((A \otimes_k R)_p/R)$. It is easy to see that this implies $\text{ord}(y, Y/k) \leq \text{ord}(x, X/k)$.

(1.8) The following remark will not be used in the sequel. For proofs and details we refer to [13].

Remark. Let $A$ be a noetherian local $k$-algebra. Then $\text{ord}(A/k)$ is a partition. It is equal to $\text{ord}(\hat{A}/k)$ where $\hat{A}$ is the completion of $A$. If $k$ is noetherian regular and $A$ is of essentially finite type over $k$, then $\text{ord}(A/k) = \text{ord}(A/Z)$. $A$ is regular if and only if $\text{ord}(A/Z) = 0$. If $A = R/I$ where $I$ is an ideal in a noetherian regular local ring $R$, then $\text{ord}(A/Z)$ is determined by the sequence $\nu(E)$, cf. [14, p. 209].

2. The nilpotent scheme.

(2.1) Consider an action $h$ of an affine group scheme $G$ on an affine scheme $X$ over $k$. We have the morphisms $h, pr_2 : G \times_k X \to X$. The orbit $Gx$ of $x \in X$ is defined as the subset $h(pr_2^{-1}(x))$ of $X$. Let $V$ be a subscheme of $X$. Let $U$ be the open set where the induced morphism $h^V : G \times_k V \to X$ is smooth. $V$ is called a cross section at $x$ if $x \in V$ and $e^V(\alpha) \in U$. Here $e^V : V \to G \times_k V$ is induced by the unit $e \in G(k)$. The subscheme $V$ is called a global cross section if $U \to \text{Spec}(k)$ is surjective. $V$ is called an invariant subscheme if the morphism $h^V$ factorizes over $V$.

Let $A(X)^G$ be the equalizer of the comorphisms $A(X) \Rightarrow A(G) \otimes_k A(X)$. If $Y$ is an affine scheme, a $G$-invariant $k$-morphism $f : X \to Y$ corresponds to a comorphism $A(Y) \to A(X)$ factorizing over $A(X)^G$. We define the affine quotient of the action by $[X/G] := \text{Spec}(A(X)^G)$. It is called universal if the induced morphism $[X_R/G(R)] \to [X/G]_R$ is an isomorphism for any $k$-algebra $R$.

Remarks. (a) Let $G$ be smooth over $k$. Then $pr_2$ and $h$ are smooth morphisms. If $x' \in Gx$ then $(X, x) \sim (X, x')$, cf. (1.7). If $V$ is a cross section at $x$ then $(X, x) \sim (V, x)$.

(b) The condition, that the affine quotient $[X/G]$ is universal, is a local condition on $\text{Spec}(k)$ for the topology $(f p q c)$, cf. [8, IV], see [13, p. 38]. If $k$ is a field any affine quotient is universal.

(c) Other types of quotients are discussed in [17, p. 3].

(2.2) Proposition. Assume in (2.1) that the morphism $X \to \text{Spec}(k)$ is smooth and irreducible, cf. [12, IV 4.5.5], and that $V$ is affine and a global cross section.

(a) The morphism $A(X)^G \to A(V)$ is injective.

(b) If $A(X)^G \to A(V)$ is bijective then $[X/G]$ is universal.
Proof. (a) Consider a nonzero \( f \in A(X)^G \). Assume that \( f|V = 0 \). There is a commutative diagram (i), so we have \( f \circ h^V = 0 \).

\[
\begin{array}{ccc}
G \times_k V & \xrightarrow{pr_2} & V \\
\downarrow h^V & & \downarrow f|V \\
X & \xrightarrow{f} & \text{Spec}(k[T])
\end{array}
\]

The morphism \( h^V \) is flat on \( U \), so \( h^V(U) \) is an open subset of \( X \) with \( f|h^V(U) = 0 \). Since \( f \neq 0 \), there is a generic point \( x \) of \( \text{Supp}(f|_x) \). Let \( p \in \text{Spec}(k) \) be the image of \( x \). Let \( \xi \) be the unique generic point of \( X \otimes_k k(p) \). As \( h^V(U) \to \text{Spec}(k) \) is surjective we have \( \xi \in h^V(U) \) and hence \( x \neq \xi \). Since \( \mathcal{O}_{X,x} \otimes_k k(p) \) is regular there is an \( \mathcal{O}_{X,x} \otimes_k k(p) \)-regular element \( t \in m_x \). By [12, IV 11.3.7], \( t \) is \( \mathcal{O}_{X,x} \)-regular. It is easy to see that this contradicts the choice of \( x \).

The argument used here was suggested by P. Deligne.

(b) Let \( R \) be a \(*\)-algebra. We have to prove that \( u: A(X)^G \otimes R \to A(X_R)^G(R) \) is bijective. As the assumptions of (a) are stable under base-change, the morphism \( v: A(X_R)^G(R) \to A(V) \otimes R \) is injective by (a). So it suffices to observe that \( v \circ u \) is bijective.

(2.3) Let \( G \) be a smooth affine group scheme over \( * \). Recall that the Lie-algebra \( \text{Lie}(G) \) is defined as the group functor such that \( \text{Lie}(G)(R) \) is the (additively written) kernel of the morphism \( G(R\{\delta\}) \to G(R) \) induced by \( \delta \mapsto 0 \) where \( R \) is a \(*\)-algebra. \( \text{Lie}(G) \) is a smooth affine group scheme, in fact a vector bundle. There is a canonical action of \( G \) on \( \text{Lie}(G) \). If \( R \) is a \(*\)-algebra then \( \text{Lie}(G)(R) = \text{Lie}(G(R)) \). See [8, II 4]. Usually we write \( g := \text{Lie}(G) \).

If \( K \) is a field over \( k \), a section \( x \in g(K) = \text{Lie}(G(K))(K) \) is nilpotent if and only if its image is a nilpotent endomorphism of \( F \) for some (or any) immersion of \( G(K) \) in a \( K \)-group \( \text{Gl}(F) \), cf. [2, p. 151]. A point \( x \in g \) is called nilpotent if the corresponding section \( x \in g(k(x)) \) is nilpotent.

(2.4) Definition. Let \( G \) be a reductive group scheme over \( k \), cf. [8, XIX 2.7]. If the affine quotient \( [g/G] \) is universal, cf. (2.1), then we define the nilpotent scheme \( N(G) := p^{-1}p(0) \) where \( 0 \in g(k) \) is the zero section and \( p: g \to [g/G] \) is the quotient morphism.

Proposition. Let \( N(G) \) be defined.

(a) \( N(G) \) is a \( G \)-invariant closed subscheme of \( g \).

(b) If \( R \) is a \(*\)-algebra then \( N(G_R)(k) = N(G_R)(k) \).

(c) A point \( x \in g \) is nilpotent if and only if \( x \in N(G) \).

Proof. (a) is trivial. (b) is a consequence of the assumption that \( [g/G] \) is universal. (c) By (b) we may assume that \( k \) is a field and that \( x \in g(k) \). Now it is well known. The "if-part" follows from Cayley-Hamilton by an embedding.
of $G$ in some $GL(F)$. The "only-if-part" is a consequence of the following

**Lemma.** Let $G$ be a reductive $k$-group over a field $k$. If $x \in g(k)$ has the additive Jordan decomposition $x = x_s + x_n$, then $x_s$ is in the closure of the orbit $Gx$.

**Proof.** Adapt [22, (4.4)] or [21, p. 92].

(2.5) Let $G$ be a reductive group scheme over $k$. By (2.1)(b) the existence of a nilpotent scheme is a local condition on Spec$(k)$ for the topology $(f p q c)$. So we assume that $G$ is of constant type (cf. [8, XXII 2.7]) with specified root system $R = (M, R, \rho)$, i.e. a root system $R$ in a given lattice $M$ (cf. [7, p. 287]). Let $t$ be the torsion index (cf. [7, p. 294]). Let $f$ be the connection index (cf. [4, p. 167]). Consider the following conditions:

(i) $r^{-1} \in k$ and if $R \cap 2M \neq \emptyset$ then $1/2 \in k$, cf. [7, p. 296].

(ii) $r^{-1}f^{-1} \in k$.

(iii) If $R$ has a component of type $A_l$ then $(l + 1)^{-1} \in k$, of type $B_l, C_l, D_l, G_2$ then $1/2 \in k$, of type $E_6, E_7, F_4$ then $1/6 \in k$, of type $E_8$ then $1/30 \in k$.

The conditions (ii) and (iii) are equivalent and imply (i).

(2.6) **Theorem.** Let $G$ be as in (2.5) satisfying condition (i).

(a) The affine quotient $[g/G]$ is universal. The quotient morphisms $p: g \to [g/G]$ is flat. $N(G)$ is defined and flat over $k$.

(b) Let $T$ be a maximal torus of $G$ with Weyl group $W$, cf. [8, XXII 3]. Put $t := \text{Lie}(T)$. The affine quotient $[t/W]$ is universal. The canonical morphism $[t/W] \to [g/G]$ is an isomorphism.

