NONREGULAR ULTRAFILTERS AND LARGE CARDINALS

BY

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ABSTRACT. The relationship between the existence of nonregular ultrafilters and large cardinals in the constructible universe is studied.

1. Introduction. Our notation and terminology follows that of the most recent set-theoretic literature: for example $|x|$ denotes the cardinality of the set $x$, small Greek letters $\alpha, \beta, \gamma, \ldots$ denote ordinals, cardinals are initial ordinals, the Greek letters $\kappa, \lambda, \ldots$ are reserved for denoting cardinals, and so on. Needless to say, the word 'ultrafilter', the central object of our study, refers to a maximal filter. In particular, to simplify terminology, the phrase 'U is an ultrafilter over $\kappa$' will always refer to a uniform ultrafilter over $\kappa$, that is, for any set $x$ we require $x \in U \rightarrow |x| = \kappa$.

A central notion in the study of structural theory of countably incomplete ultrafilters over uncountable cardinals is the degree of regularity of the ultrafilter.

1.1. Definition. An ultrafilter $D$ is $(\kappa, \lambda)$-regular if there is a set $S \subseteq D$ of cardinality $\lambda$ such that $T \subseteq S \wedge |T| = \kappa \rightarrow \bigcap T = 0$.

An ultrafilter is $\lambda$-regular if it is $(\omega, \lambda)$-regular.

This notion is due to Keisler. He showed that ultrapowers taken using fairly regular ultrafilters have a great model-theoretic significance. Keisler also showed that for every cardinal $\lambda$ there is a $\lambda$-regular ultrafilter over $\lambda$. (For details, see for example Chang and Keisler [2].) The reverse direction here, namely the existence of suitably nonregular ultrafilters was left completely open. It is obvious that every ultrafilter over a cardinal $\lambda$ is $(\lambda, \lambda)$-regular. Beyond this, our usual set-theoretic axioms do not seem to tell too much. On the other hand, in the constructible universe, K. Prikry [13] has shown that every ultrafilter over a successor cardinal $\kappa^+$ is $(\kappa, \kappa^+)$-regular. R. Jensen [3] extending the work of Prikry showed that every ultrafilter over cardinals of type $\omega_n (n < \omega)$ is regular.

The introduction of large cardinal axioms seems to be the most natural approach to the problem of existence of nonregular ultrafilters.
1.2. Definition. A cardinal \( \lambda \) is measurable if there is a \( \lambda \)-complete ultrafilter \( D \) over \( \lambda \); i.e.

\[ X \subseteq D \wedge |X| < \lambda \rightarrow \cap X \in D. \]

It is then easy to see that every such \( D \) is not \( (\rho, \lambda) \)-regular for any \( \rho < \lambda \). The cardinal \( \lambda \) here is huge indeed; using ordinary forcing techniques we can produce a weakly inaccessible cardinal \( \lambda' \) below the continuum carrying an ultrafilter which is not \( (\rho, \lambda') \)-regular for any \( \rho < \lambda' \) (for details, see Prikry [12]). However, the question of Keisler and Gillman, namely the existence of a nonregular ultrafilter over \( \omega_1 \), is completely open. As was mentioned before, Prikry [13] showed that in the constructible universe, every uniform ultrafilter over \( \omega_1 \) is regular. Benda and Ketonen [1] extended this further to show that Prikry’s result actually follows from Kurepa’s Hypothesis. Here we have the first inklings of the ‘large cardinality nature’ of nonregular ultrafilters over \( \omega_1 \); it then follows immediately that \( \omega_2 \) must be inaccessible in \( L \) whenever such ultrafilters exist. In this paper we shall show that \( \omega_1 \) itself must then be a very large cardinal in \( L \).

We shall mainly work with the problem of \((\kappa, \kappa^+)\)-regularity of ultrafilters over \( \kappa^+ \). A great deal of progress has been made in this area recently, the following being the main results:

1.3. Theorem (Benda and Ketonen [1]). If \( D \) is a non-\((\kappa, \kappa^+)\)-regular ultrafilter over \( \kappa^+ \), then \( D \) is a P-point, i.e. if \( f: \kappa^+ \rightarrow \kappa^+ \) is unbounded \((\text{mod } D)\), then there is a set \( X \in D \) such that for every \( \alpha < \kappa^+ \):

\[ |f^{-1}(\{\alpha\}) \cap X| < \kappa. \]

1.4. Definition. Given two ultrafilters \( D, U \) over a cardinal \( \kappa \), say that \( D \) is less than \( U \) in the Rudin-Keisler order, in symbols, \( D \leq_{RK} U \), if there is a function \( f: \kappa \rightarrow \kappa \) such that for any \( X \subseteq \kappa \):

\[ X \in D \leftrightarrow f^{-1}(X) \in U. \]

In this case we also denote: \( D = f_*(U) \). Similarly, given two functions \( f, g: \kappa \rightarrow \kappa \) say \( f \leq_{RK} g \) \((\text{mod } D)\) if there is a function \( h: \kappa \rightarrow \kappa \) so that \( f = h \circ g \) \((\text{mod } D)\).

