SETS DEFINABLE OVER FINITE FIELDS:
THEIR ZETA-FUNCTIONS

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ABSTRACT. Sets definable over finite fields are introduced. The rationality of the logarithmic derivative of their zeta-function is established, an application of purely algebraic content is given. The ingredients used are a result of Dwork on algebraic varieties over finite fields and model-theoretic tools.

1. Introduction. In [6] Dwork proved the rationality of the zeta-function of a variety over a finite field. The main result of this paper is to extend this as far as possible to sets definable over finite fields. In this case, the zeta-function need no longer be rational, as illustrated by the set defined over the finite field with $p$ elements ($p$ odd prime) by the formula

$$3x(x^2 - y = 0).$$

However, the logarithmic derivative of the zeta-function, i.e., the Poincaré series, turns out always rational.

The result is found using model-theoretic tools: an extension by definitions of the theory of finite fields in ordinary field language in given: this extension is shown to admit elimination of quantifiers (by virtue of a generalization of the Shoenfield Quantifier Elimination Theorem [8]), this yields a characterization of sets definable over finite fields, and the Poincaré series for these can now be proved to be rational by some computations; although the zeta-function need not be rational, from the computation one can conclude that it can always be expressed as the radical of a rational function.

Unexplained notation follows Shoenfield [7] and Bell and Slomson [4].

2. A semantic characterization of elimination of quantifiers. Let $\tau$ be a similarity type, $L_\tau$ the first-order language of type $\tau$; let $\Lambda$ be a theory in language $L_\tau$.

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(1) The results presented in this paper are part of the author’s doctoral dissertation, written at the State University of New York at Stony Brook, under the supervision of James Ax; the author wishes to thank Professor Ax for encouragement and advice.
Definition 1. We say that $\Lambda$ satisfies the isomorphism condition if for every two models $A$ and $A'$ of $\Lambda$ and every isomorphism $\theta$ of substructures of $A$ and $A'$, there is an extension of $\theta$ which is an isomorphism of a submodel of $A$ and a submodel of $A'$.

Definition 2. We say that $\Lambda$ satisfies the submodel condition if for every model $B$ of $\Lambda$, every submodel $A$ of $B$, and every closed simply existential formula $\varphi$ of $L_{r,A}$, we have

$$A \models \varphi \iff B \models \varphi.$$ 

The following theorem is well known [8, p. 85]:

Quantifier Elimination Theorem. If $\Lambda$ satisfies the isomorphism condition and the submodel condition, then $\Lambda$ admits elimination of quantifiers.

The Quantifier Elimination Theorem gives a sufficient condition for a theory to admit elimination of quantifiers. However, this condition is not necessary, as is established by the following counterexample, due to Allan Adler.

Counterexample. Let $\Gamma$ denote the "theory of independent events", described as follows:

Language of $\Gamma$: no constant symbols
no function symbols
a countable set $\{\rho_n \mid n \in \omega\}$ of unary predicate symbols.

Axioms of $\Gamma$: for every ordered pair $(S, T)$ of finite subsets of $\omega$ such that $S \cap T$ is empty we have an axiom

$$A_{(S,T)}: (\exists x) \left( \bigwedge_{n \in S} \rho_n(x) \land \bigwedge_{n \in T} \neg \rho_n(x) \right).$$

$\Gamma$ admits elimination of quantifiers as can be proved by applying Lemma 3 in [8, p. 83]. To establish the counterexample one shows that $\Gamma$ does not satisfy the isomorphism condition: indeed, we define two subsets $M, N$ of $[0, 1]$ as follows:

First, we define sequences $\{M_n\}_{n \in \omega}, \{N_n\}_{n \in \omega}$ by $M_0 = N_0 = \{0\}$, if $M_0, \ldots, M_n, N_0, \ldots, N_n$ are known, choose $\xi_1, \ldots, \xi_{2^n+1}, \eta_1, \ldots, \eta_{2^n+1}$ in $[0, 1]$ such that all are irrational,

$$\xi_j, \eta_j \in [(j-1)/2^n+1, j/2^{n+1}] \quad (j = 1, \ldots, 2^n+1),$$

all are distinct, and none are contained in $M_n$ or $N_n$. We put $M_{n+1} = M_n \cup \{\xi_1, \ldots, \xi_{2^n+1}\}, N_{n+1} = N_n \cup \{\eta_1, \ldots, \eta_{2^n+1}\}$.

We now define $M = \bigcup_{n \in \omega} M_n, N = \bigcup_{n \in \omega} N_n$.

We make $M, N$ models of $\Gamma$ by interpreting $\rho_n(x)$ to mean that the $n$th
binary digit of x is 1. The axioms then simply require that M and N should each have nonempty intersection with each dyadic interval \([j/2^n, (j+1)/2^n]\), and are satisfied by construction.

\[ M_0 = N_0 = \{0\} \] are isomorphic substructures of M and N. However, any isomorphism of submodels of M and N must take an irrational number into itself. Since \( M \cap N = \{0\} \), the isomorphism condition fails.

The Quantifier Elimination Theorem is now going to be extended to a necessary and sufficient condition, therewith yielding a semantic characterization of the elimination of quantifiers. We need

**Definition 3.** We say that A satisfies the weak isomorphism condition if for every two models A and A' of A and every isomorphism \( \theta \) of a substructure of A and a substructure of A', there is an elementary extension A'' of A' and an extension of \( \theta \) which is an isomorphism of a submodel of A and a submodel of A''.

