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ON BOUNDED FUNCTIONS SATISFYING AVERAGING CONDITIONS. II

BY

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ABSTRACT. Let $S(f)$ denote the subspace of $L^\infty(R^n)$ consisting of those real valued functions f for which

$$\lim_{r\to 0} \frac{1}{|B(x, r)|} \int_{B(x,r)} f(y)\, dy = f(x)$$

for all x in R^n and let $L(f)$ be the subspace of $S(f)$ consisting of the approximately continuous functions. A number of results concerning the existence of functions in $S(f)$ and $L(f)$ with special properties are obtained. The extreme points of the unit balls of both spaces are characterized and it is shown that $L(f)$ is not a dual space. As a preliminary step, it is shown that if E is a G_δ set of measure 0 in R^n, then the complement of E can be decomposed into a collection of closed sets in a particularly useful way.

Introduction. Let $L_R^\infty(R^n)$ denote the space of all real valued $L^\infty(R^n)$ functions. If f is in $L_R^\infty(R^n)$ and if E is a measurable subset of R^n, let $J(f, E)$ denote $\int_E f$. For each f in $L_R^\infty(R^n)$ define:

$$L(f) = \left\{ x \in R^n \,\middle|\, \lim_{r\to 0} (J(|f - f(x)|, B(x, r))/|B(x, r)|) = 0 \right\}$$

where $B(x, r) = \{y \in R^n \mid |y - x| < r\}$, i.e. $L(f)$ is the Lebesgue set of f.

$$S(f) = \left\{ x \in R^n \,\middle|\, \lim_{r\to 0} (J(f, B(x, r))/|B(x, r)|) = f(x) \right\}.$$

Let $S(n, T)$ be the subspace of $L_R^\infty(R^n)$ consisting of those functions f for which $S(f) = R^n$, and let $L(n, T)$ be the subspace of $L_R^\infty(R^n)$ consisting of those functions for which $L(f) = R^n$.

A function f in $L_R^\infty(R^n)$ is defined to be approximately continuous at x if x is a point of density of $\{y \mid |f(y) - f(x)| < \epsilon\}$ for every $\epsilon > 0$. It is easy to see that $L(n, T)$ consists precisely of those functions in $L_R^\infty(R^n)$ which are approximately continuous at each point of R^n. An example of a function which is in $S(n, T)$ but not in $L(n, T)$ is the function

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$$f(x) = \begin{cases} \sin(1/|x|^n), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

The same example shows that $S(n, T)$ is not an algebra, whereas it is readily shown that $L(n, T)$ is an algebra.

In this paper a number of results will be obtained about the existence of functions in $S(n, T)$ and $L(n, T)$ which have special properties. The extreme points of the unit balls of these spaces will also be characterized. In the case of $L(n, T)$ it will be shown that there are only two such extreme points.

The proofs depend primarily on the fact that if E is a G_δ subset of measure 0 contained in R^n, then E', the complement of E, can be decomposed in a special way into a collection of closed sets $\{\Phi_\lambda\}_{\lambda>1}$ so that the function μ defined in R^n by

$$\mu(x) = \begin{cases} 0, & x \in E, \\ 1/\inf_\lambda \{\lambda | x \in \Phi_\lambda\}, & x \notin E, \end{cases}$$

is approximately continuous and has a number of other useful properties. It will first be shown how to obtain such a decomposition of E'. The procedure used generalizes a method developed by Zygmunt Zahorski for obtaining a decomposition of the complement of a G_δ set of measure 0 contained in the open interval (0, 1) [2].

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Inverse Zahorski functions.

LEMMA 1. *Let M_1 and M_2 be two bounded measurable subsets of R^n with measures u_1 and u_2 respectively. Suppose that M_2 is a closed subset of M_1 consisting only of points of density of M_1. Then for every positive number p, there is a closed set M_p with $M_2 \subset M_p \subset M_1$ satisfying:*

(1) *Every point of M_2 is a point of density of M_p and every point of M_p is a point of density of M_1.*

(2) $|M_p| \geq u_2 + (1 - 2^{-1-p})(u_1 - u_2)$.