Hence, if \( f \leq_{RK} g \) \((\text{mod } D)\), then \( f_*(D) \leq_{RK} g_*(D) \). For more on this order, see for example Kunen [10].

1.5. Theorem (Kanamori [5]). If \( D \) is a non-\((\omega, \lambda)\)-regular ultrafilter over a regular cardinal \( \lambda \), then there is an ultrafilter \( U \) below \( D \) in the Rudin-Keisler order which extends the closed unbounded filter on \( \lambda \).

Combining Theorems 1.3 and 1.5, we have:

1.6. Theorem (Kanamori [5]). If \( D \) is a non-\((\kappa, \kappa^+)\)-regular ultrafilter over \( \kappa^+ \), then \( D \) has a first function \( f: \kappa^+ \rightarrow \kappa^+ \); i.e., every function \( f < f \) \((\text{mod } D)\)
is bounded by a constant \( < \kappa^+ \pmod{D} \) and \( f \) itself is not bounded by a constant \( \pmod{D} \).

1.7. **Definition.** An ultrafilter \( D \) over a cardinal \( \lambda \) is weakly normal if every pressing down function (i.e., a function \( f \) such that for any \( \alpha > 0: f(\alpha) < \alpha \)) on \( \lambda \) has range of cardinality \( < \lambda \) on a set of \( D \)-measure 1.

As a corollary to Theorem 1.6. we have:

1.8. **Theorem (Kanamori [5]).** If \( D \) is a non-(\( \kappa, \kappa^+ \))-regular ultrafilter over \( \kappa^+ \), then there is a weakly normal ultrafilter below \( D \) in the Rudin-Keisler order.

For weakly normal ultrafilters we have the following characterization of nonregularity:

1.9. **Theorem (Ketonen [7]).** If \( D \) is a weakly normal ultrafilter over a regular cardinal \( \lambda \), then \( D \) is \( (\mu, \lambda) \)-regular if and only if
\[
\{ \alpha | cf(\alpha) < \mu \} \in D.
\]

Combining Theorems 1.8 and 1.9, we get:

1.10. **Theorem (Kanamori [5]).** If \( \kappa \) is singular, then every ultrafilter over \( \kappa^+ \) is \( (\kappa, \kappa^+) \)-regular.

Here again we wish to point out the similarities with large cardinals: If \( D \) is a \( \lambda \)-complete ultrafilter over \( \lambda \), then it is a well-known result (Scott [15]) that there is an ultrafilter \( U \) below \( D \) which is actually normal: every pressing down function is constant \( \pmod{U} \). In this connection we wish to note the following result which has an analogue (due to Scott [15]) in the measurable case.

1.11. **Theorem (Benda and Ketonen [1]).** If there is a non-(\( \kappa, \kappa^+ \))-regular ultrafilter over \( \kappa^+ \), then
\[
2^\kappa = \kappa^+ \rightarrow 2^{\kappa^+} = \kappa^{++}.
\]

1.12. **Theorem (Benda and Ketonen [1]).** (1) Suppose \( f_\alpha: \lambda \rightarrow \mu (\alpha < \lambda) \) is a family of eventually different functions \( \pmod{F} \), where \( F \) is a \( \lambda \)-complete filter over \( \lambda \). Then an ultrafilter \( D \supseteq F \) is non-(\( \mu, \lambda \))-regular if and only if the \( f_\alpha \) are cofinal in the ultrapower of \( \mu \).

(2) If there is a non-(\( \kappa, \kappa^+ \))-regular ultrafilter over \( \kappa^+ \), then \( \kappa^{++} \) is inaccessible in \( L \).

Our methods in this paper are based on ideas of J. Silver [16] with an infusion of Benda-style techniques. Also, the paper of Vopěnka and Hrbáček [20] is relevant.

Our main results are the following:

1.13. **Theorem.** If there is a uniform, non-(\( \kappa, \kappa^+ \))-regular ultrafilter over \( \kappa^+ \), then \( 0^\# \) exists.
1.14. Theorem. If there is a uniform, weakly normal ultrafilter over a regular cardinal \( \kappa \) which is non-(\( \gamma, \kappa \))-regular for all \( \gamma < \kappa \), then \( 0^\# \) exists.

Here we shall not bother to give the formal definition of \( 0^\# \); we only need a statement equivalent to the existence of \( 0^\# \).

1.15. Definition (Kunen \[10\]). Suppose \( M \) is a transitive class model of ZFC and \( \kappa \) is a cardinal in \( M \). Then \( D \) is an \( M \)-ultrafilter on \( \kappa \) if:

(I) \( D \) is a proper subset of \( P(\kappa) \cap M \) containing no singletons.

(II) \( \forall x, y: x \subseteq y \in P(\kappa) \cap M \land x \in D \rightarrow y \in D. \)

(III) \( \forall x \in P(\kappa) \cap M: x \in D \lor \kappa - x \in D. \)

(IV) If \( \eta < \kappa \) and \( \langle x_\xi | \xi < \eta \rangle \in M \) and each \( x_\xi \in D \), then \( \bigcap \{ x_\xi | \xi < \eta \} \in M. \)

(V) If \( \langle x_\xi | \xi < \kappa \rangle \in M \), then \( \{ x_\xi \in D \} \in M. \)

The following result will then be used.