We then have

**Theorem 1.** A admits elimination of quantifiers if and only if A is model-complete and A satisfies the weak isomorphism condition.\(^{(2)}\)

**Proof.** \( \Leftarrow \): The techniques used in [8] to prove the Quantifier Elimination Theorem can easily be adapted to prove that quantifiers can be eliminated even with these weaker hypotheses.\(^{(2)}\)

\( \Rightarrow \): Model-completeness follows trivially.

3. A language in which the theory of finite fields admits elimination of quantifiers. We now describe a language and theory of finite fields in this language which admits elimination of quantifiers:

**Language:** function symbols: + (addition)
- (multiplication)
- (subtraction)

constant symbols: 1 (unity)
0 (additive identity)

predicate symbols: = (equality).

This language is the ordinary field language; henceforth, we denote it \( L_r \).

Now, we introduce for every positive integer \( n \) an \( n + 1 \)-ary predicate symbol: \( \varphi_n \). \( L_r' \) denotes the language obtained by adjoining the predicate symbols \( \{ \varphi_n \mid n \in \mathbb{Z}_{>0} \} \) to \( L_r \).

\( (2) \) Conversely, the necessity of these hypotheses follows easily by, e.g., an application of Frayne's Lemma \([4, \text{p. 161}]\).

It has been brought to my attention that Theorem 13.1 of \([7, \text{p. 63}]\) yields a characterization of elimination of quantifiers very close to this one. However, the one presented here appears to be somewhat more convenient for the purpose of this paper.
We now denote

$\Sigma$—the theory of finite fields in $L_\tau$ (i.e., the set of sentences of $L_\tau$

satisfied by all finite fields)

$\pi$—the theory of pseudo-finite fields in $L_\tau$ (i.e., the set of sentences

of $L_\tau$ satisfied by all the infinite models of $\Sigma$).

In [2, p. 255, Theorem 5], a recursive axiomatization for $\pi$ can be found.

Naturally, $\Sigma \subseteq \pi$, i.e., $F \models \pi \Rightarrow F \models \Sigma$.

Now, we let $\pi'$ and $\Sigma'$ be the theories in the language $L_{\tau'}$, obtained by taking

for axioms respectively

$$\pi \cup \{ \forall x_0 \cdots \forall x_n (\varphi_n(x_0, \ldots, x_n) \leftrightarrow \exists y(x_n y^n + \cdots + x_0 = 0) | n \in \mathbb{Z}_{>0} \}$$

and

$$\Sigma \cup \left\{ \forall x_0 \cdots \forall x_n \left( \left( \forall y_1 \cdots \forall y_n \left( \bigwedge_{i=1}^{n} x_i \neq y_i \land \forall y \left( \bigvee_{i=1}^{n} y = y_i \right) \right) \right) \right. \right.$$  

$$\left. \left. \rightarrow (\varphi_n(x_0, \ldots, x_n) \leftrightarrow \exists y(x_n y^n + \cdots + x_0 = 0)) \right) \left( \exists y_1 \cdots \exists y_n \left( \bigwedge_{i=1}^{n} y_i \neq y_i \land \forall y \left( \bigvee_{i=1}^{n} y = y_i \right) \right) \right. \right.$$  

$$\left. \left. \rightarrow (\varphi_n(x_0, \ldots, x_n) \leftrightarrow \forall y \left( y = 0 \lor \bigvee_{i=1}^{n-1} y = x_0 \right) \right) \right) \}$$  

Remarks. (a) $\Sigma'$ is an extension by definitions of $\Sigma$; given $F \models \Sigma$, $F$

becomes a model of $\Sigma'$ in a canonical way:

Case 1. $F$ is infinite—then we define the $n + 1$-ary relation $\varphi^F_n$ by

$$(a_0, \ldots, a_n) \in \varphi^F_n \iff \text{the polynomial } a_n y^n + \cdots + a_0 \text{ has a root in } F.$$  

Case 2. $F$ is finite with $k$ elements—then $\varphi^F_n$ is defined as before if $n \neq k$,

and $\varphi^k_F$ is defined by

$$(a_0, \ldots, a_k) \in \varphi^k_F \iff a_0 \text{ is a generator of } F^* \iff \text{(multiplicative subgroup of } F).$$  

(b) $F \models \pi' \iff F \models \Sigma'$ and $F$ is infinite,

(c) $F \models \Sigma' \Rightarrow (F \text{ finite with } k \text{ elements } \iff (0, 0, \ldots, 0, 1) \notin \varphi^k_F).$

Lemma 1. $\pi'$ admits elimination of quantifiers $\iff \Sigma'$ admits elimination

of quantifiers.
Proof. \( \Leftarrow \) : obvious, since \( \Sigma' \subseteq \pi' \).

\( \Rightarrow \) : by Theorem 1, it suffices to show that

(i) \( \pi' \) model-complete \( \Rightarrow \Sigma' \) model-complete, and
(ii) \( \pi' \) satisfies weak isomorphism condition \( \Rightarrow \Sigma' \) satisfies weak isomorphism condition.

(i) Let \( F_j \models \Sigma' \) \( (j = 1, 2) \) and \( F_1 \subseteq F_2 \).

If \( F_1 \) is infinite, \( F_j \models \pi' \) \( (j = 1, 2) \) and \( F_1 \subseteq F_2 \) follows from hypothesis.

If \( F_1 \) is finite with \( k \) elements,

\[
(1, 0, \ldots, 0, 1) \notin \varphi^{F_1}_k = \varphi^{F_2}_k \cap F_1^k
\]

\( \Rightarrow \) \( (1, 0, \ldots, 0, 1) \notin \varphi^{F_2}_k \Rightarrow F_2 \) finite \( k \) elements \( \Rightarrow F_1 = F_2 \).