(3) *Let $x \in M_2$, and let ϵ be an arbitrary number in* (0, 1). *If r is any positive number for which $(|M_1 \cap B(x, r)|/|B(x, r)|) \geq 1 - \epsilon$, then*

$$(|M_p \cap B(x, r)|/|B(x, r)|) \geq 1 - \epsilon - 2^{-m-p+c_n}$$

for every positive integer m for which $r \leq 1/m$, where c_n is a constant which depends only on the dimension n.

PROOF. H. Whitney has shown that since M_2 is closed, M_2' is a countable union of closed cubes Q_k with disjoint interiors, where these cubes may be chosen so that the following conditions hold:

(1) diam $Q_k \leqslant \operatorname{dist}(Q_k, M_2) \leqslant 4$ diam Q_k.

(2) If Q_k^* is the cube with the same center as Q_k and expanded by a factor $1 + \epsilon$ ($0 < \epsilon < 1/4$, ϵ fixed), then Q_k^* is contained in the union of all the cubes which touch Q_k.

(3) For each cube Q_k there are at most $N = (12)^n$ cubes which touch Q_k [1, pp. 167–169].

A cube Q_k will be said to be of class m, m a positive integer, if either $(1/(m + 1)) <$ diam $Q_k \leqslant (1/m)$ or $m <$ diam $Q_k \leqslant m + 1$. If Q_k is of class m and if $|Q_k \cap M_1| > 0$, let F_k be a closed subset of $Q_k \cap M_1$ consisting only of points of density of M_1, with $|F_k| \geqslant |Q_k \cap M_1|(1 - 2^{-m-p})$. Set $M_p = M_2 \cup_k F_k$.

It will be shown that M_p satisfies all the required conditions. First, M_p is closed, for if $\{q_m\}_{m\geqslant 1}$ is a convergent sequence in M_p, say $q_m \longrightarrow q$, and if $q \notin M_2$, then q is in some cube Q_s and some neighborhood of q is contained in Q_s^*. Since Q_s^* is contained in the union of at most N cubes Q_k, this neighborhood is contained in the union of at most N cubes. Thus for m sufficiently large, say $m \geqslant M$, $\{q_m\}_{m\geqslant M}$ is contained in at most N of the sets F_k. Since this union is closed, q is in some F_k and hence M_p is closed.

By construction, each point of M_p is a point of density of M_1. It will now be shown that each point of M_2 is a point of density of M_p. The proof will be such that (3) will be proved simultaneously.

Let x be in M_2. Let ϵ be an arbitrary number in $(0, 1)$ and let m be an arbitrary positive integer. Since, by assumption, x is a point of density of M_1, there is a $0 < \delta \leqslant 1/m$ such that for $r \leqslant \delta$, $(|B(x, r) \cap M_1|/|B(x, r)|) > 1 - \epsilon$. Set

$$d(x, r) = (|M_1 \cap B(x, r)|/|B(x, r)|) - (|M_p \cap B(x, r)|/|B(x, r)|).$$

It will be shown that $d(x, r) \leqslant 2^{-m-p+c_n}$. From this it follows that $(|M_p \cap B(x, r)|/|B(x, r)|) > 1 - \epsilon - 2^{-p+c_n-m}$, which verifies (1) since ϵ and m were arbitrary.

The proof that $d(x, r) \leqslant 2^{-m-p+c_n}$ will depend only on the fact that m is a positive integer for which $r < 1/m$. Thus (3) will also be proved.