1.16. Theorem (Kunen \[9\]). \( 0^\# \) exists if and only if there is an \( L \)-ultrafilter \( D \) over a cardinal \( \lambda \) in \( L \) such that every countable intersection of elements of \( D \) is nonempty, if and only if there is an ultrafilter over some \( \mathcal{P}(\lambda) \cap L, (\lambda \text{ a cardinal in } L) \) such that the ultrapower of \( L \) with respect to \( D \) is well founded.

For more on \( 0^\# \), see Solovay \[18\].

By Theorem 1.8, it clearly suffices to prove only 1.13. In \S \textbf{2} we prove that under hypotheses of Theorem 1.13, \( \kappa \) is weakly compact in \( L \) and \( \kappa^+ (L) < \kappa^+ \). Using these two facts, we then prove the existence of \( 0^\# \) in \S \textbf{3}.

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2. Weakly normal ultrafilters and the constructible universe. In the following, suppose that \( D \) is a uniform ultrafilter over a regular cardinal \( \lambda \).

2.1. Definition. (1) If \( f, g: \lambda \rightarrow V \), then

\[ f \sim g \iff \{ \alpha | f(\alpha) = g(\alpha) \} \in D. \]

(2) If \( f: \lambda \rightarrow V \), then

\[ [f]_D = \{ g | g \sim f \text{ and } \forall h (h \sim f \rightarrow \text{rank}(h) \geq \text{rank}(g)) \}. \]

Here the rank of a set \( x \) is the usual set-theoretic rank. We can now define the ultrapowers we need. Let \( C \) be a transitive classmodel of set theory.

2.2. Definition. (1) The ultrapower of the class \( C \) with respect to \( D \) is the class \( \Pi_D C = \{ [f]_D | f: \lambda \rightarrow C \}. \)

(2) The restricted ultrapower of the class \( C \) with respect to \( D \) is the class \( \Pi^\#_D C = \{ [f]_D | f: \lambda \rightarrow C \text{ and } \text{range}(f) < \lambda \}. \)

(3) In either case, we define the e-relation on the ultrapower to be

\[ [f]_D E [g]_D \iff \{ \alpha | f(\alpha) \in g(\alpha) \} \in D. \]
The idea of using a restricted ultrapower in set-theory appears in the paper of Vopěnka-Hrbáček [20]. It is actually a special case of Keisler’s notion of a limit-ultrapower (see Keisler [6]). A word of caution: All the models which we will construct will most often be not well founded.

2.3. Proposition (Keisler [6]). Define an embedding $i: C \rightarrow \Pi_D^* C$ by setting $i(x) = [x]_D$, the equivalence class of the constant function $x$, and let $j: \Pi_D^* C \rightarrow \Pi_D C$ be the map induced by the inclusion map. Then both $i$, $j$ are elementary embeddings, and they induce a commutative diagram.

Now assume that $D$ is weakly normal. Let $\lambda^* = [\lambda]_D$ and $\rho^* = [id]_D$. The following result is essentially due to Vopěnka and Hrbáček.

2.4. Proposition. The map $j$ is an onto map when restricted to $\lambda^*$ in $\Pi_D^* C$:

$$j: \{[f]_D \mid \text{rng} f < \lambda \text{ and } [f]_D \in \lambda^* \} \text{ onto } \{[f]_D \mid [f]_D \in \rho^* \}.$$

This is clear, since every function $< [id] \text{ (mod } D) \text{ has range of cardinality } < \lambda(\text{mod } D)$.

Thus we have an order-isomorphism via the map $j$ between the predecessors of $\lambda^*$ in $\Pi_D^* C$ and predecessors of $\rho^*$ in $\Pi_D C$. The ‘ordinal’ $\lambda^*$ gets mapped into a bigger ordinal than $\rho^*$ by $j$; thus in a sense $\lambda^*$ is the first ordinal moved. Here is the basic idea of Silver [16]: Well foundedness will be replaced by isomorphisms between structures. The next result is essentially due to Kunen [9] and in its present context to Silver [16]. It follows directly from Proposition 2.4.

2.5. Proposition. Define a collection $U$ of ‘subsets’ of $\lambda^*$ as follows:

$$U = \{[f]_D \mid \text{rng} f < \lambda \text{ and } [f]_D \subseteq \lambda^* \text{ and } \Pi_D C \models [\rho]_D \in [f]_D \text{.}\}.$$

Then the following statements hold:

I. If $x \in U$ then

$$\Pi_D^* C \models |x| = \lambda^*.$$

II. For $x$, $y \in \Pi_D C$,

$$\Pi_D^* C \models x \cup y = \lambda^* \rightarrow x \in U \text{ or } y \in U,$$

$$\Pi_D^* C \models x \cap y = 0 \rightarrow x \notin U \text{ or } y \notin U.$$

III. If $F \in \Pi_D C$ and $\Pi_D C \models F: \lambda^* \rightarrow \lambda^*$ is pressing down, then there is a $y \in \Pi_D^* C$ such that there is a $z \in U$ with

$$\Pi_D^* C \models y < \lambda^* \land z = f^{-1}(\{y\}).$$

IV. If $F \in \Pi_D C$ and $\Pi_D C \models F: y \rightarrow V \land y < \lambda^*$ and for any $a$ such that
Thus, $U$ is a \('\Pi^*_D$-ultrafilter' in the sense of Definition 1.14 with the possible exception of the following property.