(c) Assume that (2.5) (ii) holds. Let $\pi: G \to \text{ad}(G)$ be the projection onto the adjoint group, cf. [8, XXII 4.3]. Then $N(\text{ad}(G))$ is defined and equal to $N(G)$.

**Proof.** (1) We may assume that $G$ is split with respect to a (resp. the) maximal torus $T$, cf. [8, XXII 2.3]. Now $T = D_S(M)$ and $A(t) = S(M) \otimes k$. The group scheme $W$ is the constant group scheme associated to the abstract Weyl group of $R$. By (2.5) (i) and [7, pp. 295, 296] the affine quotient $[t/W]$ is universal and the quotient morphism $t \to [t/W]$ is flat.

(2) By [8, XIII 5.1] and [12, IV 17.8.3] the subscheme $t$ is a global cross section for the action of $G$ on $g$, cf. (2.1). By (2.2) this implies that $A(g)^G \to A(t)^W$ is injective.

(3) We may assume that $k = Z[1/m]$, cf. [8, XXV 1]. It follows from [20, II 3.17'] and [22, p. 220] that $A(g)^G \otimes_k Q \to A(t)^W \otimes_k Q$ is bijective. Consider $a \in A(t)^W$. There is $a_1 \in A(g)^G$ and a nonzero $s \in k$ with $a_1 | t = sa$. Put $R = k/(s)$. Now $a_1 \otimes 1_R | t_R = 0$, so by (2) we have $a_1 \otimes 1_R = 0$ in
$A(g) \otimes R$. So there is $a_2 \in A(g)$ with $a_1 = sa_2$. Since $s$ is $A(G) \otimes A(g)$-regular we have $a_2 \in A(g)^G$. Since $s$ is $A(t)$-regular we have $a = a_2|t$. This proves that $A(g)^G \rightarrow A(t)^W$ is bijective. So we have proved (b).

(4) With (b) and (2) one proves that $[g/G]$ is universal in the same way as in (2.2) (b). Let $U$ be the open subset of $g$ where $p$ is flat. Since $t \rightarrow [g/G]$ is flat by (1) and (b), and $t \subset g$ is a regular immersion, we have $t \subset U$ by [12, 0IV 15.1.16]. As $U$ is $G$-invariant this implies $U = g$ by the lemma in (2.4). The other assertions of (a) follow immediately.

(5) In the notations of [8, XXII], condition (2.5) (ii) implies that the central isogenies $G \rightarrow \text{corad}(G) \otimes \text{ss}(G)$ and $\text{ss}(G) \rightarrow \text{ad}(G)$ are étale morphisms, by [8, VIII 2.1] and [8, XXI 6.5]. So we have an isomorphism

$$A(\text{Lie}(\text{ad}(G)))^{\text{ad}(G)} \otimes A(\text{Lie}(\text{corad}(G))) \cong A(g)^G.$$ With this isomorphism one proves (c).

**Remarks.** (i) Assume that the order of the Weyl group is invertible in $k$. By [22, (6.9)] the morphism $p$ is normal cf. [12, IV 6.8.1]. (ii) If $l \geq 2$ there is a semisimple group scheme $G$ of type $D_l$ over $Z$ such that $[g/G]$ is not universal.

(2.7) **Corollary.** Let $G$ be as in (2.6). Let $d_1, \ldots, d_r$ be the degrees of $R$. Consider the partition $\lambda$ defined by $\lambda_i := d_{r+1-i} - 1$ if $i \leq r$ and $\lambda_{r+1} := 0$. Let $x$ be a point of the zero section of $g$. Then $\text{ord}(x, N(G)/k) = \lambda$.

**Proof.** By (1.7) we may assume that $G$ is split with maximal torus $T$. Let $A(g)^G = k[a_1, \ldots, a_r]$ where $a_1, \ldots, a_r$ are algebraically independent and $a_i$ is homogeneous of degree $d_{r+1-i} = 1 + \lambda_i$, cf. [7, Theorem 3]. We have $\partial_{N(G),x} = \partial_{g,x}(a)$. Since $\partial_{g,x}$ is flat over $A(g)^G$ the sequence $a$ is $\partial_{g,x}$-regular. By (1.6) (a) this implies that $\text{ord}(x, N(G)/k) \geq \lambda$. Let $p \in \text{Spec}(k)$ be the image of $x$. By (1.5) we may replace $k$ by the residue field $k(p)$. Now the assertion follows from the example in (1.4).

**Remark.** If $k$ is noetherian regular the multiplicity of the local ring $\partial_{N(G),x}$ is equal to $\Pi_{i=1}^r d_i$, i.e. the order of the Weyl group. This is proved in [13, p. 55] using the methods of [18]. Compare [16, p. 386].

3. In the classical Lie-algebras.

(3.1) We fix a free $k$-module $F$ of rank $n$. The scheme $\text{End}(F)$ is defined by $\text{End}(F)(R) := \text{End}_R(F \otimes_k R)$, cf. [11, I 9]. The group scheme $G_l(F)$ (resp. $\text{SI}(F)$) is the open (resp. closed) subscheme of $\text{End}(F)$ where the function $\det \in A(\text{End}(F))$ is invertible (resp. where $\det = 1$). $G_l(F)$ and $\text{SI}(F)$ are reductive group schemes over $k$ of type $A_{n-1}$, cf. [6] and [8]. $\text{End}(F)$ is identified with $\text{Lie}(G_l(F))$ by $x \leftrightarrow 1 + 5x$ where $x \in \text{End}(F)(R)$, see (2.3) or [8, II 4]. Now $\text{Lie}(\text{SI}(F))$ consists of the endomorphisms with zero trace.
Assume $1/2 \in k$. Let $e$ be $0$ or $1$. An $e$-form $\phi$ on $F$ is a nondegenerate bilinear form $\phi: F \times F \rightarrow k$ which is symmetric if $e = 0$, alternating if $e = 1$. By “nondegenerate” we mean that the mapping $F \rightarrow F^*$ defined by $f \mapsto \phi(f, -)$ is bijective. Let $\phi$ be an $e$-form. The subgroup functor $G'(F, \phi)$ of $G(F)$ is defined by $x \in G'(F, \phi)(R)$ if and only if

$$\phi(\xi f, x g) = \phi(f, g) \quad (f, g \in F \otimes R).$$

We define $G(F, \phi) := G'(F, \phi) \cap SL(F)$. If $e = 0$ then $G(F, \phi)$ is the special orthogonal group scheme. If $e = 1$ then $G(F, \phi) = G'(F, \phi)$; it is the symplectic group scheme. Put $l := \lfloor \frac{n}{2} \rfloor$ and $\xi := n - 2l$. So $\xi$ is $0$ or $1$. If $e = 1$ then $\xi = 0$. Now $G(F, \phi)$ is a semisimple group scheme of type $B_l$ if $e = 0$, $\xi = 1$, of type $C_l$ if $e = 1$, $\xi = 0$, of type $D_l$ if $e = \xi = 0$, cf. [6] and [8]. The common Lie-algebra of $G(F, \phi)$ and $G'(F, \phi)$ is denoted by $\mathfrak{g}(F, \phi)$. For $x \in End(F)(R)$ we have $x \in \mathfrak{g}(F, \phi)(R)$ if and only if

$$\phi(\xi f, g) + \phi(f, \xi g) = 0 \quad (f, g \in F \otimes R).$$

**Convention.** In the rest of this paper we consider two cases.

**Case I.** $G := G' := G(F), l := n.$

**Case II.** $(e, l)$ where $e, l \in \{0, 1\}, e + l \leq 1$: $1/2 \in k$, $\phi$ is an $e$-form on $F$, $G := G(F, \phi)$, $G' := G'(F, \phi), n = 2l + \xi$.

In both cases $l$ is the reductive rank of $G$. We put $\mathfrak{g} := \text{Lie}(G)$. While considering Case II it is convenient to label concepts introduced for Case I with the index $l$, e.g. $\mathfrak{g} \subseteq \mathfrak{g}_l = \text{End}(F)$.

**3.2 Lemma.** Case II. Let $\phi_1$ be another $e$-form on $F$. Then there is a faithfully flat étale $k$-algebra $R$ such that $\phi_1$ and $\phi$ induce equivalent forms on $F \otimes R$.

**Proof.** By [15, pp. 34, 35] the scheme $\text{Isom}(\phi_1, \phi)$ is smooth over $k$. If $K$ is an algebraically closed field over $k$ then $\text{Isom}(\phi_1, \phi)(K) \neq \emptyset$. Hence by [12, IV 17.16.3] there is a faithfully flat étale $k$-algebra $R$ with $\text{Isom}(\phi_1, \phi)(R) \neq \emptyset$.