Let K be the set of all integers for which Q_k has nonempty intersection with the boundary of $B(x, r)$ and set

$$A = \bigcup_{k\in K} Q_k; \quad A_1 = A \cap B(x, r); \quad A_2 = A \cap B(x, r)';$$

$$\xi = |M_p \cap A|/|M_1 \cap A| \text{ if } |M_1 \cap A| > 0, \qquad \xi = 1 \text{ if } |M_1 \cap A| = 0;$$

$$\xi_1 = |M_p \cap A_1|/|M_1 \cap A_1| \text{ if } |M_1 \cap A_1| > 0, \quad \xi_1 = 1 \text{ if } |M_1 \cap A_1| = 0;$$

$$\xi_2 = |M_p \cap A_2|/|M_1 \cap A_2| \text{ if } |M_1 \cap A_2| > 0, \quad \xi_2 = 1 \text{ if } |M_1 \cap A_2| = 0.$$

We have

$$d(x, r) = (1/|B(x, r)|)\left\{\sum_{Q_k \subset B(x,r)} |(Q_k \cap M_1) - F_k| + |(M_1 - M_p) \cap A_1|\right\}.$$

Since diam $Q_k < \text{dist}(Q_k, M_2) \leqslant r < 1/m$ for each cube Q_k which intersects $B(x, r)$, $|(Q_k \cap M_1) - F_k| < 2^{-m-p}|Q_k \cap M_1|$, and thus

$$d(x, r) \leqslant (1/|B(x, r)|)\left\{2^{-m-p} \sum_{Q_k \subset B(x,r)} |Q_k \cap M_1| + |(M_1 - M_p) \cap A_1|\right\}.$$

Thus if $|M_1 \cap A_1| = 0$, $d(x, r) \leqslant 2^{-m-p}$.

Suppose $|M_1 \cap A_1| > 0$. Observe that

$$d(x, r) \leqslant (1/|B(x, r)|)\left\{2^{-m-p} \sum_{Q_k \subset B(x,r)} |Q_k \cap M_1| + (1 - \xi_1)|M_1 \cap A_1|\right\}.$$

The object of the calculations which follow is to show that $1 - \xi_1 \leqslant 2^{-m-p}\{1 + (|A_2|/|M_1 \cap A_1|)\}$.

By solving the equation $\xi|M_1 \cap A| = \xi_1|M_1 \cap A_1| + \xi_2|M_1 \cap A_2|$ for ξ_1 and observing that $|M_1 \cap A_2| = |M_1 \cap A_1| + |M_1 \cap A_2|$, we obtain

$$\xi_1 = \xi - (\xi_2|M_1 \cap A_2| - \xi|M_1 \cap A_2|)/|M_1 \cap A_1|.$$

Since $|F_k| \geqslant |Q_k \cap M_1|(1 - 2^{-m-p})$ for each Q_k which intersects $B(x, r)$, $|M_p \cap A| \geqslant (1 - 2^{-m-p})|M_1 \cap A|$ and $\xi > 1 - 2^{-m-p}$. Thus

$$\xi_1 > 1 - 2^{-m-p} - \{(\xi_2 - (1 - 2^{-m-p}))/|M_1 \cap A_1|\}|M_1 \cap A_2|.$$

Since $0 < \xi_2 \leqslant 1$ and $|M_1 \cap A_2| \leqslant |A_2|$,

$$\xi_1 > 1 - 2^{-m-p} - 2^{-m-p}|A_2|/|M_1 \cap A_1|,$$

and

$$1 - \xi_1 \leqslant 2^{-m-p}\{1 + (|A_2|/|M_1 \cap A_1|)\}.$$

It follows that

$$\begin{aligned} d(x, r) &\leqslant 2^{-m-p}\{1 + \{|M_1 \cap A_1|(1 + (|A_2|/|M_1 \cap A_1|))\}/|B(x, r)|\} \\ &\leqslant 2^{-m-p}\{2 + (|A_2|/|B(x, r)|)\}. \end{aligned}$$

Since diam $Q_k \leqslant r$ for each Q_k which intersects $B(x, r)$, $A_2 \subset B(x, 2r) - B(x, r)$. Thus,

$$(|A_2|/|B(x, r)|) \leqslant (1/|B(x, r)|)(|B(x, 2r)| - |B(x, r)|).$$

The number $d_n = (1/|B(x, r)|)(|B(x, 2r)| - |B(x, r)|)$ depends only on n and $d(x, r) \leqslant 2^{-m-p}(2 + d_n) \leqslant 2^{-m-p+c_n}$, where $2^{c_n} = 2 + d_n$. Therefore, (1) and (3) both hold.