(ii) Let \( F_j \models \Sigma' \) \( (j = 1, 2) \) and \( \theta \) an isomorphism of nonempty-substructures:

If both \( F_1 \) and \( F_2 \) are infinite, \( F_j \models \pi' \) and \( \theta \) can be extended by hypothesis.

If \( F_1 \) is finite with \( k \) elements, \( (1, 0, \ldots, 0, 1) \notin \varphi^{F_1}_k \Rightarrow (1, \ldots, 0, 1) \notin \varphi^{F_2}_k \) (because \( \theta \) is an isomorphism) \( \Rightarrow F_2 \) is finite with \( k \) elements. Hence \( \theta \) is an isomorphism of two subrings of two fields with \( k \) elements, the subrings containing the prime fields; so, obviously, \( \theta \) can be extended to the fields with \( k \) elements.

If \( F_2 \) is finite with \( k \) elements a similar reasoning holds.

Theorem 2. \( \pi' \) admits elimination of quantifiers.

Proof. By Theorem 1, this proof is immediately reduced to the proof of the following two lemmas:

Lemma 2. \( \pi' \) is model-complete.

Lemma 3. \( \pi' \) satisfies the weak isomorphism condition.

For the proofs of Lemmas 2 and 3 we need

Lemma 4. Let \( F_i \models \pi' \) \( (i = 1, 2) \), and assume that \( F_1 \) is a subfield of \( F_2 \); then \( F_1 \subseteq F_2 \) (i.e., for all \( n \in \mathbb{Z}_{>0} \), \( \varphi^{F_1}_n = \varphi^{F_2}_n \cap F_1^{n+1} \) \( \iff \) \( F_1 \) is relatively algebraically closed in \( F_2 \).

We also use

Lemma 5. Let \( \Lambda \) be a theory without finite models in a language of cardinality \( \aleph_0 \). Then: \( \Lambda \) model-complete \( \iff \) for any model \( A \models \Lambda \) of cardinality \( \aleph_0 \),

\[
\Lambda \cup \text{Diagram of } A \text{ is complete.}
\]
Proof. \(\Rightarrow\): obvious, from one of the current definitions of model-completeness.

\(\Leftarrow\): let \(\mathcal{B}_1, \mathcal{B}_2 \models \Lambda, \mathcal{B}_1 \subseteq \mathcal{B}_2\).

By Robinson's test for model-completeness; it suffices to show that if \(\phi\) is a primitive sentence in the language of \(\mathcal{B}_1\) and \(\mathcal{B}_2 \models \phi\), then \(\mathcal{B}_1 \models \phi\). Indeed: in \(\phi\) occur only a finite set \(S\) of constants designating elements of \(|\mathcal{B}_1|\). By Skolem-Loewenheim, we can extend \(S\) to a model \(\mathcal{B}_3 \models \Lambda\) such that \(S \subseteq |\mathcal{B}_3|\) and \(\mathcal{B}_3 \subseteq \mathcal{B}_1 \subseteq \mathcal{B}_2\) and \(\text{card} |\mathcal{B}_3| = \aleph_0\). By hypothesis, \(\text{Diag} \mathcal{B}_3 \cup \Lambda\) is complete. But

\[\mathcal{B}_2 \models \text{Diag} \mathcal{B}_3 \cup \Lambda, \quad \text{and}\]

\[\mathcal{B}_2 \models \phi, \quad \text{so}\]

\[\text{Diag} \mathcal{B}_3 \cup \Lambda \models \phi, \quad \text{hence} \mathcal{B}_3 \models \phi\]

and \(\mathcal{B}_3 \subseteq \mathcal{B}_1 \Rightarrow \mathcal{B}_1 \models \phi\). Q.E.D.

Proof of Lemma 2. Since \(\pi'\) has no finite models, by Lemma 5, to prove that \(\pi'\) is model-complete it suffices to show that \(\mathcal{F} \models \pi'\) and \(\text{card} \mathcal{F} = \aleph_0 \Rightarrow \pi' \cup \text{Diag} \mathcal{F}\) complete: Let \(\mathcal{F}_1, \mathcal{F}_2 \models \pi' \cup \text{Diag} \mathcal{F}\); we want to show that

\[\mathcal{F}_1 \equiv \mathcal{F}_2\]

(in language \(L_{\pi'}\) of \(\pi' \cup \text{Diag} \mathcal{F}\)).

We may assume that \(\mathcal{F}_i \subseteq \mathcal{F}_i\) \((i = 1, 2)\), and by Loewenheim-Skolem, we may assume \(\text{card} \mathcal{F}_i = \aleph_0\) \((i = 1, 2)\).

Now let \(D\) be a nonprincipal ultrafilter on the set of positive integers \(I\); let

\[e_i = \mathcal{F}_i/D \quad (i = 1, 2),\]

since \(e_i\) is pseudo-finite, \(e_i\) is hyper-finite; (cf. definition in [2, p. 246]) so we have \(\mathcal{F} \subseteq e_i \leq e_i\), with \(e_i\) hyper-finite; by Lemma 4, \(\mathcal{F}\) is relatively algebraically closed in \(e_i\) \((i = 1, 2)\); and also \(\text{card} e_1 = \text{card} e_2 > \text{card} \mathcal{F}\). Hence, by [2, p. 247, Theorem 1], \(e_1\) and \(e_2\) are isomorphic as fields over \(\mathcal{F}\); but this implies that they are isomorphic as structures of type \(\tau''\), since the \(\varphi_{e_i}\) relations are "algebraic", i.e., preserved under field-isomorphisms. Hence

\[\mathcal{F}_1 \leq e_1 \cong e_2 \geq \mathcal{F}_2, \quad \text{so}\]

\[\mathcal{F}_1 \equiv \mathcal{F}_2\]. Q.E.D.