**3.3 Definition.** In Case I, $z \in \mathfrak{g}(R)$ is called a standard nilpotent with base-data $(f, \lambda)$ if $f = (f_1, \ldots, f_r)$ is a sequence in $F \otimes R$ and $\lambda$ is a partition, such that $\lambda^1 = r$, that the set $\{z^a f_i\}$, where $1 \leq i \leq r$ and $0 \leq a < \lambda_i$, is a basis of $F \otimes R$ and that $z^a f_i = 0$ if $a \geq \lambda_i$.

In Case II, $z \in \mathfrak{g}(R)$ is called a standard nilpotent with base-data $(f, \lambda, \beta, \alpha)$ if $z \in \mathfrak{g}_l(R)$ is a standard nilpotent with base-data $(f, \lambda)$, $\beta$ is a permutation of $\{1, \ldots, r\}$ where $r = \lambda^1$, and $\alpha: \{1, \ldots, r\} \rightarrow R$ is a mapping such that

$$\begin{cases} 
\phi(z^a f_i, z^b f_j) = (-1)^a \alpha(i) & \text{if } j = \beta i \text{ and } a + b + 1 = \lambda_i, \\
\phi(z^a f_i, z^b f_j) = 0 & \text{otherwise}. 
\end{cases}$$

(1)
REMARK. Clearly $|\lambda| = n$. In Case II the assumptions imply

\[
\alpha(f)^{-1} \in R, \quad \beta^2 = \text{id}, \quad \lambda_{\beta f} = \lambda_f, \quad \alpha(\beta) = (-1)^{\lambda_{f^{-1} + \epsilon}} \alpha(f).
\]

(3.4) The set $P_e$ is defined as the subset of $P$ consisting of the partitions $\lambda$ such that for any $m \gg 1$ with $m \equiv \epsilon (2)$ the number of indices $i$ with $\lambda_i = m$ is even. These partitions are called orthogonal, resp. symplectic; in [10, p. 556]. We define $P(n)$ as the set of partitions $\lambda$ with $|\lambda| = n$, and $P_e(n) := P_e \cap P(n)$. We write $P_{(e)}$ to denote $P$ in Case I and $P_e$ in Case II. So in (3.3) we have $\lambda \in P_{(e)}(n)$.

(3.5) If $x \in g$ is nilpotent, cf. (2.3), then the section $x \in g(k(x))$ is a standard nilpotent by [20, IV]. Let $\lambda \in P(n)$. We define $\mathcal{O}(\lambda)$ as the set of points $x \in g$ such that the section $x$ is a standard nilpotent with partition $\lambda$. In case II we have $\mathcal{O}(\lambda) = \mathcal{O}_\lambda(\lambda) \cap g$, and $\Sigma(\lambda) \neq \emptyset$ if and only if $\lambda \in P_e(n)$.

Let $k$ be a field and $x \in \mathcal{O}(\lambda)$. By [20, IV] we have $\mathcal{O}(\lambda) = G'x$, and $\mathcal{O}(\lambda) \neq Gx$ if and only if we are in the very-even case: Case II $(0, 0)$ with $\lambda_i$ even for all $i$.

(3.6) LEMMA. Case I. If $\lambda \in P(n)$, there is a standard nilpotent element $z \in g(k)$ with partition $\lambda$.

Case II. If $\lambda, \beta$ and $\alpha$ satisfy the conditions (3.3)(2), then there is an $e$-form $\phi_1$ on $F$ and a standard nilpotent element $z \in g(F, \phi_1)(k)$ with base-data $(f, \lambda, \beta, \alpha)$ for some sequence $f$ in $F$.

PROOF. Case I is trivial.

Case II. Choose a standard nilpotent $z \in g_f(k)$ with base-data $(f, \lambda)$. Let $\phi_1: F \otimes F \rightarrow k$ be the bilinear form defined by (3.3)(1). One verifies that $\phi_1$ is an $e$-form on $F$ with $z \in g(F, \phi_1)(k)$.

(3.7) The standard cross section. Let $z \in g(k)$ be a standard nilpotent element with base-data $(f, \lambda)$, resp. $(f, \lambda, \beta, \alpha)$. Below we construct a linear subscheme $L \subset g$ such that $g(R) = [g(R), x] \oplus L(R)$ for any $k$-algebra $R$. This implies that the subscheme $z + L \subset g$ is a cross section for the adjoint action of $G$ in all points of the section $z$, cf. (2.1). In fact the tangent morphism of $\text{Ad}: G \times (z + L) \rightarrow g$ at the section $(e, z)$ is the surjective morphism $g \oplus L \rightarrow g$ given by $(x, y) \mapsto [x, z] + y$. So smoothness of $\text{Ad}$ at $(e, z)$ follows from [12, IV 17.11.1].

Let $\Psi$ be the set of pairs $(i, a)$ with $0 \leq a \leq \lambda_i$. Put $f(i, a) := z^a f_i$. Then $\{f(\psi) | \psi \in \Psi\}$ is a basis of $F$. Let $\{u(\psi)\}$ be the dual basis of $F^*$. This means that $\{u(\psi)\}$ is the basis of $F^* = \text{Hom}(F, k)$ with

$$\langle u(\psi), f(\psi') \rangle = \delta_{\psi, \psi'} \quad (\text{Kronecker delta}).$$

The coordinates $\xi(\psi; \psi')$ on $g_f$ are defined by $\xi(\psi; \psi'(x) = \langle u(\psi), xf(\psi') \rangle$. 

Clearly \( \{ \xi(\psi; \psi') \psi, \psi' \in \Psi \} \) is a basis of \( \mathfrak{g}_f(k) \). Let \( \{ e(\psi; \psi') \} \) be the dual basis of \( \mathfrak{g}_f(k) \). We have

\[
e(\psi; \psi') f(\psi') = \delta_{\psi', \psi} f(\psi),
\]

\[
[e(i, a; j, b), z] = e(i, a; j, b) - e(i, a + 1; j, b)
\]

where \( e(i, a; j, b) = 0 \) if \( a \geq \lambda_i \) or \( b < 0 \). In Case I let \( \mathfrak{g}_{ij} \), \( L_{ij} \) and \( L \) be the linear subschemes of \( \mathfrak{g} \) defined by

\[
\mathfrak{g}_{ij}(R) := \sum_{a,b} R e(i, a; j, b),
\]

\[
L_{ij}(R) := \sum_{a,b} R e(i, a; j, \lambda_j - 1), \quad 0 \leq a < \min(\lambda_i, \lambda_j),
\]

\[
L(R) := \sum_{i,j} L_{ij}(R).
\]

Then we have \( \mathfrak{g}_{ij} = [\mathfrak{g}_{ij}, z] \oplus L_{ij} \) and \( \mathfrak{g} = [\mathfrak{g}, z] \oplus L \).

Case II. The coordinates \( \eta(\psi; \psi') \) on \( \mathfrak{g} \) are defined by \( \eta(\psi; \psi')(x) = \phi(f(\psi), x f(\psi')) \). Since \( \eta(\psi; \psi') = (-1)^{1+e} \eta(\psi'; \psi) \) we have a basis of \( \mathfrak{g}(k)^* \) consisting of the \( \eta(i, a; j, b) \) with \( i < j \), or \( i = j \) and \( a < b + e \). Let \( y(\psi; \psi') \) be the dual basis of \( \mathfrak{g}(k) \). One shows that

\[
y(i, a; j, b), z] = y(i, a; j, b - 1) + y(i, a - 1; j, b)
\]

if \( i < j \), or \( i = j \) and \( a < b - 1 \),

\[
y(i, a; a + 1), z] = y(i, a + 1; i, a + 1) + 2y(i, a; a),
\]

\[
y(i, a; i, a), z] = y(i, a - 1; i, a) \quad \text{if } e = 1,
\]

where \( y(\psi; \psi') = 0 \) if not yet defined. For \( i \leq j \) let \( \mathfrak{g}_{ij} \), \( L_{ij} \) and \( L \) be the linear subschemes of \( \mathfrak{g} \) defined by

\[
\mathfrak{g}_{ij}(R) := \sum_{a,b} R y(i, a; j, b),
\]

\[
L_{ij}(R) := \sum_{a,b} R y(i, \lambda_i - 1; j, b) \quad \text{if } i < j,
\]

\[
L_{ij}(R) := \sum_{a,b} R y(i, \lambda_i - 2 + e - a; i, \lambda_i - 1 - a), \quad 0 \leq a \leq \frac{1}{2}(\lambda_i - 2 + e),
\]

\[
L(R) := \sum_{i,j} L_{ij}(R).
\]

Then we have \( \mathfrak{g}_{ij} = [\mathfrak{g}_{ij}, z] \oplus L_{ij} \) and \( \mathfrak{g} = [\mathfrak{g}, z] \oplus L \).