Finally,

$$|M_p| = |M_2| + \sum_k |F_k| \geqslant u_2 + \sum_k |Q_k \cap M_1|(1 - 2^{-m-p})$$
$$> u_2 + (1 - 2^{-1-p}) \sum_k |Q_k \cap M_1| = u_2 + (1 - 2^{-1-p})(u_1 - u_2),$$

so that (2) also holds. Q.E.D.

COROLLARY 1. *For each G_δ set E, of measure 0 in R^n, there is an increasing sequence of compact sets $\{F_k\}_{k>1}$ with $|\Phi_k| > k$ such that $E' = \bigcup_k \Phi_k$ and $|B(x, r) \cap \Phi_{k+1}|/|B(x, r)| > 1 - 2^{-m-k+c_n}$ whenever $x \in \Phi_k$ and $r \leqslant 1/m$, m a positive integer.*

PROOF. Since E is a G_δ of measure 0, there exists an increasing sequence of closed sets $\{F_k\}_{k\geqslant 1}$ with $E' = \bigcup_{k\geqslant 1} F_k$. Let $\{a_k\}_{k\geqslant 1}$ be a strictly increasing sequence of positive numbers for which $|B(0, a_k)| > (1/(1 - 2^{-k})) \cdot (k - 2^{-k})$ and for which $a_{k+1} - a_k$ is greater than 1 for all k. Let P_1 be any closed subset of $E' \cap B(0, a_1)$ for which $|P_1| > 1$ and set $\Phi_1 = P_1 \cup (F_1 \cap B(0, a_1))$.

Since $\Phi_1 \subset E' \cap B(0, a_2)$ and $|\Phi_1| + (1 - 2^{-2})(|B(0, a_2)| - |\Phi_1|) > 2$, the preceding lemma implies that there is a closed set P_2 of measure greater than 2 with $\Phi_1 \subset P_2 \subset E' \cap B(0, a_2)$ which satisfies conditions (1), (2) and (3) of the lemma, with $M_2 = \Phi_1$, $M_1 = E' \cap B(0, a_2)$ and $p = 1$.

For each x in Φ_1 and $r < 1$, $B(x, r) \subset B(0, a_2)$ since $a_2 - a_1 > 1$. Thus $|E' \cap B(0, a_2) \cap B(x, r)|/|B(x, r)| = 1$ and by (3) of Lemma 4.1,

$$|P_2 \cap B(x, r)|/|B(x, r)| \geqslant 1 - 2^{-m-1+c_n}$$

for every positive integer m such that $r \leqslant 1/m$. Set $\Phi_2 = P_2 \cup (F_2 \cap B(0, a_2))$.

Continue inductively. Having defined Φ_k for $k \leqslant s$ so that $\Phi_k \subset F_k \cap B(0, a_k)$, $|\Phi_k| > k$ and $|\Phi_k \cap B(x, r)|/|B(x, r)| > 1 - 2^{-m-(k-1)+c_n}$ for $x \in \Phi_{k-1}$ and $r \leqslant 1/m$, let P_{s+1} be a closed set of measure greater than $s + 1$ for which $\Phi_s \subset P_{s+1} \subset E' \cap B(0, a_{s+1})$ and for which (1), (2) and (3) of Lemma 4.1 hold with $M_2 = \Phi_s$, $p = s$ and $M_1 = E' \cap B(0, a_{s+1})$. Since $|E' \cap B(0, a_{s+1}) \cap B(x, r)|/|B(x, r)|$ equals 1 for each x in Φ_s and $r < 1$, (3) of the lemma implies that $|P_{s+1} \cap B(x, r)|/|B(x, r)| > 1 - 2^{-s-m+c_n}$ for each positive integer m for which $r < 1/m$. Set $\Phi_{s+1} = P_{s+1} \cup (F_{s+1} \cap B(0, a_{s+1}))$.