Proof of Lemma 3. Let \(e_i \models \pi'\) \((i = 1, 2)\), \(\mathcal{D}_i \subseteq e_i\) and \(\theta: \mathcal{D}_1 \rightarrow \mathcal{D}_2\) be an isomorphism (of structures of type \(\tau'\)).

\(\mathcal{D}_i\) is a substructure of \(e_i\), hence an integral domain. Let \(\mathcal{F}_i\) be the quotient field of \(\mathcal{D}_i\): \(\mathcal{F}_i \subseteq e_i\), and certainly \(\theta\) extends to a field-isomorphism \(\theta: \mathcal{F}_1 \rightarrow \mathcal{F}_2\). \(\theta\) is also an isomorphism of structures of type \(\tau'\), as can be easily checked; so \(\theta\)
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has the following property:

\[ a_n x^n + \cdots + a_0 \in F_1[x] \text{ has a zero in } e_1 \]
\[ \iff \theta(a_n x^n + \cdots + a_0) \in F_2[x] \text{ has a zero in } e_2. \]

Now let \( \widetilde{F}_1^r \) be the relative algebraic closure of \( F_1 \) in \( e_1 \). Of course, we again have that

\[ a_n x^n + \cdots + a_0 \in F_1[x] \text{ has a zero in } \widetilde{F}_1^r \]
\[ \iff \theta(a_n x^n + \cdots + a_0) \in F_2[x] \text{ has a zero in } \widetilde{F}_2^r. \]

Hence by [1, p. 172, Lemma 5], we can extend \( \theta \) to a field-isomorphism \( \theta: \widetilde{F}_1^r \to \widetilde{F}_2^r \). \( \theta \) is still an isomorphism of structures of type \( r' \) because now

\[ (a_0, \ldots, a_n) \in \varphi_n \widetilde{F}_1^r = \varphi_n \widetilde{F}_1^{r+n+1} \iff a_n x^n + \cdots + a_0 \text{ has a zero in } \widetilde{F}_1^r \]
\[ \iff \theta(a_n) x^n + \cdots + \theta(a_0) \text{ has a zero in } \widetilde{F}_2^r \]
\[ \iff \theta(a_n) x^n + \cdots + \theta(a_0) \text{ has a zero in } e_2 \]
\[ \iff (\theta(a_0), \ldots, (a_n)) \in \varphi_n^{e_2} \cap F_1^{r+n+1} = \varphi_n F_1^r. \]

Let \( \alpha = \text{card } e_2 \). By upward Loewenheim-Skolem, let \( H'_2 \) be such that \( e_2 < H'_2 \) and card \( H'_2 = \alpha^+ \). Now, let \( H_2 \) be such that \( e_2 < H_2 < H_2, \) card \( H_2 = 2\alpha \) and \( H_2 \) is \( \alpha^+ \)-saturated [4, Theorem 11.1.7].

Then we have that \( e_2 < H_2, H_2 \) is hyper-finite, card \( H_2 = 2\alpha \) and \( \widetilde{F}_2^r \) is relatively algebraically closed in \( H_2 \) (because \( e_2 < H_2 \)).

Let \( \beta = \text{card } \widetilde{F}_1^r = \text{card } \widetilde{F}_2^r < \alpha < 2\alpha; \) by downward Loewenheim-Skolem, let \( H_1 \) be such that \( \widetilde{F}_1^r \subseteq H_1 < e_1 \) and card \( H_1 = \beta \). Then we know that \( H_1 \) is quasi-finite (because \( H_1 < e_1 \Rightarrow H_1 \vdash \pi' \), card \( H_1 < \text{card } H_2 \), and \( \widetilde{F}_1^r \) is relatively algebraically closed in \( H_1 \). So by [2, Lemma 2] we can extend \( \theta \) to a field-monomorphism \( \theta: H_1 \to H_2 \) such that \( \theta(H_1) \) is relatively algebraically closed in \( H_2 \).

If we take \( \varphi_n^{\theta(H_1)} \) to be defined on \( \theta(H_1) \) through \( \theta \), we get, since \( H_1 \vdash \pi' \), that \( \theta(H_1) \vdash \pi' \). But now \( H_2, \theta(H_1) \vdash \pi', \theta(H_1) \) is a subfield of \( H_2 \), and is relatively algebraically closed in \( H_2 \). Then Lemma 4 applies to show that \( \theta(H_1) \subseteq H_2 \), i.e., with \( \varphi_n^{\theta(H_1)} \) defined as above, \( \theta(H_1) \) is a submodel of \( H_2 \). Hence we have proved the weak isomorphism condition. Q.E.D.

4. Sets definable over a finite field: the rationality of their Poincaré series.

In this section, we shall use the following

**Notation.** \( L_o \)—ordinary field language, as described in §3.

\( L_r \)—ordinary field language with all the \( n + 1 \)-ary predicate symbols \( \varphi_n \)

adjoined (\( n \in \mathbb{Z}_{>0} \)).
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Σ-theory of finite fields in \( L_\tau \).

\( \Sigma' \)-theory of finite fields with defining axioms for \( \varphi_n \) adjoined (as in §3).

\( k \)-finite field of cardinality \( q \).

\( L_{r,k} \) with \( q \) new constant symbol adjoined.

\( k_s \)-unique extension of \( k \) of degree \( s \).

\( k \)-algebraic closure of \( k \).