Let \( F^\times \) be identified with \( F \) in such a way that \( \langle u, f \rangle = \phi(u, f) \). Putting \( |\alpha| = (-1)^{2\alpha(i)}^{-1} \), we get the following glossary:
$|l|_{i} f(i, a) = u(\beta i, \lambda_{i} - 1 - a),$

$\eta(i, a; j, b) = \xi(\beta j, \lambda_{j} - 1 - a; j, b) |g|_{i}$

$y(i, a; j, b) = |l|_{i} e(\beta i, \lambda_{i} - 1 - a; j, b) - (-1)^{e} |l|_{b} e(\beta j, \lambda_{j} - 1 - b; i, a)$

if $i < j$, or $i = j$ and $a < b$,

$y(i, a; j, b) = 0$ otherwise.

**Remark.** In Case I our $z + L$ is one of the cross sections of Arnold [1].

(3.8) An elementary calculation shows that $L(k)$ is a free $k$-module of rank $l + \gamma(\epsilon)(\lambda)$ if we write $\gamma(\epsilon)(\lambda) : = \gamma(\lambda)$ in Case I and $\gamma(\epsilon)(\lambda) : = \gamma_{e}(\lambda)$ in Case II where

$$
\gamma(\lambda) : = 2 \sum (i - 1)\lambda_{i} \quad \text{if } \lambda \in P(n),
$$

$$
\gamma_{e}(\lambda) : = \sum (i - 1)\lambda_{i} + (2e - 1)[\frac{1}{2} \# \{ i | \lambda_{i} \equiv 1 (2) \}] \quad \text{if } \lambda \in P_{e}(n).
$$

Now the centralizer of $z$ in $g(k)$ is also a free $k$-module of rank $l + \gamma(\epsilon)(\lambda)$. By [20, I 5.6] we have the following:

**Corollary.** Assume that $k$ is a field.

(a) If $x \in \Sigma(\lambda)$ then $\dim(Gx) = \dim(g) - l - \gamma(\epsilon)(\lambda)$.

(b) There is a unique nilpotent orbit $C_{\text{reg}}$ of maximal dimension.

$C_{\text{reg}} = \Sigma(v)$ where $v_{*} = (n)$ in the Cases I and II ($e, 1 - e$) and $v_{*} = (n - 1, 1)$ in Case II ($0, 0$). We have $\dim(C_{\text{reg}}) = \dim(g) - l$. If $C$ is another nilpotent orbit in $g$ then $\dim(C) \leq \dim(g) - l - 2$.

See also [1], [20, IV 2.28] and [21, p. 136].

(3.9) The mapping $\Sigma: P(n) \to P$ is defined by $(\Sigma \lambda)^{m} : = \Sigma_{i>m} \lambda_{i} (m \in N)$. As the corresponding propositions in [10, p. 567] are false, we shall prove the following:

**Proposition.** Let $\lambda, \mu \in P_{(\epsilon)}(n)$ be such that

$$
\{ \mu \} = \{ v \in P_{(\epsilon)}(n) | \Sigma \lambda > \Sigma v \geq \Sigma \mu \}.
$$

Then there are $\rho, \sigma, \tau \in P_{(\epsilon)}$ with $\lambda = \rho + \sigma, \mu = \rho + \tau$ and $\sigma, \tau$ as described in the following table.
Case | $\sigma_*$ | $\tau_*$ | Res*ictions
--- | --- | --- | 
I | $(p, q)$ | $(p + 1, q - 1)$ | $p > q \geq 1$
| $(a)$ | $(p, p)$ | $(p + 1, p - 1)$ | $p > 1$ and $p \equiv 0 \pmod{2}$
| $(b_1)$ | $(p, q)$ | $(p + 2, q - 2)$ | $p > q \geq 2$ and $p \equiv q \equiv 0 \pmod{2}$
| $(b_2)$ | $(p, p, q)$ | $(p + 1, p + 1, q - 2)$ | $p > q > 2$
| $(b_3)$ | $(p, q, q)$ | $(p + 2, q - 1, q - 1)$ | $p > q \geq 1$
| $(b_4)$ | $(p, p, q, q)$ | $(p + 1, p + 1, q - 1, q - 1)$ | $p > q \geq 1$

Proof. See (1.1) for the addition of partitions. Case I may be left to the reader. Case II. It is easy to see that we may assume disjointness: if $\lambda_i = \mu_j$ then $\lambda_i = 0$. Now we have to prove $\lambda = \sigma, \mu = \tau$ as in the table.

(a) Assume that there is a minimal $l \in \mathbb{N}$ with $\lambda_l \neq 0$ and $\lambda_l \equiv 0 \pmod{2}$. There is a maximal $m \in \mathbb{N}$ with $\lambda_m = \lambda_l$. Define $\nu \in P_e(n)$ by $\nu_i = \lambda_i + 1, \nu_m = \lambda_m - 1$ and $\nu_i = \lambda_i$ otherwise. Clearly $\Sigma \lambda > \Sigma \nu$. Using disjointness one proves $\Sigma \nu \geq \Sigma \mu$, so that $\nu = \mu$ and, again by disjointness, we are in case (a).

(b) Now $\lambda_i \equiv 0 \pmod{2}$ whenever $l > 0$. By disjointness there is an $m \in \mathbb{N}$ with $\mu_m > \lambda_1 > \mu_{m+1}$. It is easy to see that we can define $\nu \in P_e(n)$ satisfying $\Sigma \nu > \Sigma \mu$ as follows:

- If $\mu_m \equiv 0 \pmod{2}$, then $\nu_m = \mu_m - 2$ and $\nu_i = \mu_i$ if $i < m$;
- if $\mu_m \equiv 0 \pmod{2}$, then $\nu_{m-1} = \nu_m = \mu_m - 1$ and $\nu_i = \mu_i$ if $i < m - 1$;
- if $\mu_{m+1} \equiv 0 \pmod{2}$, then $\nu_{m+1} = \mu_{m+1} + 2$ and $\nu_i = \mu_i$ if $i > m + 1$;
- if $\mu_{m+1} \equiv 0 \pmod{2}$, then $\nu_{m+1} = \nu_{m+2} = \mu_{m+1} + 1$ and $\nu_i = \mu_i$ if $i > m + 2$.

One proves that $\Sigma \lambda > \Sigma \nu$, so that $\lambda = \nu$ and we are in one of the four cases (b).

(3.10) Theorem. Let $k$ be a field. Consider $z \in \mathfrak{O}(\lambda)$ and $x \in \mathfrak{O}(\mu)$. We have $z \in \mathfrak{G}x - Gx$ if and only if $\Sigma \lambda > \Sigma \mu$.

Remark. This theorem is due to Gerstenhaber, see [9, p. 327] and [10, pp. 567–569]. His proof for Case II is incomplete, see (3.9). Our proof seems to be more explicit.

Proof. We may assume that $z$ and $x$ are rational points. So $z$ is a standard nilpotent in $g(k)$ with partition $\lambda$. If $i \in \mathbb{N}$ then the endomorphism $z^i$ of $F$ has rank $(\Sigma D\lambda)^i$, see (1.1) for the definition of $D$.

Assume that $z \in \mathfrak{G}x - Gx$. The rank of $z^i$ is less than or equal to the rank of $x^i$. This implies $\Sigma D\lambda \leq \Sigma D\mu$ and hence $\Sigma \lambda > \Sigma \mu$ by [9, p. 327]. As $\lambda \neq \mu$ it is easy to see that $\Sigma \lambda > \Sigma \mu$.

Assume that $\Sigma \lambda > \Sigma \mu$. We have to prove that $z \in \mathfrak{G}x$. We may assume that $\lambda$ and $\mu$ are as in (3.9). So $\lambda$ and $\mu$ are not both very-even, cf. (3.5), and it suffices to prove that $z \in \mathfrak{O}(\mu)$. Using the notations of (3.7) we shall construct $y \in g(k)$ and a sequence $f(t) \ (t \in k)$ in such a way that $z(t) = z + ty \in \mathfrak{G}x$.
$\mathfrak{g}(k)$ is a standard nilpotent in $\mathfrak{g}_{s}(k)$ with base-data $(f(t), \mu)$ if $t \neq 0$. This will prove $z \in \mathfrak{C}(\mu)$.

Using a direct sum decomposition we may assume $\rho = 0$, $\lambda = \sigma$, $\mu = \tau$; cf. (3.9).

Case I. We have $\lambda_* = (p, q)$ and $\mu_* = (p + 1, q - 1)$. Let $(f_1, f_2, \lambda)$ be base-data for $z$. Put $y := e(1, q - 1; 2, q - 1)$. Put $f_1(t) := f_2$ and, if $q > 1$, $f_2(t) := tf_1 - zf_2$. We have

$$0 < a < q - 1 \Rightarrow z(t)^a f_1(t) = z^a f_2,$$

$$q < a < p \Rightarrow z(t)^a f_1(t) = tz^{a-1} f_1,$$

$$0 < a < q - 2 \Rightarrow z(t)^a f_2(t) = tz^a f_1 - z^{a+1} f_2,$$

$$z(t)^{p+1} f_1(t) = 0 \text{ and } z(t)^{q-1} f_2(t) = 0.$$

This implies that $z(t) \in \mathfrak{C}(\mu)$ if $t \neq 0$.