V. If $F \in \Pi^*_D C$ and $\Pi^*_D C \models F: \lambda^* \rightarrow V$, then there is a $Z \in \Pi^*_D C$ such that

$$F(a) \in U \iff \Pi^*_D C \models a \in Z.$$ 

To accomplish this, we need to extend the isomorphism of Proposition 2.4 to subsets of $\lambda^*$ in $\Pi^*_D C$. Some kind of 'smoothness' of the model $C$ is required. From now on we shall assume that $C$ is the constructible universe.

2.6. Proposition. If $D$ is a weakly normal ultrafilter over $\lambda$, then there exists an isomorphism $G$ such that for any relation $R \subseteq \lambda \times \lambda$,

$$G: \Pi_D <L_\alpha, e, R \cap (\alpha \times \alpha)> \cong \Pi^*_D <L_\lambda, e, R>.$$ 

2.7. Proposition. If $D$ is a weakly normal ultrafilter over $\lambda$, then there exists an isomorphism $H$ between the structure $\Pi^*_D <L_\lambda^+, e>$, where $\lambda^+$ is the (real) successor of $\lambda$, and an initial $E$-segment of the structure $\Pi_D <L_\lambda^+, e>$. 

These two propositions are directly modeled after those of Silver [16] and their proofs are similar. The proof of Proposition 2.6 is a straightforward application of weak normality. For the sake of completeness, we shall include the proof of Proposition 2.7.

Proof of Proposition 2.7. Given an ordinal $\alpha < \lambda^+$, let $R^\alpha \subseteq \lambda \times \lambda$ be a (possibly nonconstructible) relation coding $<L_\alpha, e>$; i.e., the structures $<\lambda, R^\alpha>$ and $<L_\alpha, e>$ are isomorphic. Then the set

$$C^\alpha = \{ \gamma | \langle \gamma, R^\alpha \cap (\gamma \times \gamma) \rangle \text{ is an elementary substructure of } \langle \lambda, R^\alpha \rangle \}$$

is a closed unbounded set and therefore belongs to $D$. Let $g^\alpha$ be a function so that for $\gamma \in C^\alpha$, $g^\alpha(\gamma) < \gamma^+$ and

$$<L_{g^\alpha(\gamma)}, e> \cong <\gamma, R^\alpha \cap (\gamma \times \gamma)>.$$ 

We have by Proposition 2.6,

$$\Pi^*_D <L_\alpha, e> \cong \Pi^*_D <\lambda, R^\alpha> \cong \Pi^*_D <\gamma, R^\alpha \cap (\gamma \times \gamma)> \cong \Pi^*_D <L_{g^\alpha(\gamma)}, e>.$$ 

Thus we have a canonical isomorphism for $\alpha < \lambda^+$

$$H^\alpha: \Pi^*_D <L_\alpha, e> \cong \Pi^*_D <L_{g^\alpha(\gamma)}, e>.$$
It remains to show that for $\alpha < \beta$,

$$H^\alpha = H^\beta \quad \text{on } \prod^\beta_\alpha (L_\alpha, \varepsilon).$$

This is an immediate consequence of the following fact: For any $\tau < \lambda^+$, let $P_\tau$ be the isomorphism

$$P_\tau: \langle \lambda, R^\tau \rangle \cong \langle L_\tau, \varepsilon \rangle.$$ 

Then for any $\alpha < \beta$ the set

$$\{ y \in L_\alpha \cap P_\beta^\alpha (\delta \uparrow \delta < \gamma) \}$$

contains a closed unbounded subset of $\lambda$. □

The functions $g^\alpha (\gamma)$ constructed in the above proof have an important property:

**2.8. Proposition.** If $\alpha < \beta < \lambda^+$, then there is a closed unbounded set $C \subseteq \lambda$ such that $\gamma \in C \rightarrow g^\alpha (\gamma) < g^\beta (\gamma)$.

**Proof.** If $\alpha < \beta < \lambda^+$, then there is an isomorphism $\iota$ from $\langle \lambda, R^\alpha \rangle$ onto a $R^\beta$-proper initial segment of the structure $\langle \lambda, R^\beta \rangle$. Hence there is an ordinal $\delta < \lambda$ s.t. $i(\gamma)R^\beta \delta$ for $\gamma < \lambda$. Let $C = \{ \gamma | \gamma > \delta \text{ and } \gamma \in C^\alpha \cap C^\beta \}$. This set satisfies our requirements. □

**2.9. Theorem.** If $D$ is a weakly normal ultrafilter over $\lambda$ such that $D$ is not $(\gamma, \lambda)$-regular for any $\gamma < \lambda$, then there is an isomorphism

$$H: \prod^\beta_\alpha (L_{\lambda^+}, \varepsilon) \cong \prod^\beta_\alpha (L_{\alpha^+}, \varepsilon)$$

extending the isomorphism $G^{-1}$ of Proposition 2.6.