**DEFINITION 4.** Let \( U = \{ U_s \}_{s \in \mathbb{Z}_+} \) with \( U_s \subset k_s^r, \forall s \in \mathbb{Z}_+ \); then \( U \) is called a **definable** \( r \)-set over \( k \) \( \iff \) there exists a formula \( \varphi \) in \( L_{r,k} \) with \( r \) free variables such that 

\[
U_s = \{ (a_1, \ldots, a_r) \in k_s^r | k_s \models \varphi[a_1, \ldots, a_r] \}, \quad \forall s \in \mathbb{Z}_+.
\]

We then say that \( U \) is **defined** by \( \varphi \).

**REMARK.** If \( U \) is definable over \( k \), the formula defining \( U \) is not unique: in fact, every formula representing the same element in the \( r \)th Lindenbaum algebra of \( \Sigma \) will also define \( U \).

**DEFINITION 5.** Say \( U \) is a definable \( r \)-set, defined by \( \varphi \). We have \( U_s = \{ (a_1, \ldots, a_r) \in k_s^r | k_s \models \varphi[a_1, \ldots, a_r] \} \); the **zeta-function** of \( U \) is defined to be the formal power series in \( t \)

\[
\zeta_U(t) = \exp \sum_{s=1}^{\infty} \frac{N_s(U)}{s} t^s,
\]

where \( N_s(U) = \#U_s \) = cardinality of \( U_s \). Following terminology used in \[5, p. 47\] we let the **Poincaré series** of \( U \) be defined by

\[
\pi_U(t) = t \frac{d}{dt} \log \zeta_U(t) = \sum_{s=1}^{\infty} N_s(U) t^s.
\]

The main result of this section is

**THEOREM 3.** The Poincaré series of a definable set is rational. \((3)\)

**DEFINITION 6.** A definable \( r \)-set \( V \) over \( k \) will be called a **variety** over \( k \) if it can be defined by a formula of type

\[
\bigwedge_{i=1}^{n} p_i(x_1, \ldots, x_r) = 0, \quad \text{with}
\]

\[
p_i(x_1, \ldots, x_r) \in k[x_1, \ldots, x_r] \quad (i = 1, \ldots, n).
\]

**DEFINITION 7.** A definable \( r \)-set will be called **primitive** if it can be defined by a formula of type

\[(3) \text{ As usual, a formal power series is called rational when it is the quotient of two polynomials.}\]
A definable set will be called constructible if it can be defined by a formula which is quantifier free in $L_{r,k}$.

Definition 9. Let $U = \{ U_s \}_{s \in \mathbb{Z}_{>0}}$ and $V = \{ V_s \}_{s \in \mathbb{Z}_{>0}}$ be definable $r$-sets. We define the union, intersection and difference of $U$ and $V$ “pointwise”, i.e., by

\[
(U \cup V)_s = U_s \cup V_s, \quad (U \cap V)_s = U_s \cap V_s, \quad (U - V)_s = U_s - V_s, \quad \forall s \in \mathbb{Z}_{>0}.
\]

Lemma 6. If $U$ is a constructible set, then $\xi_U(t)$ is a rational function. Hence, so is $\pi_U(t)$.

Proof. Dwork [6] showed that $\xi_{V - W}(t)$ is rational, for $V, W$ varieties.

Any primitive set $P_n$ is a difference of varieties: in fact, if $P$ is defined by $\bigwedge_{i=1}^n p_i(\overline{x}) = 0 \land \bigwedge_{j=1}^m q_j(\overline{x}) \neq 0$, we have that

\[
\Sigma \vdash \left( \bigwedge_{i=1}^n p_i(\overline{x}) \land \bigwedge_{j=1}^m q_j(\overline{x}) \neq 0 \right) \leftrightarrow \left( \bigwedge_{i=1}^n p_i(\overline{x}) = 0 \land \prod_{j=1}^m q_j \neq 0 \right).
\]

So if $V$ is defined by $\bigwedge_{i=1}^n p_i(\overline{x}) = 0$ and $W$ is defined by $(\prod_{j=1}^m q_j(\overline{x})) = 0$, then $P = V - W$. So the Lemma holds for primitive sets.

Now observe that the intersection of primitive sets is primitive; on the other hand, any constructible set is the union of primitive sets, i.e., if $U$ is constructible, there exist primitive sets $P_1, \ldots, P_n$ such that $U = \bigcup_{i=1}^n P_i$ and so $U_s = \bigcup_{i=1}^n (P_i)_s$; it is easily verified that

\[
\# \left( \bigcup_{i=1}^n (P_i)_s \right) = \sum_{\phi \subseteq \{1, \ldots, n\}} (-1)^{\#B+1} \# \left( \bigcap_{i \in B} (P_i)_s \right), \quad \text{i.e.,}
\]

\[
N_s(U) = \sum_{\phi \subseteq \{1, \ldots, n\}} (-1)^{\#B+1} N_s \left( \bigcap_{i \in B} P_i \right) = \sum_{\phi \subseteq \{1, \ldots, n\}} (-1)^{\#B+1} N_s(P_B),
\]

where $P_B = \bigcap_{i \in B} P_i$, for all $B \subseteq \{1, \ldots, n\}$. But $P_B$ is a primitive set, hence $\xi_{P_B}(t)$ is rational, so

\[
\xi_U(t) = \prod_{\phi \subseteq \{1, \ldots, n\}} \xi_{P_B}(t)(-1)^{\#B+1}
\]

is rational. Q.E.D.