Case II. Of the five possibilities, cf. (3.9), we only treat (b_3) and (b_4) with $q \geq 2$. The other cases are easier, see [13, (4.3.7)], and already settled in [10, pp. 568, 569]. We choose convenient base-data $((f_1, \ldots, f_r), \lambda, \beta, \alpha)$ for $z$. The verifications are left to the reader.

(b_3) $\lambda_* = (p, q, q)$, $p \equiv q \equiv \neq (2)$, $r = 3$, $\beta = \text{id}$, $\mu_* = (p + 2, q - 1, q - 1)$. Choose

$$y := y(1, p - 1, 2, 0) + y(1, p - 1, 3, 0)$$

$$= e(1, 0; 2, 0) + e(1, 0; 3, 0) + e(2, q - 1; 1, p - 1) - e(3, q - 1; 1, p - 1),$$

$$f_1(t) := f_2, f_2(t) := zf_2 \text{ and } f_3(t) := z^{p-q+1} f_1 - tf_2 + tf_3.$$

(b_4) $\lambda_* = (p, p, q, q)$, $p \equiv q \equiv \neq (2)$, $r = 4$, $\beta = \text{id}$, $\mu_* = (p + 1, p + 1, q - 1, q - 1)$. Choose

$$y := y(1, p - 1, 3, 0) + y(1, p - 1, 4, 0)$$

$$= e(1, 0; 3, 0) + e(1, 0; 4, 0) + e(2, 0; 3, 0) + e(2, 0; 4, 0)$$

$$+ e(3, q - 1; 1, p - 1) - e(4, q - 1; 1, p - 1)$$

$$- e(3, q - 1; 2, p - 1) + e(4, q - 1; 2, p - 1),$$

$$f_1(t) := f_1, f_2(t) := f_3, f_3(t) := zf_3 \text{ and } f_4(t) := z^{p-q+1} f_1 - tf_3 + tf_4.$$

4. The classical nilpotent scheme, singularities.

(4.1) The symmetrical polynomials $\sigma_1, \ldots, \sigma_n \in A(\text{End}(F))$ are defined by the equation

$$\det(x + T \cdot \text{id}) = T^n + \sum_{m=1}^{n} T^{n-m} \sigma_m(x).$$
in $R[T]$ where $R$ is a $k$-algebra and $x \in \text{End}(F)(R)$. They are invariant under the adjoint action of $G(F)$ on $\text{End}(F)$. Let $X = (x_{ij})$ be the matrix of $x$ with respect to some basis $f_1, \ldots, f_n$ of $F$. Then

$$\sigma_m(x) = \sum \det(x_{ij})_{i,j \in I}$$

where the summation is over all subsets $I$ of $\{1, \ldots, n\}$ with $|I| = m$.

Case II. Clearly $\sigma_m|g \in A(\mathfrak{g})^G$. Let $\Phi$ be the matrix $\phi(f_{ij})$. We have $x \in \mathfrak{g}(R)$ if and only if $\Phi = -\Phi \Phi^{-1}$. This implies that $\sigma_m|g = 0$ if $m$ is odd. Assume $\epsilon = \xi = 0$. We define $\tau_1 \in A(\mathfrak{g})$ by $\tau_1(x) := \text{Pf}(\Phi X)$, where Pf denotes the Pfaffian, cf. [3, §5, no. 2]. Using loc. cit. one proves that $\tau_1^2 = \det(\Phi)\sigma_n$ and that $\tau_1 \in A(\mathfrak{g})^G$.

We define the sequence $a = (a_1, \ldots, a_l)$ in $A(\mathfrak{g})$ as follows. In Case I we put $a_1 := \sigma_1$. In Case II $(\epsilon, 1 - \epsilon)$ we put $a_1 := \sigma_2$. In Case II $(0, 0)$ we put $a_1 := \sigma_2$ if $i < l$, and $a_1 := \tau_1$.

**Theorem.** (a) $A(\mathfrak{g})^G$ is the free polynomial ring $k[a_1, \ldots, a_l]$.
(b) The sequence $a$ is $A(\mathfrak{g})$-regular (in any order).
(c) $N(G) = \text{Spec}(A(\mathfrak{g})/(a_l))$, it is flat over $k$.
(d) $N(G)$ is smooth over $k$ in the points of $\Sigma(v)$ where $v$ is, cf. (3.8)(b).
(e) If $k$ is a normal ring then $N(G)$ is a normal scheme.

**Proof.** (a) Let $u: k[T_1, \ldots, T_l] \to A(\mathfrak{g})^G$ be defined by $T_i \mapsto a_i$. We have to prove that $u$ is bijective. Replacing $k$ by a faithfully flat $k$-algebra (cf. (3.6) and (3.2)), we may assume the existence of a standard nilpotent $z \in \mathfrak{g}(k)$ with partition $\nu$, cf. (3.8)(b). Let $z + L$ be the cross section of (3.7). By (2.2) the morphism $\nu: A(\mathfrak{g})^G \to A(z + L)$ is injective. Case by case one shows that $\nu \circ u$ is bijective, so that $u$ is bijective.

(b) and (c). By [7], Theorem (2.6) applies. So $A(\mathfrak{g})$ is flat over $A(\mathfrak{g})^G$.
So we have (b) and (c).

(d) We may use the cross section of (a). Now $(z + L) \cap N(G)$ is a cross section at $z$ for the action of $G$ on $N(G)$, and the assertion follows from $(z + L) \cap N(G) \cong \text{Spec}(k)$.

(e) By (c) and [12, IV 6.14.1] we may assume that $k$ is a field. Now $N(G)$ is nonsingular in codimension one, by (d) and (3.8)(b). So $N(G)$ is normal by Serre’s criterion [12, IV 5.8.6].

**Remarks.** (i) There are other ways to prove the theorem, either avoiding (3.7) or avoiding (2.6) and [7]. (ii) It can be shown that $N(SL(F))$ exists and is equal to $N(Gl(F))$, where $k$ is arbitrary. Here (2.6) does not apply.

(4.2) If $\lambda \in P(n)$, the partition $\Sigma\lambda$ is defined in (3.9). Case II $(\epsilon, 1 - \epsilon)$. If $\lambda \in P_\epsilon(n)$ where $n = 2l + 1 - \epsilon$, then we define the partition $\Sigma_\epsilon\lambda$ by $(\Sigma_\epsilon\lambda)_l := (\Sigma\lambda)_{2l-\epsilon}$. Case II $(0, 0)$. If $\lambda \in P_0(n)$ where $n = 2l$, then we define $\Sigma_0\lambda$
WIM HESSELINK

\[ 16 \]

\[ \theta + \nu \text{ where } \theta, \nu \in P \text{ are given by } \theta_i := (\Sigma \lambda)_{2i+1} \text{ and } \nu_i := \left( \frac{1}{2} \lambda^1 - 1 \right). \]

Note: in the last case \( \lambda^1 \) is even and \( (\Sigma \lambda)_1 = \lambda^1 - 1 \). We write \( \Sigma_{(e)} \) to denote \( \Sigma \) in Case I and \( \Sigma_{(e)} \) in Case II. \( \Sigma_{(e)} \) means that \( e = 0 \) is excluded in Case II.

**THEOREM.** Consider \( x \in \mathfrak{D}(\lambda) \). Then \( \text{ord}(x, N(G)/k) = \Sigma_{(1)} \lambda \) in Cases I and II \((1, 0), \) and \( \text{ord}(x, N(G)/k) > \Sigma_0 \lambda \) in Case II \((0, \xi) \).

**Proof.** (1) By (1.7) we may replace \((N(G), x)\) by a smoothly equivalent pointed scheme. So by (3.6) and (3.2) we may assume the existence of a standard nilpotent \( z \in \mathfrak{g}(k) \) with partition \( \lambda \). By (2.1) (a) we may assume that \( x = z(p) \) for some \( p \in \text{Spec}(k) \). Put \( A := \mathfrak{o}_{g, x} \) and \( B := \mathfrak{o}_{N(G), x} \). We have \( B = A/\langle a \rangle \) where \( a \) is the \( A \)-regular sequence of (4.1), or rather its image in \( A \).

(2) Let \( J \) be the ideal in \( A(\mathfrak{g}) \) corresponding to the section \( z \). So \( x \) corresponds to the prime ideal \( J + \mathfrak{p}A(\mathfrak{g}) \). We claim

(a) If \( 1 < i < n \) and \( m := 1 + (\Sigma \lambda)^{n+1-i} \) then \( \sigma_i \in J^m \).

(b) In Case II \((0, 0)\) we have \( \tau_i \in J^m \) where \( m := \frac{1}{2} \lambda^1 \).