**Proof.** By Proposition 2.7 it suffices to show that if the functions $g^\alpha (\alpha < \lambda^+)$ are not cofinal (mod $D$) in the ultrapower

$$\prod^\beta_\alpha (L_{\alpha^+}, \varepsilon), \quad \alpha < \lambda$$

then $D$ is $(\gamma, \lambda)$-regular for some $\gamma < \lambda$. If the $g^\alpha$ are not cofinal, there is a function $h: \lambda \rightarrow \lambda$ s.t. $h(\gamma) < \gamma^+$ and for all $\gamma < \lambda$ and for all $\alpha < \lambda^+$ we have: $g^\alpha < h(\text{mod } D)$. Now, let $k_\gamma$ be a one-to-one function from $\lambda \rightarrow \lambda$ which maps $h(\gamma) \rightarrow \gamma$. Define for $\alpha < \lambda^+$, $\gamma < \lambda$: $h^\alpha (\gamma) = k_\gamma (g^\alpha (\gamma))$. Each $h^\alpha$ is a pressing down function (mod $D$). Therefore we may, without loss of generality, assume that there is a $\xi < \lambda$ such that $h^\alpha < \xi$ (mod $D$) for all $\alpha < \lambda^+$. But, by Proposition 2.8, the functions $g^\alpha$ are mutually eventually different modulo the closed unbounded filter. Therefore, by Benda’s Theorem 1.2, $D$ is $(\xi, \lambda)$-regular. □

The following result is then a straightforward analog of Silver’s Theorem 1.5 in [16].
2.10. **Theorem.** If $D$ is not-$(\gamma, \lambda)$-regular for any $\gamma < \lambda$ and $D$ is weakly normal, then the ultrafilter $U$ satisfies condition V.

**Proof.** Suppose that $F \in \Pi^*_D C$ and $\Pi^*_D C \models F \colon \lambda^+ \rightarrow V$. Let $y \in \Pi_D C$ so that

$$\Pi_D C \models y = \{s < p^* \mid p^* E(f)(s)\}.$$  

Then $H^{-1}(y)$ will satisfy the requirements of condition V. ∎

The way is finally clear for large cardinality results:

2.11. **Theorem.** If $D$ is weakly normal over $\lambda$ and $D$ is non-$(\gamma, \lambda)$-regular for any $\gamma < \lambda$, then $\lambda$ is $\Pi^1_n$-indescribable for every $n < \omega$ in $L$. That is, for any constructible relation $R$ on $\lambda$, $\Pi^1_n$ sentence $\phi$: If $((\lambda, <, R) \models \phi)^L$, then there is an $\alpha < \lambda$ such that $((\alpha, <, R|\omega) \models \phi)^L$. As a matter of fact,

$$\{\alpha \mid ((\alpha, <, R|\omega) \models \phi)^L\} \in D.$$  

**Proof.** This is clear from Theorem 2.9, since $\phi$ becomes a first order statement in $L^{\lambda^+}$. ∎

Thus, we now know that $\lambda$ is weakly compact under the hypotheses of Theorem 2.11. We shall prove that $\lambda^+(L) < \lambda^+$ in this situation by contradiction.

2.12. **Proposition.** If $X$ is a function $\lambda \rightarrow P(\lambda) \cap L$ with range of cardinality $< \lambda$, then

$$[X] \in U \leftrightarrow \{\alpha \mid x \in X(\alpha)\} \in D.$$  

This is immediate from the definition of $U$.

2.13. **Proposition.** For any sequence $\langle m(\alpha) \mid \alpha < \lambda \rangle$, where $m(\alpha)$ is a constructible subset of $\alpha$, there exists a function $A : \lambda \rightarrow P(\lambda) \cap L$ with range of cardinality $< \lambda$ such that

$$\{\alpha \mid m(\alpha) = A(\alpha) \cap \alpha\} \in D.$$  

**Proof.** The 'set' $[m]_D \in \Pi_D L$. Let $A : \lambda \rightarrow P(\lambda) \cap L$ so that $H([A]_D) = [m]_D$. Then $A$ satisfies our requirements. Note that by 2.9, if $x \subseteq \lambda$ and $x \cap \alpha \in L$ for all $\alpha < \lambda$, then $x \in L$. ∎

There is a useful modification of our basic construction: Instead of looking at all functions, we can restrict our attention to constructible functions: Define

$$\Pi^*_D L = \{[f]_D \mid f \in L \land f : \lambda \rightarrow L\},$$

$$\Pi^{**}_D L = \{[f]_D \mid f \in L \land f : \lambda \rightarrow L \land |\text{rng}(f)| < \lambda\}.$$  