We shall now reduce the proof of Theorem 3 to
**Lemma 8.** Let $U \subseteq k'$ be definable, defined by an atomic formula in $L_{r,k}$ of type

$$\varphi_n(p_0(x_1, \ldots, x_r), \ldots, p_n(x_1, \ldots, x_r)),$$

with $p_i(x_1, \ldots, x_r) \in k[x_1, \ldots, x_r]$ ($i = 1, \ldots, n$) (obviously, we mean that $U$ is defined by a formula of $L_{r,k}$ equivalent to $\varphi_n(p_0(x), \ldots, p_n(x))$; then $\pi_U(t)$ is rational.

Before we prove Lemma 8, we shall reduce the proof of Theorem 3 to it, i.e., show that Theorem 3 follows from Lemmas 7 and 8.

Let $U$ be a definable set; it has been proved in §3 that $\Sigma'$ admits elimination of quantifiers, hence we may assume $U$ defined by a quantifier-free formula $\varphi$ in the language $L_{r,k}$, i.e., $U$ is the union of sets defined by formulae of type

$$\bigwedge_{i=1}^\mu p_i(x) = 0 \land \bigwedge_{j=1}^\nu \varphi_{n_j}(p_{n_j,0}(x), \ldots, p_{n_j,n_j}(x)) \land \bigwedge_{k=1}^\xi q_k(x)$$

(\*)

Again, since intersections of sets defined by formulae of type (\*) are again defined by formulae of type (\*), it will suffice to prove that the $\xi$-functions of sets defined by formulae of type (\*) have the required property.

We are now reduced to sets $U$ defined by formulae of type (\*). To proceed, we start by freeing ourselves from the restrictions imposed by the defining axiom for $\varphi_m$ in case we are interpreting this relation in a field with $m$ elements.

**Lemma 9.** Let $U$ be defined by a formula $\varphi$ of type (\*). Let $\psi'$ be obtained from $U$ by replacing each occurrence of $\varphi_m(p_{m,0}(x), \ldots, p_{m,m}(x))$ by $3 z(p_{m,0}(x) + \cdots + p_{m,m}(x)z^m = 0)$. Let $U'$ be the set defined by $\psi'$. Then, if $\pi_U(t)$ is rational, so is $\pi_{U'}(t)$.

**Proof.** Let

$$A = \{ m \in \mathbb{Z}_{>0} \mid \varphi_m \text{ occurs in } \varphi \text{ and } m = q^s, \text{ for some } s \in \mathbb{Z}_{>0} \},$$

$$B = \{ s \in \mathbb{Z}_{>0} \mid q^s = m, \text{ for some } m \in A \}.$$

If $B = \emptyset$, $\forall s \in \mathbb{Z}_{>0}$, $U_s = U'_s$ hence $N_s(U) = N_s(U')$ and the result is obvious. But if $B \neq \emptyset$, it certainly is finite. Also, $\forall s \in \mathbb{Z}_{>0}, s \notin B \Rightarrow N_s(U) = N_s(U')$. Hence $\pi_U(t) = \Sigma_{s=1}^\infty N_s(U) t^s = \Sigma_{s=1}^\infty N_s(U') t^s + \Sigma_{s \in B} N_s(U') t^s$. From the finiteness of $B$ and the rationality of $\Sigma_{s=1}^\infty N_s(U') t^s$
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we immediately conclude the rationality of \( \pi_U(t) \). Q.E.D.

So in everything that follows we may replace \( \varphi_m(p_{m,0}, \ldots, p_{m,m}) \) by

\[
\exists z(p_{m,0} + \cdots + p_{m,m}z^m = 0).
\]

As before, in formulae of type (**) we may assume \( \xi \leq 1 \) by replacing

\[
\wedge_{k=1}^{\xi} q_k(\overline{x}) \not= 0 \text{ by } \prod_{k=1}^{\xi} q_k(\overline{x}) \not= 0; \text{ similarly. We may assume } \eta \leq 1; \text{ indeed:}
\]

\[
\Sigma \vdash \bigwedge_{m=1}^{\eta} \exists z(p_{nm,0}(\overline{x}) + \cdots + p_{nm,n}(\overline{x})z^n = 0)
\]

\[
\iff \exists z\left(\prod_{m=1}^{\eta} (p_{nm,0}(\overline{x}) + \cdots + p_{nm,n}(\overline{x})z^n) = 0.\right.
\]

Furthermore, we can always assume \( \xi = 0 \):

\[
\Sigma \vdash q(\overline{x}) \not= 0 \wedge \forall \varphi_n(p_0(\overline{x}), \ldots, p_n(\overline{x})) \iff q(\overline{x})
\]

\[
\not= 0 \wedge \exists z(p_0(\overline{x}) + \cdots + p_n(\overline{x})z^n = 0),
\]

\[
\Sigma \vdash q(\overline{x}) \not= 0 \wedge \exists z(p_0(\overline{x}) + \cdots + p_n(\overline{x})z^n = 0)
\]

\[
\iff \exists z(q(\overline{x})(p_n(\overline{x})z^n + \cdots + p_0(\overline{x}))),
\]

\[
\Sigma \vdash \exists z(q(\overline{x})(p_n(\overline{x})z^n + \cdots + p_0(\overline{x})) = 0)
\]

\[
\iff \neg \varphi_n(q(\overline{x}), \ldots, q(\overline{x})p_n(\overline{x})).
\]

Should \( \eta = 0 \), we can always introduce the conjunct \( \neg \varphi_1(1.0) \). So, we may assume \( \xi = 0, \eta \leq 1 \). We are now reduced to showing our result for sets defined by formulae of type

\[
(**) \quad \bigwedge_{i=1}^{\mu} p_i(\overline{x}) = 0 \wedge \bigwedge_{j=\mu+1}^{\nu} \varphi_{n_j}(p_{n_j,0}(\overline{x}), \ldots, p_{n_j,n}(\overline{x})).
\]