**Proof of (a).** It suffices to consider Case I. Let \((f, \lambda)\) be base-data for \( z \).

Using the notation of (3.7) we define

\[ \sigma_P := \det \xi(\psi; \psi')_{\psi, \psi' \in P} \]

if \( \emptyset \neq P \subset \Psi \). So \( \sigma_i = \Sigma \sigma_P \) where the summation is over all \( P \) with \#\( P \) = \( i \). If \( \xi(\psi; \psi') \notin J \) then we have \( \psi' = (j, a), \psi = (j, a + 1) \) for some \( j \) and \( a \). Consider \( P \) with \#\( P \) = \( i \). If \( \pi \) is a permutation of \( P \) then one verifies that

\[ \# \{ (j, a) \in P | \pi(j, a) \neq (j, a + 1) \} \geq 1 + (\Sigma \lambda)_{n+1-i} = m. \]

This implies \( \sigma_P \in J^m \), proving (a).

**Proof of (b).** We may assume that \( k \) is reduced. Now the assertion follows from \( \tau_i^2 = \det(\phi)_{\sigma \in J^{2m}}, \) cf. (a).

(3) By (1.6)(a) it follows from (2)(a), (b) that \( \text{ord}(B/k) > \Sigma_{(e)} \lambda \). This proves the theorem in Case II \((0, \xi)\). In the rest of the proof Case II \((0, \xi)\) is excluded. It suffices to prove

\[ \text{ord}(B/k) \leq \Sigma_{(1)} \lambda. \]

By (1.5) and (4.1)(c) we may replace \( k \) by an algebraic closure of the field \( k(p) \). So henceforth \( k \) is an algebraically closed field. Now \( x \) and \( z \) may be identified.

Let \((f, \lambda, \beta, \alpha)\) be its base-data.

(4) We prove (3) \((*)\) by induction on \( n = |\lambda| \). The cases with \( n \leq 1 \) are trivial. So assume \( n \geq 2 \). Put \( r := \lambda^1 \). Let \( \rho \) be the partition with \( \rho_\ast = (r - 1) \). The partition \( \mu \) is defined as follows.

**Case I.** \( \mu_r := \lambda_r - 1, \mu_i := \lambda_i \) if \( i \neq r \).

**Case II.** If \( \lambda_r \) is even then \( \mu_r := \lambda_r - 2 \) and \( \mu_i := \lambda_i \) otherwise. If \( \lambda_r \) is
odd so that $\lambda_{r-1} = \lambda_r$, then $\mu_{r-1} := \mu_r = \lambda_r - 1$ and $\mu_i := \lambda_i$ otherwise.

One verifies that $\mu \in \mathcal{P}_{(1)}$ and that $\Sigma_{(1)} \lambda = \rho + \Sigma_{(1)} \mu$.

We have $F = \Sigma k f(\psi), \psi \in \Psi$, cf. (3.7). Let $P$ be the subset of $\Psi$ containing $(r, 0)$ and in Case II also $(\beta_r, \lambda_r - 1)$. Put $F' := \Sigma k f(\psi), \psi \not\in P$, and $F'' := \Sigma k f(\psi), \psi \in P$. Clearly $F = F' \oplus F''$. In Case II the form $\psi' := \phi | F''$ is nondegenerate and hence a 1-form on $F'$. We put $G' := GL(F')$ in Case I and $G' := G(F', \phi')$ in Case II. So the convention (3.1) concerning $G'$ is not applied here.

We put $g' := \text{Lie}(G')$, etc.

Let

$$
\begin{pmatrix}
  x' & x_2 \\
  x_1 & x_3
\end{pmatrix}
$$

be the matrix of $x$ with respect to the decomposition $F = F' \oplus F''$. Now $x'$ is a standard nilpotent in $g'(k)$ with partition $\mu$. Consider the ring

$$B' := \mathcal{O}_{N(G'), x'} = \mathcal{O}_{\text{g'}, x'/<(\sigma_i)_{i<n}'>}.$$

By induction we have $\text{ord}(B'/k) \leq \Sigma_{(1)} \mu$. One verifies that

$$w \mapsto \begin{pmatrix} w & x_2 \\ x_1 & x_3 \end{pmatrix}$$

defines a regular immersion $u: g' \to g$ such that $u(x') = x$, $u^0(\sigma_n) = 0$ and $u^0(\sigma_i) = \sigma_i'$ if $i < n$, where $u^0: A(g) \to A(g')$ is the comorphism. Put $R := A/\langle(\sigma_i)_{i<n}\rangle$ so that $B = R/(f')$ where $f$ is the image of $\sigma_n$ in $R$. Now there is an $R$-regular sequence $x$ in $R$ such that $B' \cong R/(x)$ and $f \in (x)$. By (1.6)(b) this implies

$$\text{ord}(B/k) \leq \rho + \text{ord}(B'/k) \leq \rho + \Sigma_{(1)} \mu = \Sigma_{(1)} \lambda$$

provided that $f \not\in m_R^{r+1}$. So in order to prove the theorem it suffices to prove that

$$\sigma_\Psi \notin \langle(\sigma_P)_{P \neq \Psi}\rangle + m_A^{r+1}$$

where we have used the notation of (2).

(5) In Case II we normalize the base-data of $x$ as follows: $\beta_i \neq i$ if and only if $\lambda_i$ is odd; $|\beta_i - i| < 1$ for all $i$; if $i > \beta_i$ then $\alpha(i) = 1$. Now $i \leq \beta_i$ implies $\alpha(i) = (-1)^{\lambda_i}$. With the notation of (3.7) we define a linear subvariety $M$ of $g$.

**Case I.** $M := \Sigma \text{ke}(i, 0; j, \lambda_i - 1) (1 \leq i, j \leq r)$.

**Case II.** $M := \Sigma k v(i, \lambda_i - 1; j, \lambda_j - 1)$ where the summation is over all pairs $(i, j)$ such that $i = j$ or $i \leq \beta_i < j \leq \beta_j$. So in this case $M \subset M_I$.

The ring $A(x + M)$ is considered as a graded $k$-algebra such that $x$ corresponds to the augmentation ideal. The functions $\sigma_{p|x + M}$ are homogeneous,
\( \sigma_p | x + M \) is homogeneous of degree \( r \). So it suffices to prove

\((*)\) \hspace{1cm} \sigma_p | x + M \notin \langle (\sigma_p | x + M)_{P \neq \Psi} \rangle.

(6) Case I. It is easy to see that \( x + M \) has a subvariety \( x_1 + M_1 \) such that

\[ \sigma_p | x_1 + M_1 \neq 0 \text{ if and only if } P = \Psi. \]

This proves (5)(*) and the theorem.

Case II. Consider the subvariety \( x_1 + M_1 \) of \( x + M \) where

\[ x_1 := x + \sum_{i > \beta_1} y(i, \lambda_i - 1; i, \lambda_i - 1), \]

\[ M_1 := \sum_{i > \beta_1} k y(i, \lambda_i - 1; i, \lambda_i - 1) \text{ if } i \leq \beta_1, i \leq j \leq \beta). \]

Now \( x_1 \) is a standard nilpotent in \( g_f(k) \) with base-data \( (f', \lambda') \) such that

\[ M_1 = \sum_{i < j} k (e'(i, 0; j, \lambda_j - 1) + e'(j, 0; i, \lambda_i - 1)) \]

with respect to the new base-data. In order to prove (5)(*) and hence the theorem, it suffices to show that

\[ \sigma_p | x_1 + M_1 \notin \langle (\sigma_p | x_1 + M_1)_{P \neq \Psi} \rangle. \]

This is a consequence of the following:

**Lemma.** Assume \( \text{char}(k) \neq 2 \). Let \( r \in \mathbb{N} \). Consider the ring \( k[T_{ij}] \) where \( 1 \leq i \leq j \leq r \). Put \( T_{ij} := T_{ji} \text{ if } i > j \). Put \( Q := \{1, \ldots, r\} \). If \( \emptyset \neq P \subset Q \) define \( \sigma_P := \det(T_{ij})_{i,j \in P} \). Then \( \sigma_Q \notin \langle (\sigma_P)_{P \neq Q} \rangle \).

**Proof.** We may assume \( r \geq 3 \). Let \( I \) be the ideal generated by all \( T_{ij} \) such that \( 1 \neq |i - j| \neq r - 1 \), and all \( T_{ij}^2 \). It is easy to see that \( \sigma_P \notin I \) if and only if \( P = Q \).

(4.3) The following facts are not proved here, see [13, pp. 11–13].
(i) The mapping \( \Sigma(1) : P_{(1)}(n) \to P \) is injective.
(ii) If \( \Sigma_1 \lambda \leq \Sigma_1 \mu \) where \( \lambda, \mu \in P_1(n) \) then \( \Sigma \lambda \leq \Sigma \mu \).
(iii) If \( \lambda \in P_{(1)}(n) \) then \( \gamma_{(1)}(\lambda) = 2|\Sigma_{(1)}\lambda| \).