2.14. **Proposition.** If $\lambda^{+L} = \lambda^+$ and $D$ is weakly normal and not $(\gamma, \lambda)$-regular for any $\gamma < \lambda$, then

$$\Pi^{**}_D(L_{\lambda^+}, e) \cong \Pi^{**}_D(L_{\alpha^+}, e).$$
PROOF. For $\alpha < \lambda^+$ we can then require the relations $R^\alpha$ of Proposition 2.6 to be constructible. Theorem 2.9 then implies that

$$\Pi^*_D(L^+, \epsilon) \cong \prod_{\alpha < \lambda} (L^+, \epsilon)$$

from which our claim follows since the $g^\alpha(\alpha < \lambda^+)$ are then cofinal in the ultrapower. \[\square\]

2.15. PROPOSITION. Under the hypotheses of Proposition 2.15: If $\langle X_\xi \mid \xi < \lambda \rangle$ is a constructible sequence of subsets of $\lambda$, then there is a constructible function $A: \lambda \rightarrow P(\lambda) \cap L$ with range $< \lambda$ such that for any constructible $f: \lambda \rightarrow \lambda$ with range $< \lambda$ we have

$$\{\alpha \mid \alpha \in X_f(\alpha)\} \in D \iff \{\alpha \mid f(\alpha) \in A(\alpha)\} \in D.$$ 

PROOF. This follows immediately from the constructible analogues of Theorem 2.10 and Proposition 2.12. \[\square\]

Let $n$ be a positive integer. For a set $x$, $[x]^n$ denotes the set of all unordered $n$-tuples from the set $x$.

2.16. PROPOSITION. Under the hypotheses of Proposition 2.15: If $F: [\lambda]^n \rightarrow \{0, 1\}$ is constructible, then there is a function $X: \lambda \rightarrow \hat{P}(\lambda)$ with range of cardinality $< \lambda$ such that $X \in L$ and

(a) $\forall \alpha < \lambda, X(\alpha)$ is homogeneous for $F$; i.e., $F''[X(\alpha)]^2$ is a singleton.

(b) $\{\alpha \mid \alpha \in X(\alpha)\} \in D$.

PROOF. Using the constructible analogue of Theorem 2.10 and standard techniques of, say, Kunen [10], we can show that there is a $[X]^D \in U \cap \Pi^*_D$ such that $\Pi^*_D \models [X]^D$ is homogeneous for $i(F)$. \[\square\]

We shall present the proof of the entirely analogous 2.20 in more detail.

2.17. DEFINITION. $F: [\lambda]^n \rightarrow \lambda$ is pressing down if for all $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_n$,

$$F(\langle \alpha_1 \ldots \alpha_n \rangle) < \alpha_1.$$ 

2.18. THEOREM. Under the hypotheses of Proposition 2.14: If $F: [\lambda]^n \rightarrow \lambda$ is a constructible pressing down function, then there is a set $X \in D \cap L$ and a $\xi < \lambda$ such that $F''[X]^n \subseteq \xi$.

PROOF. Take $n = 2$. For each $\alpha < \lambda$ define

$$g_\alpha(\beta) = F(\langle \alpha, \beta \rangle) \quad (\beta < \alpha).$$

Thus $g_\alpha \subseteq \alpha \times \alpha$. By the constructible analog of 2.14, there is a constructible map $T: \lambda \rightarrow \lambda^\lambda$ with range of cardinality $\xi < \lambda$ so that for any $\alpha < \lambda$, $T(\alpha)$ is a pressing down function and
By a variant of Theorem 1.8 the set \( Y = \{ \alpha \mid cf^\delta(\alpha) > (\xi^+)^L \} \in D \). This can be seen for example as follows: Suppose that \( Z = \{ \alpha \mid cf^\delta(\alpha) < (\xi^+)^L \} \in D \). Then we can find a constructible sequence \( \{ A_\alpha \mid \alpha < \lambda \} \) of sets such that \( \lambda = \forall \alpha < \lambda : A_\alpha \subseteq \alpha \) has order type \( (\xi^+)^L \), and if \( \alpha \in Z \), then \( A_\alpha \) is cofinal in \( \alpha \). But then, by Proposition 2.13, we can find a constructible \( A \subseteq \lambda \) such that the set \( \{ \alpha \in Z \mid A_\alpha = A \cap \alpha \} \) has cardinality \( \lambda \), contradiction by Theorem 2.11. Hence, if

\[
K(\gamma) = \sup\{ T(\alpha(\gamma)) \mid \alpha < \lambda \},
\]

\( K \) is pressing down on \( Y \). By weak normality, there is a \( \xi < \lambda \) so that

\[
Z = \{ \gamma \in Y \mid K(\gamma) < \xi \} \in D \cap L.
\]

It then follows that \( F < \xi \) on \( [Z \cap U]^2 \). □

2.19. Theorem. Under the hypotheses of Proposition 2.15: Suppose that \( \{ f_\alpha \mid \alpha < \lambda \} \) is a constructible family of bounded functions \( \lambda \rightarrow \lambda \). Then there is a constructible function \( g : \lambda \rightarrow \xi \) such that for every \( \alpha < \lambda \)

\[
f_\alpha \preceq_R K g \quad (\text{mod } D \cap L).
\]