Indeed, if we get it for this case, then if we consider the set \( U \) defined by

\[
\bigwedge_{i=1}^{\mu} p_i(\overline{x}) = 0 \wedge \bigwedge_{j=1}^{\nu} \varphi_{n_j}(\cdots, \cdots) \wedge \forall \varphi_n(\cdots, \cdots),
\]

we observe that \( U = V - W \), where \( V \) is defined by a formula of type (**) and \( W \) by \( \varphi_n(\cdots, \cdots) \), so \( N_s(U) = N_s(V) - N_s(V \cap W) \), where \( V \cap W \) is again defined by a formula of type (**).

Now to prove the result for a set \( U \) defined by (**) it will suffice to establish the following:

**Claim.** Let \( V_i \) be defined by \( p_i(\overline{x}) = 0 \) \( (i = 1, \ldots, \mu) \) and by

\[
\varphi_{n_i}(p_{n_i,0}(\overline{x}), \ldots, p_{n_i,n}(\overline{x})) \text{ for } i = \mu + 1, \ldots, \nu. \text{ Then for all } B \subseteq \{1, \ldots, \nu\}, \quad V_B = \bigcup_{i \in B} V_i \text{ is a set such that } d/dt \log \xi_{V_B}(t) \text{ is rational.}
\]

Suppose we have proved the Claim: then
\[ N_s(U) = \# \left( \bigcap_{i=1}^{\nu} (V_i)_s \right) = \sum_{B \subseteq \{1, \ldots, \nu\}} (-1)^{|B|} \# (V_B)_s \]
\[ = \sum_{B \subseteq \{1, \ldots, \nu\}} (-1)^{|B|} N_s(V_B). \]

Now to prove the Claim:

Let

\[ B_1 = B \cap \{1, \ldots, \nu\}, \]
\[ B_2 = B \cap \sum_{\{\mu + 1, \ldots, \nu\}} V_B = \bigcup_{i \in B_1} V_i \cup \bigcup_{i \in B_2} V_i \]

but \( \bigcup_{i \in B_1} V_i \) can be defined by \( \prod_{i \in B_1} p_i(\vec{x}) = 0 \), and \( \bigcup_{i \in B_2} V_i \) can be defined by

\[ \exists z \left( \prod_{i \in B_2} (p_{n_j} z_j^n + \cdots + p_{n_j} 0) = 0 \right), \]

i.e., by \( \varphi_n(q_0(\vec{x}), \ldots, q_\nu(\vec{x})) \), where \( n = \sum_{j \in B_2} n_j \) and the \( q_i(\vec{x}) \) are adequately computed.

Hence \( V_B \) is defined by

\[ \prod_{i \in B_1} p_i(\vec{x}) = 0 \lor \varphi_n(q_0(\vec{x}), \ldots, q_\nu(\vec{x})), \]

hence by

\[ \exists z \left( \pi p_i(\vec{x}) q_\nu(\vec{x}) z^n + \cdots + \pi p_i(\vec{x}) q_0(\vec{x}) = 0 \right), \]

hence by

\[ \varphi_n(\pi p_i(\vec{x}) q_\nu(\vec{x}), \ldots, \pi p_i(\vec{x}) q_0(\vec{x})), \]

and the proof of Theorem 3 is actually reduced to Lemma 8.

**Proof of Lemma 8.** Let \( U \) be defined by

\[ \varphi_n(p_0(x_1, \ldots, x_r), \ldots, p_n(x_1, \ldots, x_r)); \]

by Lemma 9 we may assume \( n > q \):

\[ U_s = \{(a_1, \ldots, a_r) \in k^n_s \mid \text{there exists } b \in k_s \text{ such that } p_n(b^n + \cdots + p_0(\bar{a}) = 0). \}

Let \( f(x_1, \ldots, x_r, z) = p_0(x_1, \ldots, x_r) + \cdots + p_n(x_1, \ldots, x_r) z^n \in k[x_1, \ldots, x_r, z] \). Let \( V \) be the variety in \( k^{r+1} \) defined by \( f(\vec{x}, z) = 0 \):

\[ V_s = \{(\bar{a}, b) \in k^{r+1}_s \mid f(\bar{a}, b) = 0 \}. \]

Let
\[ V_{s,i} = \{ (\bar{a}, b) \in k_s^{r+1} | p_n(\bar{a})z^n + \cdots + p_0(\bar{a}) \text{ has } i \text{ distinct roots in } k_s \text{ and } b \text{ is one of them} \} \]

\[(i = 1, \ldots, n); \text{ obviously, we have } V_s = \bigcup_{i=1}^{n} V_{s,i} \text{ and we observe that} \]

\[ N_s(U) = \#U_s = \sum_{i=1}^{n} \frac{\#V_{s,i}}{i}. \]

Now let \( H_i \) be the constructible \( r + i \) set defined by

\[ f(\bar{x}, z_1) = 0 \land \cdots \land f(\bar{x}, z_i) = 0 \land \bigwedge_{k,m=1}^{i} z_k - z_m \neq 0. \]