Using (1.7), (2.1)(a), (3.5), (3.8), (3.10), (4.2) we get the following

**Corollary.** Case I and II (1, 0). Let \( x \in \mathfrak{G}(\lambda) \) and \( y \in N(G) \).
(a) \( y \in \mathfrak{G}(\lambda) \) if and only if \( \text{ord}(y, N(G)/k) = \text{ord}(x, N(G)/k) \).
(b) \( y \in Gx \) if and only if \( (N(G), y) \sim (N(G), x) \), cf. (1.7).

Assume that \( k \) is a field.
(c) \( \text{codim}(Gx, N(G)) = 2|\text{ord}(x, N(G)/k)| \).
(d) \( y \in Gx \) if and only if \( \text{ord}(y, N(G)/k) \geq \text{ord}(x, N(G)/k) \).

(4.4) **Remark.** In (4.2) Case II (0, \( \zeta \)) inequality occurs if \( \lambda_1 \) is even and also if \( \lambda_* = (3, 3, 2, 2) \), but we have equality if \( \lambda_* = (3, 3, 2, 2, 1) \). In the last case we have
The nilpotent scheme of a classical group

\[ \text{codim}(G_x, N(G)) = \gamma_0(\lambda) > 2|\Sigma_0\lambda| = 2|\text{ord}(x, N(G)/k)| \]

if \( k \) is a field, compare (4.3)(c) and (4.9) table \( B_5 \).

(4.5) The polynomials \( f_a \) are defined by \( f_a := 0 \) if \( a < 0 \), \( f_0 := 1 \) and \( f_a := \sum_{i \geq 1} X_i f_a-i \) if \( a > 0 \). They are determined by the generating function

\[ \sum_{a=0}^{\infty} T^a f_a = \left( 1 - \sum_{i \geq 1} X_i T^i \right)^{-1}. \]

Clearly \( f_a(X_1) = X_a^a \) if \( a \geq 0 \). One can prove that

\[ f_a(X_1, X_2) = \sum \binom{a-i}{i} X_1^{a-2i} X_2^i \quad (0 \leq i \leq \lfloor a/2 \rfloor). \]

Let \( A^m \) denote the affine space over \( k \) of rank \( m \), say with coordinate ring \( k[X_1, \ldots, X_m] \). It is pointed in some point of the origin section. The Kleinian singularities \( A_i \) and \( D_i \) are the pointed subschemes of \( A^3 \) given by one equation:

- \( A_p, l \geq 1, \) by \( X_1^{l+1} + X_2 X_3 = 0 \),
- \( D_p, l \geq 3, \) by \( X_1^{l-1} - X_1 X_2^2 + X_3^2 = 0 \), if \( 1/2 \in k \).

We define the following singularities.

If \( l \geq 3 \), \( AA_l \) in \( A^2 \times A^4 \) by

\[ \begin{align*}
    f_l(X_1, X_2) + Y_1 Y_3 + Y_2 Y_4 &= 0, \\
    X_2 f_{l-1}(X_1, X_2) - Y_4(X_1 Y_2 - X_2 Y_1) + Y_2 Y_3 &= 0.
\end{align*} \]

If \( 1/2 \in k \) and \( l \geq 3 \), \( BB_l \) in \( A^2 \times A^4 \) by

\[ \begin{align*}
    f_{l-1}(2X_1, -X_2^2) - 2Y_1 Y_3 + Y_2^2 - Y_4^2 &= 0, \\
    X_2^2 f_{l-2}(2X_1, -X_2^2) + (Y_3 - X_1 Y_2^2 - X_2^2 Y_1^2 - 2Y_4(X_1 Y_4 - X_2 Y_2) &= 0.
\end{align*} \]

If \( 1/2 \in k \) and \( l \geq 2 \), \( CC_l \) in \( A^3 \times A^2 \) by

\[ (X_3^2 - X_1 X_2)^l + X_1 Y_1^2 + 2X_3 Y_1 Y_2 + X_2 Y_2^2 = 0. \]

If \( 1/2 \in k \) and \( l \geq 5 \), \( CD_l \) in \( A^2 \times A^4 \) by

\[ \begin{align*}
    f_{l-2}(X_1, X_2) + X_1 Y_2^2 - X_2 Y_1^2 - Y_3^2 + 2Y_2 Y_4 &= 0, \\
    X_2 f_{l-3}(X_1, X_2) + X_2(X_1 Y_1^2 + Y_2^2 - 2Y_1 Y_3) + Y_4^2 &= 0.
\end{align*} \]

If \( 1/2 \in k \) and \( l \geq 3 \), \( DD_l \) in \( A^3 \times A^3 \) by

\[ \begin{align*}
    (x_1^2 + x_2^2 + x_3^2)^{l-1} + y_1^2 + y_2^2 + y_3^2 &= 0, \\
    x_1 y_1 + x_2 y_2 + x_3 y_3 &= 0.
\end{align*} \]
(4.6) Proposition. Assume in Case II that \( l + e + \xi \geq 3 \). Consider \( \lambda \in P(\epsilon)(n) \) with \( 0 < \gamma(\epsilon)(\lambda) < 6 \), cf. (3.8). If \( x \in \Sigma(\lambda) \) then \((N(G), x)\) is smoothly equivalent (cf. (1.7)) to the singularity (cf. (4.5)) given in the following table.

<table>
<thead>
<tr>
<th>( G )</th>
<th>( \lambda )</th>
<th>Dynkin diagram</th>
<th>( \gamma(\epsilon)(\lambda) )</th>
<th>singularity</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G_1 )</td>
<td>( n \geq 2 ) ( (n-1,1) )</td>
<td>( 2 )</td>
<td>( 2 )</td>
<td>( A_{n-1} )</td>
</tr>
<tr>
<td>( SO_{2\ell+1} )</td>
<td>( \ell \geq 2 ) ( (2\ell-1,1,1) )</td>
<td>( 2 )</td>
<td>( 2 )</td>
<td>( A_{2\ell-1} )</td>
</tr>
<tr>
<td>( Sp_{2\ell} )</td>
<td>( \ell \geq 2 ) ( (2\ell-2,2) )</td>
<td>( 2 )</td>
<td>( 2 )</td>
<td>( D_{\ell+1} )</td>
</tr>
<tr>
<td>( SO_{2\ell} )</td>
<td>( \ell \geq 3 ) ( (2\ell-3,3) )</td>
<td>( 2 )</td>
<td>( 2 )</td>
<td>( D_{\ell} )</td>
</tr>
<tr>
<td>( G_1 )</td>
<td>( n \geq 4 ) ( (n-2,2) )</td>
<td>( 4 )</td>
<td>( 4 )</td>
<td>( A_{n-1} )</td>
</tr>
<tr>
<td>( SO_{2\ell+1} )</td>
<td>( \ell \geq 3 ) ( (2\ell-3,3,1) )</td>
<td>( 4 )</td>
<td>( 4 )</td>
<td>( B_{\ell+1} )</td>
</tr>
<tr>
<td>( Sp_{2\ell} )</td>
<td>( \ell \geq 2 ) ( (2\ell-2,1,1) )</td>
<td>( 4 )</td>
<td>( 4 )</td>
<td>( C_{\ell} )</td>
</tr>
<tr>
<td>( SO_{2\ell} )</td>
<td>( \ell \geq 3 ) ( (3,3) )</td>
<td>( 4 )</td>
<td>( 4 )</td>
<td>( D_{\ell} )</td>
</tr>
<tr>
<td>( SO_{2\ell} )</td>
<td>( \ell \geq 4 ) ( (2\ell-4,4) )</td>
<td>( 4 )</td>
<td>( 4 )</td>
<td>( CD_{\ell+1} )</td>
</tr>
</tbody>
</table>

Remark. We have \( \gamma(\epsilon)(\lambda) = \text{codim}(Gx, N(G)_{(k(x))}) \). For the singularities with \( \gamma(\epsilon)(\lambda) = 2 \), compare [5] and [21, pp. 140–158]. In the table we have added the Dynkin diagram of the section \( x \in \mathfrak{g}(k(x)) \), cf. [20, III, IV], where \( \leftarrow \) means a string with numbers 2 attached to the nodes.

Proof. The classification of all possibilities for \( \lambda \) is easy. By the sequence of reductions used in (4.2)(1) we may assume that \( x = z(p) \) where \( z \) is a standard nilpotent with partition \( \lambda \) and \( p \in \text{Spec}(k) \). In Case II the base-data for \( z \) may be prescribed within the bounds set by (3.3)(2). Let \( z + L \) be the cross section of (3.7). Then \( (z + L) \cap N(G) \) is a cross section at \( z \) for the action of \( G \) on \( N(G) \). So \((N(G), x)\) is smoothly equivalent to \((z + L) \cap N(G), z(p))\) by (2.1)(a). The two singularities to be determined for \( G_1^l \) will be examples in (4.7) and (4.8). We do not give the tedious calculations needed to settle Case II, see [13, p. 79] for some indications.