The sequence $\{\Phi_k\}_{k\geqslant 1}$ satisfies the conditions of the theorem. Q.E.D.

We observe that by suitable choice of a_1 and P_1, Φ_1 can be made to contain a specified compact subset of E'.

If E is a G_δ set of measure 0 in R^n, an increasing sequence of compact subsets of R^n satisfying the conditions of Corollary 1 will be called a Zahorski sequence for E.

THEOREM 1. *Let E be a G_δ set of measure* 0 *in R^n. There exists a real valued, measurable function u defined on R^n having the following properties*:

(1) $0 \leq u \leq 1$.

(2) *u is* 0 *precisely on E.*

(3) *u is continuous at each point of E.*

(4) *For every x_0 in R^n and every $\epsilon > 0$, there is an $r > 0$ such that $u(x) \leq (1/(1-\epsilon))u(x_0)$ whenever x is in $B(x_0, r)$.*

(5) *Every x in R^n is a Lebesgue point of u.*

PROOF. Let $\{\Phi_k\}_{k \geq 1}$ be a Zahorski sequence for E. A closed set Φ_r will now be defined for each number r of the form $m/2^s$, where m and s are positive integers and $m > 2^s$. These closed sets will satisfy these two conditions:

(a) $\Phi_{s'} \subset \Phi_s$ if $s > s'$,

(b) $\Phi_{s'}$ consists only of points of density of Φ_s if $s > s'$.

For each odd integer $k > 2$, $k = 2m + 1$, let $\Phi_{k/2}$ be a closed set with $\Phi_m \subset \Phi_{k/2} \subset \Phi_{m+1}$ and $|\Phi_{k/2}| > \frac{1}{2}(|\Phi_{m+1}| + |\Phi_m|)$, for which every point of Φ_m is a point of density of $\Phi_{k/2}$ and every point of $\Phi_{k/2}$ is a point of density of Φ_{m+1}. Such a set exists by Lemma 1. Having defined $\Phi_{r/2^k}$ for all $r > 2^k$ and all $k \leq s$, let $\Phi_{r/2^{s+1}}$, $r > 2^{s+1}$, $r = 2t + 1$, be a closed set with

$$\Phi_{t/2^s} \subset \Phi_{r/2^{s+1}} \subset \Phi_{(t+1)/2^s} \quad \text{and} \quad |\Phi_{r/2^{s+1}}| > \tfrac{1}{2}(|\Phi_{(t+1)/2^s}|),$$

for which each point of $\Phi_{t/2^s}$ is a point of density of $\Phi_{r/2^{s+1}}$ and each point of $\Phi_{r/2^{s+1}}$ is a point of density of $\Phi_{(t+1)/2^s}$.

Now let λ be any real number greater than or equal to 1 and define $\Phi_\lambda = \bigcap_{m \geq \lambda 2^k} \Phi_{m/2^k}$. The collection of closed sets $\{\Phi_\lambda\}_{\lambda \geq 1}$ also satisfies (a) and (b).

Define the function u on R^n by

$$u(p) = \begin{cases} 1/\inf\{\lambda | p \in \Phi_\lambda\} & \text{if } p \notin E, \\ 0 & \text{if } p \in E. \end{cases}$$

Properties (1) and (2) from the statement of the theorem follow immediately from the definition of u. (3)–(5) will now be verified.

Let p be in E, and let $\epsilon > 0$ be arbitrary. If r is less than dist$(p, \Phi_{1/\epsilon})$, then $B(p, r) \cap \Phi_{1/\epsilon}$ is empty and $u(x)$ is less than ϵ for x in $B(p, r)$. Thus u is continuous on E.