Proof. Define a pressing down function \( F \) by:

\[
F(\{ \alpha, \beta \}) = \begin{cases} 
0 & \text{if } \forall \gamma < \alpha < \beta : f_\gamma(\alpha) = f_\gamma(\beta), \\
\text{least } \gamma < \alpha \text{ so that } f_\gamma(\alpha) \neq f_\gamma(\beta) & \text{otherwise.}
\end{cases}
\]

By Theorem 2.20 we can find an \( \eta < \lambda \) and a set \( X \in D \cap L \) such that \( F < \eta \) on \( [X]^2 \), i.e., for \( \alpha, \beta \in X \) either \( \forall \gamma < \alpha < \beta : f_\gamma(\alpha) = f_\gamma(\beta) \) or there is a \( \mu < \eta \) such that \( f_\mu(\alpha) \neq f_\mu(\beta) \). By Theorem 2.19 there is a constructible partitioning \( \{ Y_\xi \mid \xi < \theta \} \) (\( \theta < \lambda \)) such that \( Y = U_\xi \{ Y_\xi \mid \xi < \theta \} \in D \), \( Y \subseteq X \) and for all \( \xi < \theta \) either \( \alpha, \beta \in Y_\xi \rightarrow \forall \gamma < \alpha < \beta : f_\gamma(\alpha) = f_\gamma(\beta) \) or \( \alpha, \beta \in Y_\xi \rightarrow \exists \mu < \eta : f_\mu(\alpha) \neq f_\mu(\beta) \). We can without a loss of generality assume that, for any \( \xi < \theta \), \( |Y_\xi| = \lambda \). This rules out the second possibility listed above since \( \lambda \) is inaccessible in \( L \).

Hence, if we set \( g = \xi \) on \( Y_\xi \), then for \( \gamma < \lambda \):

\[
f_\gamma \preceq_R K g \quad \text{on } Y - \gamma. \quad \Box
\]

2.20. Corollary. Under the hypotheses of Proposition 2.14: For every \( \gamma < \lambda^+ \) there is a \( \xi_\gamma < \lambda \) and a constructible function \( g_\gamma : \lambda \rightarrow \xi_\gamma \) such that for any bounded function \( h : \lambda \rightarrow \lambda \) with \( h \in L_\gamma \) we have

\[
h \preceq_R K g_\gamma \quad (\text{mod } D \cap L).
\]

The following is a trivial modification of the fundamental result of Silver [16].
2.21. Theorem (Silver [16]). If $D$ is a weakly normal ultrafilter over $\lambda$ which is not $(\gamma, \lambda)$-regular for any $\gamma < \lambda$ such that

$$\prod_\alpha^{\ast}(L^{\lambda+}, e) \equiv \prod_\alpha^{D}(L^{\lambda+}, e)$$

and

$$|\prod_\alpha^{D}\omega| < \lambda,$$

then $\text{cof}(\lambda^{+(L)}) < \lambda$.

We can finally prove:

2.22. Theorem. If there is a weakly normal ultrafilter over $\lambda$ which is not $(\gamma, \lambda)$-regular for any $\gamma < \lambda$, then $\lambda^{+(L)} < \lambda^+$. 

Proof. For suppose that $\lambda^{+(L)} = \lambda^+$. By Theorem 2.22: $|\prod_\alpha^{D}\omega| \geq \lambda$. By Corollary 2.21, there is a fixed $\xi < \lambda$ and constructible $g_\gamma: \lambda \to \xi$ ($\gamma < \lambda^+$) such that for any $f \in L_\gamma$

$$f \equiv R_K g_\gamma \mod D \cap L.$$

Therefore

$$|\{[f]_D \mid f \in L_\gamma \text{ and } f: \lambda \to \omega\}| \leq (\xi^+)\.\L.$$ 

Since $\lambda^{+(L)} = \lambda^+$ by assumption and $\lambda$ is inaccessible in $L$,

$$|\prod_\alpha^{D}\omega| < \lambda;$$

a contradiction. $\Box$

3. The main results. We shall now prove our main results. As was remarked before, it suffices to prove Theorem 1.14.

From now on, assume that $D$ is a weakly normal ultrafilter over a regular cardinal $\lambda$ such that $D$ is not $(\gamma, \lambda)$-regular for any $\gamma < \lambda$. By the results of §2, $\lambda$ is inaccessible in $L$ and $\lambda^{+(L)} < \lambda^+$.

As before, for any $\tau < \lambda^+$ pick $R^\tau \subseteq \lambda \times \lambda$ coding $(L_\tau, e)$ and let $P_\tau$: $(\lambda, R^\tau) \to (L_\tau, e)$ be the isomorphism. Let $P = \{\tau|L_\tau < L_\lambda^+ \text{ and } \tau > (\lambda^+)^L\}$. As usual, the symbol '$A < B'$ means that $A$ is an elementary substructure of $B$.