By Lemma 6, \( \xi_{H_i} \) is rational. We also have \( (H_i)_s = \{(\bar{a}, \bar{b}) \in k_s^{r+1} | f(\bar{a}, \bar{b}, z_k) = 0 \text{ for } k = 1, \ldots, i \text{ and } b_k \neq b_m \text{ if } k \neq m\} \). Our aim is to compute \( \#V_{s,i} \)
from \( N_s(H_i) \). For this purpose, let

\[ E_{s,i} = \{ (\bar{a}, b) \in (H_i)_s | f(\bar{a}, \bar{b}, z_k) \text{ has exactly } i \text{ distinct roots in } k_s \}, \]

\[ F_{s,i} = \{ (\bar{a}, b) \in (H_i)_s | f(\bar{a}, \bar{b}, z_k) \text{ has } > i \text{ distinct roots in } k_s \}. \]

Of course, \( (H_i)_s = E_{s,i} \cup F_{s,i} \) and also

\[ \#\{ \bar{a} \in k_s^r | f(\bar{a}, \bar{b}, z_k) \text{ has exactly } i \text{ roots in } k_s \} = \frac{1}{i!} \#E_{s,i} = \frac{\#V_{s,i}}{i}, \]

hence \( \#V_{s,i} = \#E_{s,i} / (i - 1)! \), and if we can compute \( \#E_{s,i} = N_s(H_i) - \#F_{s,i} \)
adequately, we are through.

Indeed, consider the map

\[ \pi_i : \bigcup_{k=i+1}^{n} E_{s,k} \rightarrow F_{s,i}, \]

\[(\bar{a}, b_1, \ldots, b_i, \ldots, b_k) \mapsto (\bar{a}, b_1, \ldots, b_i). \]

\( \pi_i \) is certainly surjective and also

\[ k \neq k' \Rightarrow \pi_i(E_{s,k}) \cap \pi_i(E_{s,k'}) = \emptyset \]

(Indeed: \( (\bar{a}, b_1, \ldots, b_i) \in \pi_i(E_{s,k}) \Rightarrow f(\bar{a}, \bar{b}, z_k) \text{ has exactly } k \text{ roots} \)). So

\[ F_{s,i} = \bigcup_{k=i+1}^{n} \pi_i(E_{s,k}), \text{ hence} \]

\[ \#F_{s,i} = \sum_{k=i+1}^{n} \#\pi_i(E_{s,k}). \]

But for \( k = i + 1, \ldots, n, \#E_{s,k} / (k - i)! = \#\pi_i(E_{s,k}); \text{ hence } \#E_{s,i} = N_s(H_i) - \]
\[ \#E_{s,i} = N_s(H_i) - \sum_{j=i+1}^{n} \#E_{s,j}(j - i)! \] but we also know that \[ \#E_{s,n} = N_s(H_n) \] (from the definitions) and so we get
\[
\#V_{s,n} = \frac{1}{(n - 1)!} N_s(H_n),
\]
\[
\#V_{s,i} = \frac{1}{(i - 1)!} \#E_{s,i} = \frac{1}{(i - 1)!} \left( N_s(H_i) - \sum_{j=i+1}^{n} (j - i)! \#V_{s,j} \right)
\]

\[(i = 1, \ldots, n - 1).\]

This certainly determines each \( \#V_{s,i} \) as a linear combination of the \( N_s(H_j) \) \((j = 1, \ldots, n)\) with rational coefficients (independent of \( s \)); hence
\[
N_s(U) = \sum_{i=1}^{n} \frac{\#V_{s,i}}{i}
\]
is given by a linear combination of the \( N_s(H_j) \) with rational coefficients, independent of \( s \); hence the rationality of \( \sum N_s(U) t^s \) follows from the rationality of \( \sum N_s(H_j) t^s \). Q.E.D.

Remark. The proof yields that \( \pi_U(t) \) is rational for any definable set \( U \). Certainly, \( \xi_U(t) \) may not be rational. However, this proof also shows that \( \xi_U(t) \) is always algebraic, indeed, it can always be written as the radical of a rational function.

5. Application. Let us consider the following:

Definition 10. Let \( \theta: \tilde{k}^r \rightarrow \tilde{k}^t \) be a function; suppose we can find a \( t \)-tuple of polynomials \( f_1, \ldots, f_t \in \tilde{k}[x_1, \ldots, x_r] \) such that for all \( (a_1, \ldots, a_r) \in \tilde{k}^r \), \( \theta(a_1, \ldots, a_r) = (f_1(a_1, \ldots, a_r), \ldots, f_t(a_1, \ldots, a_r)) \); then \( \theta \) is called an \( r - t \)-morphism over \( k \), and the \( t \)-tuple \( (f_1, \ldots, f_t) \) is said to define \( \theta \).

We can state the following

Lemma 10. If \( U \) is a definable \( r \)-set over \( k \), and \( \theta \) is an \( r - t \)-morphism over \( k \), then \( \theta(U) \) is a definable \( t \)-set over \( k \).

Proof. Say \( U \) is defined by the formula \( \varphi(x_1, \ldots, x_r) \) of \( L_{r,k} \) and \( \theta \) by the \( t \)-tuple \( (f_1(x_1, \ldots, x_r), \ldots, f_t(x_1, \ldots, x_r)) \). Then it is trivial to check that \( \theta(U) \) can be defined by the formula \( \psi(y_1, \ldots, y_t) \) given by
\[
\begin{align*}
3 x_1 & \ldots \ 3 x_r (y_1 = f_1(x_1, \ldots, x_r) \land \cdots \land y_t) \\
& = f_t(x_1, \ldots, x_r) \land \varphi(x_1, \ldots, x_r)). \quad \text{Q.E.D.}
\end{align*}
\]

In particular, we get the following generalization of Dwork’s result:

The logarithmic derivative of the zeta-function of the image of a variety by a morphism is rational.
REFERENCES


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