Let x_0 be in E', and let $\epsilon > 0$ be arbitrary. If r is less than dist$(x_0, \Phi_{(1-\epsilon)/u(x_0)})$, then $u(x) \leq u(x_0)/(1-\epsilon)$ for all x in $B(x_0, r)$. Thus (4) holds. This property ensures that u is measurable.

Since u is continuous on E, (5) holds for every x in E. Let x_0 be in E' and let $\epsilon > 0$ be arbitrary. Since x_0 is in $\Phi_{(1+\epsilon/2)/u(x_0)}$, x_0 is a point of density of $\Phi_{(1+\epsilon)/u(x_0)}$ and thus of $\{y \mid u(y) \geq u(x_0)/(1+\epsilon)\}$. This, together with (4) and the boundedness of u, yields (5).

Thus u satisfies all the required conditions. Q.E.D.

If E is a G_δ set of measure 0 in R^n, a collection of closed sets $\{\Phi_\lambda\}_{\lambda \geq 1}$, constructed in the manner of the first part of the proof of this last theorem, will be called a Zahorski collection for E. The function

$$u(x) = \begin{cases} 1/\inf_\lambda\{\lambda \mid x \in \Phi_\lambda\}, & x \notin E, \\ 0, & x \in E, \end{cases}$$

will be called the corresponding inverse Zahorski function.

Applications to $S(n, T)$ and $L(n, T)$. An immediate consequence of Theorem 1 is

THEOREM 2. *If E is a G_δ set of measure 0 in R^n, then there is a function in $L(n, T)$ of norm 1 which vanishes precisely on E.*

PROOF. Let u be an inverse Zahorski function for E. u has norm 1 and vanishes precisely on E. Since, in addition, every point of R^n is a Lebesgue point of u, u satisfies the conditions of the theorem. Q.E.D.

If E is a G_δ of measure 0 contained in R^n and if F is a compact subset of E', then it is possible to find a Zahorski collection $\{\Phi_\lambda\}_{\lambda \geq 1}$ for E for which F is a subset of Φ_1. The corresponding inverse Zahorski function has norm 1, is 0 on E and identically 1 on F. Since every point of R^n is a Lebesgue point of u, u is in $L(n, T)$. We therefore also have

THEOREM 3. *If E is a G_δ of measure 0 in R^n and if F is a compact subset of R^n, disjoint from E, then there is a function of norm 1 in $L(n, T)$ which is 0 at each point of E and 1 at each point of F.*

COROLLARY 2. *If $\{w_k\}_{k \geq 1}$ is an arbitrary sequence of distinct points in R^n and if $\{a_k\}_{k \geq 1}$ is an absolutely summable sequence of real numbers, then there is a function g in $L(n, T)$ for which $g(w_k) = a_k$ for all k.*

PROOF. For each i, let S_i be a G_δ of measure 0 containing $\{w_k\}_{k \geq 1} - \{w_i\}$ and not containing w_i. Let u_i be an inverse Zahorski function for S_i for which $u_i(w_i) = 1$.

Since $\sum_{k=1}^\infty |a_k| < \infty$ and $\|u_i\|_\infty = 1$ for all i, every point of R^n is a Lebesgue point of the function $g = \sum_{k=1}^\infty a_k u_k$. Thus g is in $L(n, T)$. Since $u_i(w_k) = \delta_{ik}$, $g(w_k) = a_k$ for every k. Q.E.D.

COROLLARY 3. *If $\{w_k\}_{k \geq 1}$ is a convergent sequence of distinct points of*

R^n with limit $w \neq w_k$ any k and if $\{a_k\}_{k \geq 1}$ is an arbitrary sequence of 0's and 1's, then there is a function g in $L(n, T)$, with $\|g\|_\infty = 1$, for which $g(w_k) = a_k$ for all k.

The proof is similar to that of Corollary 2.