For $\tau \in P$, let

$$B^\tau = \{\alpha|\alpha < \lambda, (\alpha, R^\tau \cap \alpha) < (\lambda, R^\tau) \text{ and } P_\tau^\prime(\alpha) \cap \lambda = \alpha\}.$$

Taking the transitive collapse of $(\alpha, R^\tau \cap \alpha)$, for $\alpha \in B^\tau$ we get an elementary embedding $\Pi^{\tau}_\alpha: L_\sigma^{\tau}(\alpha) \to L_\lambda^+$ such that $\Pi^{\tau}_\alpha(\alpha) = \lambda$ and $\alpha$ is the first ordinal moved. Here the functions $g^\tau$ are constructed as in the proof of Proposition 2.7.

We can then define ultrafilters $U^\tau_\alpha$ over $P(\alpha) \cap L_\sigma^{\tau}(\alpha)$ as follows:

$$X \in U^\tau_\alpha \iff \alpha \in \Pi^{\tau}_\alpha(\lambda).$$
3.1. Lemma. (a) Let $\tau_0 = (\lambda^+)^L$. Then

$$X_0 = \{ \alpha \mid g^\tau_0(\alpha) = (\alpha^+)^L \} \in D.$$  

(b) For any $\tau < \eta < \lambda^+$, $\tau, \eta \in P$: $g^\tau \leq g^\eta$ on a closed unbounded set.

(c) For any $f: \lambda \to \lambda$ such that $f(\alpha) < |\alpha|^+$ there is a $\tau \in P$ such that $f \leq g^\tau$ (mod $D$).

(d) For any $\tau < \eta < \lambda^+$, $\tau, \eta \in P$: There is a closed unbounded set $C$ such that

$$C \cap X_0 \subseteq \{ \alpha \mid U_\alpha = U_\alpha = U_\alpha \}.$$  

Proof. To prove (a), use the fact that $(\lambda^+)^L < \lambda^+$ and Theorem 2.9.

(b) is simply a restatement of Proposition 2.8. (c) follows immediately from the proof of Theorem 2.9.

To prove (d), argue as follows: Given $\tau, \eta$, the set

$$A = \{ \alpha \mid L_\tau \cap P^\eta(\{ \delta \mid \delta < \gamma \}) = L_\eta \cap P^\eta(\{ \delta \mid \delta < \gamma \}) \}$$

is closed unbounded. By (a), (b) there is a closed unbounded set $C \subseteq A$ so that

$$(\alpha^+)^L < g^\tau(\alpha) < g^\eta(\alpha).$$

Now, $L_{g^\eta(\alpha)}$ is the transitive collapse of $P^\eta(\{ \delta \mid \delta < \alpha \})$. Since all the subsets of $\alpha$ in $L_{g^\eta(\alpha)}$ appear already in $L_{g^\tau(\alpha)}$ for all $\alpha \in C \cap X_0$, $U_\alpha = U_\alpha$. □

Thus, we can find ultrafilters $U_\alpha$ over $P(\alpha) \cap L$ for $\alpha \in X_0$ such that for every $\tau \in P$ there is a closed unbounded set $C^\tau$ such that for $\alpha \in X_0 \cap C^\tau$:

$$X \in U_\alpha \leftrightarrow \alpha \in \Pi^\alpha(X).$$

To finish off the proof, by Theorem 1.16 it suffices to prove:

3.2. Lemma. There is an $\alpha \in X_0$ such that the ultrapower

$$\text{Ult}(L, U_\alpha) = \prod^L U_\alpha L$$

is well founded.

Proof. If this was not the case, for every $\alpha \in X_0$ we can find a sequence $f^\alpha_i: \alpha \to \text{ORD}$ such that for all $i < \omega$:

$$\{ \gamma \mid \gamma < \alpha, f^\alpha_i(\gamma) > f^\alpha_i(\gamma) \} \in U_\alpha$$

and each $f_i \in L_\theta$ for some ordinal $\theta$ depending only on $\alpha$. Form the elementary substructure $M_\alpha$ of $L_\theta$ generated by the set $\alpha \cup \{ f^\alpha_i \mid i < \omega \}$. By collapsing $M_\alpha$, we get an ordinal $\beta < |\alpha|^+$ such that

$$\langle M_\alpha, e \rangle \cong \langle L_\beta, e \rangle.$$  

From this follows that we can without loss of generality assume that each $f^\alpha_i: \alpha \to |\alpha|^+$. Let $f$ be a function $\lambda \to \lambda$ such that $f(\alpha) < |\alpha|^+$ and for all $\alpha, i$:
By Lemma 3.1 there is a closed unbounded set $C$ and a $\tau \in P$ such that

$$f(\alpha) < g^\tau(\alpha) \quad (\alpha \in C \cap X_0)$$

and

$$X \in U_\alpha \iff \alpha \in \Pi^*_\alpha(X)$$

where $\Pi^*_\alpha$ is an elementary embedding $L_{g^\tau(\alpha)} \rightarrow L^{\lambda^+}$ with $\alpha$ the first ordinal moved.

But, given $\alpha \in C \cap X_0$ we then have

$$(\Pi^*_{\alpha+2})^\alpha(\alpha) > (\Pi^*_{\alpha+1})^\alpha(\alpha) > \ldots,$$

a contradiction. □

REFERENCES

3. R. Jensen, Some combinatorial principles of $L$ (Mimeographed).