(4.7) Case I with \( \lambda = (p, 1^q) \), i.e. \( (p, 1, \ldots, 1) \) with \( q \) times 1. We have \( n = p + q \) and \( r := \lambda^1 = q + 1 \). On \( z + L \) we define the coordinate functions \( \xi_{ar}, \xi_{ij} \) as follows: if \( R \) is a \( k \)-algebra and \( x \in (z + L)(R) \), then
(1) \[ x = z - \sum_{a=1}^{p} \xi_{a}(x)e(1, p - a; 1, p - 1) - \sum_{(i,j) \neq (1,1)} \xi_{ij}(x)e(i, 0; j, \lambda_j - 1). \]

So \( A(z + L) = k[\xi_a, \xi_{ij}] \). Put \( \xi_{11} = 0 \).

If \( a \geq 1 \), let \( s_a, h_a \in k[\xi_{ij}] \) be defined by

\[
\begin{cases}
    s_a = \sum \det(\xi_{ij})_{i,j \in I} \\
    h_a = \sum \det(\xi_{ij})_{i,j \in \{1\} \cup I}
\end{cases}
\]

where in both cases the summation is over the subsets \( I \) of \( \{2, \ldots, r\} \) with \( \# I = a \).

Clearly, if \( a \geq r \) then \( s_a = h_a = 0 \). The subscheme \( (z + L) \cap N(G) \) of \( z + L \) is defined by the equations \( \sigma_m(z + L) = 0 \) (\( 1 \leq m \leq n \)). One verifies that

\[
\begin{align*}
(-1)^m \sigma_m(z + L) &= \xi_m + s_m + \sum_{a=1}^{m-1} \xi_a s_{m-a} & \text{if } 1 \leq m \leq p, \\
(-1)^m \sigma_m(z + L) &= h_{m-p} + s_m + \sum_{a=1}^{p} \xi_a s_{m-a} & \text{if } p < m \leq n.
\end{align*}
\]

The first \( p \) equations can be solved inductively. With the notations of (4.5) we obtain

\( \xi_m = f_m(-s_1, -s_2, \ldots, -s_q) \) (\( 1 \leq m \leq p \)). So \( (z + L) \cap N(G) \) is isomorphic to the subscheme of \( \text{Spec } k[\xi_{ij}] \) defined by the equations

\[
h_{m-p} + \sum_{a=0}^{p} s_{m-a} f_a(-s_1, \ldots, -s_q) = 0 \quad (p < m \leq n).
\]

**Examples.**

(a) \( \lambda_* = (n - 1, 1) \). Putting \( X_1 = -\xi_{22}, X_2 = \xi_{12}, X_3 = \xi_{21} \), we get the singularity \( A_{n-1} \), cf. (4.5).

(b) \( \lambda_* = (n - 2, 1, 1) \). The scheme \( (z + L) \cap N(G) \) is isomorphic to the subscheme of \( A^8 = \text{Spec}(k[\xi_{ij}]) \), where \( 1 \leq i, j \leq 3 \leq i + j \), defined by the equations

\[
\begin{align*}
0 &= f_{n-1}(-s_1, -s_2) - h_1, \\
0 &= s_2 f_{n-2}(-s_1, -s_2) + h_2.
\end{align*}
\]

where

\[
\begin{align*}
s_1 &= \xi_{22} + \xi_{33}, \\
s_2 &= \xi_{22} \xi_{33} - \xi_{23} \xi_{32}, \quad \text{and} \quad h_2 = \begin{vmatrix}
0 & \xi_{12} & \xi_{13} \\
\xi_{21} & \xi_{22} & \xi_{23} \\
\xi_{31} & \xi_{32} & \xi_{33}
\end{vmatrix},
\end{align*}
\]

(4.8) Case I for arbitrary \( \lambda \). We use a different cross section, viz. \( z + L'' \) defined by \( L'' = \Sigma_{i,j} L''_{ij} \) where \( L''_{ij} = L_{ij} \) if \( i \neq 1 \) or \( j = 1 \) and \( L''_{1,j}(R) = \Sigma_{0 \leq b < \lambda_j} \text{Re} \{1, 0; j, b\} \) if \( j \neq 1 \), see (3.7). Again we have
Put $p := \lambda_1$, $q := n - \lambda_1$ and $\mu := (p, 1^q)$. Put $z' := \sum_{r=0}^{p-2} c(1, a + 1; 1, a)$, so that $z'$ is a standard nilpotent element in $g(k)$ with partition $\mu$. In the obvious way we define base-data $(\tau', \mu)$ for $z'$. The cross section $z' + L'$ at $z'$ used in (4.7) contains $z + L''$. So we can use the elimination in (4.7) of $\xi_{a}$, $1 \leq a \leq p$, substituting into the matrix $(\xi_{ij})$ at some places the constant functions $0$ or $-1$, cf. (4.7)(1).

**Example.** If $\lambda = (n - 2, 2), n \geq 4$, we use the matrix

$$
(\xi_{ij}) = \begin{pmatrix}
0 & Y_3 & Y_4 \\
Y_1 & -X_1 & -1 \\
Y_2 & -X_2 & 0
\end{pmatrix}
$$

and we obtain the equations (4.7)(5) where $s_1 = -X_1, s_2 = -X_2, h_1 = Y_1 Y_3 + Y_2 Y_4$ and $h_2 = Y_4(X_1 Y_2 - X_2 Y_1) - Y_2 Y_3$. So $(z + L'') \cap N(G)$ is isomorphic to the singularity $AAn_1$, cf. (4.5).

(4.9) **Tables for the orbits in $N(G)$**. We give the adjacency structure (cf. (3.10)), the Dynkin diagram (cf. [20, IV]), the codimension of the orbits $\gamma_\ell(x)$ (cf. (3.8)), and the partition $\text{ord} = \text{ord}(x, N(G)/k)$ (cf. (4.2)). The number of orbits is denoted by $\#$. In the cases $SO_2$, with even $l$, the partition $\lambda$ may represent two orbits, cf. (3.5). We give the Dynkin diagram of one of them and indicate how to get the other one by the symbol $\uparrow$.

For $SO_n$ we give $\Sigma_0 \lambda$, which is a lower bound of $\text{ord}$, cf. (4.2). Whenever there are reasons to assume $\text{ord} \neq \Sigma_0 \lambda$, we give a conjectured value of $\text{ord}$ or a question mark. As $D_2 = A_1 + A_1, B_2 = C_2$ and $D_3 = A_3$, the values of $\text{ord}$ for the cases $SO_4, SO_5$ and $SO_6$ are not conjectural.

\[
\begin{array}{ccccccc}
A_1 & Gl_1 & \lambda & Dy & \gamma(\lambda) & \text{ord} \\
\# = 2 & & 1 & 1 & 0 & 2 & 1 \\
\hline
A_2 & Gl_2 & \lambda & Dy & \gamma(\lambda) & \text{ord} \\
\# = 3 & & 1 & 1 & 0 & 6 & 2 \ 1
\end{array}
\]

\[
D_2 = A_1 + A_1 & SO_4 & \lambda & Dy & \gamma_0(\lambda) & (\Sigma_1 \lambda) & \text{ord} \\
\# = 4 & & 1 & 1 & 0 & 4 & 1 \ 1
\]
### The Nilpotent Scheme of a Classical Group

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<th>$\gamma_6(\lambda)$</th>
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\[ C_3 \quad SSp_6 \quad \lambda^* \quad \gamma_1(\lambda) \quad \text{ord}_x \]

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4 & 1 & 1 & 2 \quad 1 \quad 0 \quad 4 \quad 2 \\
3 & 3 & 0 & 2 \quad 0 \quad 4 \quad 1 \quad 1 \\
2 & 2 & 2 & 0 \quad 0 \quad 2 \quad 6 \quad 2 \quad 1 \\
2 & 2 & 1 & 1 \quad 0 \quad 1 \quad 0 \quad 8 \quad 3 \quad 1 \\
2 & 1 & 4 & 1 \quad 0 \quad 0 \quad 12 \quad 4 \quad 2 \\
1 & 8 & 0 \quad 0 \quad 0 \quad 18 \quad 5 \quad 3 \quad 1
\end{array}

\[ \# = 8 \]

\[ A_6 \quad G\text{l}_3 \quad \lambda^* \quad \gamma(\lambda) \quad \text{ord}_x \]

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2 & 1 & 8 & 1 \quad 0 \quad 0 \quad 1 \quad 12 \quad 3 \quad 2 \quad 1 \\
1 & 8 & 0 \quad 0 \quad 0 \quad 20 \quad 4 \quad 3 \quad 2 \quad 1
\end{array}

\[ \# = 7 \]

\[ D_6 \quad S\text{O}_3 \quad \lambda^* \quad \gamma_6(\lambda) \quad (\Sigma_6 \lambda)_* \quad \text{ord}_x ? \]

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\[ \# = 12 \]
THE NILPOTENT SCHEME OF A CLASSICAL GROUP

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THE NILPOTENT SCHEME OF A CLASSICAL GROUP

\[ \Lambda_\gamma \times G_1^* \]

\[ \gamma(\lambda) \]

\[ \text{ord.} \]

REFERENCES