LEMMA 2. *Let f be in $L_R^\infty(R^n)$ and let E be a G_δ of measure 0 containing $\{x \mid x \notin L(f)\}$. If u is an inverse Zahorski function for E, then uf is in $L(n, T)$.*

PROOF. It is sufficient to show that $L(uf) = R^n$. If x is in E, $u(x) = 0$ and

$$\lim_{r \to 0} J(|uf - u(x)f(x)|, B(x, r))/|B(x, r)| = \lim_{r \to 0} J(|u|, B(x, r))/|B(x, r)| = 0.$$

If $x \notin E$, then x is a Lebesgue point of both u and f and so also for the product. Q.E.D.

Thus every function in $L_R^\infty(R^n)$ can be multiplied by a suitable inverse Zahorski function so that the product is in $L(n, T)$.

THEOREM 4. *If f is in $L_R^\infty(R^n)$ and if F is a compact subset of the Lebesgue points of f, then there is a function in $L(n, T)$ whose restriction to F is f.*

PROOF. Let E be a G_δ of measure 0 disjoint from F, which contains $\{x \in R^n \mid x \notin L(f)\}$. Let $\{\Phi_\lambda\}_{\lambda \geq 1}$ be a Zahorski collection for E with $F \subset \Phi_1$ and let u be the corresponding inverse Zahorski function. uf is the required function. Q.E.D.

Consequently $L(n, T)$ is locally dense in measure in $L_R^\infty(R^n)$, i.e. if F is a compact subset of R^n, then there is a sequence of functions in $L(n, T)$ which converges in measure to f on F.

Lemma 2 may be applied to characterize the extreme points of the unit ball of $S(n, T)$.

THEOREM 5. *F is an extreme point of the unit ball of $S(n, T)$ if and only if $|F| = 1$ a.e.*

PROOF. If $|F| = 1$ a.e., then F is an extreme point of the unit ball of $L_R^\infty(R^n)$ and hence also of $S(n, T)$. Conversely, suppose F fails to have modulus 1 at each point of some subset of R^n of positive measure. Let E be a G_δ of measure 0 in R^n containing $\{x \in R^n \mid x \notin L(1 - |F|)\}$. Let u be an inverse Zahorski function for E. By Lemma 5.1, $u(1 - |F|)$ is in $L(n, T)$ and so in $S(n, T)$. Since $u(1 - |F|) \leq 1 - |F|$, $\|u(1 - |F|) - F\|_\infty \leq 1$ and $\|u(1 - |F|) + F\|_\infty \leq 1$ so that F is not extreme. Q.E.D.

It is easy to see that the same result holds for the unit ball of $L(n, T)$, i.e. F is an extreme point of the unit ball of $L(n, T)$ if and only if $|F| = 1$ a.e. If

$|F| = 1$ a.e., then F is an extreme point of $S(n, T)$ and so also of $L(n, T)$. If F is in $L(n, T)$, then it follows from the inequality

$$J(||F| - |F(x)||, B(x, r)) \leq J(|F - F(x)|, B(x, r))$$

that $1 - |F|$ is also in $L(n, T)$. Thus if $|F|$ is less than 1 on a set of positive measure, then $G = 1 - |F|$ is a function in $L(n, T)$ which satisfies $\|F - G\|_\infty \leq 1$ and $\|F + G\|_\infty \leq 1$ so that F is not extreme.

THEOREM 6. *$L(n, T)$ is not the dual of a Banach space.*

PROOF. It is sufficient to show that the only extreme points of the unit ball of $L(n, T)$ are the constant functions 1 and -1. That this is so is a consequence of the following lemma:

LEMMA 3. *If f is a function in $L(n, T)$ which assumes the value* 0 *or* 1 *a.e., then f is constant.*

PROOF. Let $g(x) = f(x)(1 - f(x))$. Since

$$g(x) = \lim_{r \to 0} J(g, B(x, r))/|B(x, r)| = 0$$

for each x in R^n, f actually assumes the values 0 or 1 everywhere.

Let $K = \{x \in R^n \mid f \text{ is discontinuous at } x\}$. It is sufficient to show that K is empty.

Suppose K is not empty.

CLAIM. If $x_0 \in K$, then every neighborhood of x_0 contains some x in K for which $f(x) \neq f(x_0)$.

PROOF OF CLAIM. Let x_0 be in K and suppose, without loss of generality, that $f(x_0) = 1$. Let $B(x_0, r)$ be an arbitrary ball in R^n with center at x_0 and having radius r. Let s be any number in $(0, r/2)$. Since f is discontinuous at x_0, there is some a in $B(x_0, s)$ for which $f(a) = 0$. If a is in K, we are done. If a is not in K, f is continuous at a and so vanishes in a neighborhood of a. Set $t_a = \sup_t\{t > 0 | f \text{ is identically } 0 \text{ in } B(a, t)\}$. $B(a, t_a)$ is a subset of $B(x_0, r)$ and is not tangent to $B(x_0, r)$ at any point. (Otherwise we would have x_0 in $B(a, t_a)$ but $f(x_0) = 1$.) Let x be an arbitrary point on the boundary of $B(a, t_a)$. We have

$$\begin{aligned} f(x) &= \lim_{r \to 0} J(f, B(x, r))/|B(x, r)| \\ &= \lim_{r \to 0} J(f, B(x, r) \cap B(a, t_a)')/|B(x, r)| \\ &\leq \lim_{r \to 0} |B(x, r) \cap B(a, t_a)'|/|B(x, r)| < 1. \end{aligned}$$

Thus $f(x) = 0$ and f vanishes on the boundary of $B(a, t_a)$. By choice of t_a and compactness of the boundary, f must have at least one discontinuity x' on the boundary of $B(a, t_a)$. Since x' is in $K \cap B(x_0, r)$ and $f(x') \neq f(x_0)$, the proof of the claim is complete.

Now let x_1 be in K with $f(x_1) = 1$ and let $0 < r_1 < \frac{1}{2}$ be such that for $0 < r < r_1$, $J(f, B(x_1, r))/|B(x_1, r)| > 1 - \frac{1}{2}$.

Let x_2 be any point in $K \cap B(x_1, r_1)$ for which $f(x_2) = 0$, and let $0 < r_2 < \frac{1}{2}^2$ be such that for $0 < r < r_2$, $J(f, B(x_2, r))/|B(x_2, r)| < \frac{1}{2}^2$ and $\bar{B}(x_2, r_2) \subset B(x_1, r_1)$.

Continue defining x_k and r_k inductively as follows: If k is odd, let x_k be any point in $K \cap B(x_{k-1}, r_{k-1})$ for which $f(x_k) = 1$ and let $0 < r_k < \frac{1}{2}^k$ be such that for $0 < r < r_k$, $\bar{B}(x_k, r) \subset B(x_{k-1}, r_{k-1})$ and $J(f, B(x_k, r))/|B(x_k, r)| > 1 - \frac{1}{2}^k$. If k is even choose x_k and r_k in a similar way except that $f(x_k) = 0$ and $J(f, B(x_k, r))/|B(x_k, r)| < \frac{1}{2}^k$ for $0 < r < r_k$. Let x be in the intersection of the $\bar{B}(x_k, r_k)$. Then

$$\overline{\lim_{k\to\infty}} |J(f - f(x), B(x_k, r_k))|/|B(x_k, r_k)|$$

$$\leqslant \{ |B(x, 2r_k)|/|B(x_k, r_k)| \}$$

$$\times \lim_{k\to\infty} J(|f - f(x)|, B(x, 2r_k))/|B(x, 2r_k)| = 0.$$

But this implies that $f(x)$ must be both 0 and 1 which is impossible. Q.E.D.

The example

$$f(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0, \end{cases}$$

shows that there are nonconstant extreme points of the unit ball of $S(n, T)$.

REFERENCES

1. E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, N. J., 1970. MR 44 #7280.

2. Z. Zahorski, *Über die Menge der Punkte in welchen die Ableitung unendlich ist*, Tôhoku Math. J. 48 (1941), 321–330. MR 10, 359.

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