A PLANCHEREL FORMULA FOR IDYLLIC NILPOTENT LIE GROUPS(1)

BY

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ABSTRACT. A procedure is developed which can be used to compute the Plancherel measure for a certain class of nilpotent Lie groups, including the Heisenberg groups, free groups, two-and three-step groups, the nilpotent part of an Iwasawa decomposition of the R-split form of the classical simple groups $A_p$, $C_p$, $G_2$.

Let $G$ be a connected, simply connected nilpotent Lie group. The Plancherel formula for $G$ can be expressed in terms of Plancherel measure of a normal subgroup $N$ and projective Plancherel measures of certain subgroups of $G/N$. To get an explicit measure for $G$, we need an explicit formula for (1) the disintegration of Plancherel measure of $N$ under the action of $G$ on $\hat{N}$, and (2) projective Plancherel measures of $G_\gamma/N$, where $G_\gamma$ is the stability subgroup at $\gamma$ in $\hat{N}$. When both $N$ and $G_\gamma/N$ are abelian, the measures (1) and (2) are obtained as special cases of more general problems. These measures combine into Plancherel measure for $G$.

0. Introduction. For a connected, simply connected, real nilpotent Lie group $G$, Dixmier [8], Kirillov [12], [13] and Pukânszky [19] have shown that the generic representations $\pi \in \hat{G}$ can be parametrized by a Zariski-open subset of a finite-dimensional real vector space $R^k$, and that Plancherel measure for $G$ (see [7], [18], [22]), $\mu_G$, is then a rational function times Lebesgue measure on $R^k - R(\gamma) d\gamma$. The main result of this paper is a technique for computing the rational function $R(\gamma)$ in terms of the structure constants of the Lie algebra of $G$.

Kleppner and Lipsman’s [14], [15] Plancherel formulation of the Mackey machine for expressing $\hat{G}$ in terms of $\hat{N}$ and irreducible projective representations of certain subgroups of $G/N$ (the little groups), for $N < G$, is used to compute $\mu_G$ for a certain class of nilpotent Lie groups $G$. The procedure obtained for computing $\mu_G$ is explicit and can be carried out without too much trouble if the

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projective measures are reasonable. The method works for those connected, simply connected, nilpotent Lie groups $G$ which have an abelian normal Lie subgroup $N$ such that for $\mu_N$ almost all $\gamma \in \hat{N}$, $G_\gamma / N$ is abelian, where $G_\gamma$ is the stability subgroup at $\gamma$ for the action of $G$ on $\hat{N}$. Such a nilpotent Lie group is called idyllic.

When $N$ is abelian, $\hat{N}$ is $\mathfrak{n}'$, the dual of the Lie algebra $\mathfrak{n}$ of $N$, and $\mu_N$ is Lebesgue measure on $\mathfrak{n}'$. The orbit space $\hat{N} / G$ is $\mathfrak{n}' / G$, the orbit space of the coadjoint representation of $G$ in $\mathfrak{n}'$. We need an explicit formulation of the disintegration of Lebesgue measure on $\mathfrak{n}'$ into a measure on $\mathfrak{n}' / G$ and measures on the orbits of $G$ in $\mathfrak{n}'$. When $G_\gamma / N$ is abelian, the projective Plancherel measure can be computed. $\gamma \in \hat{N}$ extends to an $\omega_\gamma$-representation of $G_\gamma$. When $G_\gamma / N$ is abelian, the multiplier $\omega_\gamma$ on $G_\gamma / N$ is the exponential of an alternating bilinear form on $G_\gamma / N$.

Let $H$ be a finite-dimensional real vector space, $A : H \times H \to R$ an alternating bilinear form on $H$, and $\omega_A$ the multiplier on $H$ defined by $\omega_A(x, y) = e^{i A(x, y)} / 2$. In §1, we compute the projective Plancherel measure on the space of irreducible $\omega_A$-representations of $H$ corresponding to a given Haar measure on $H$.

Let $G$ be a connected, simply connected, nilpotent Lie group with Lie algebra $\mathfrak{g}$. In §2, we define a particular Haar measure $m_G$ on $G$ and show its invariance under certain types of changes of coordinates on $G$ (Lemma 2.1). Theorem 2.1 gives a formula (2.4) expressing $m_G$ in terms of a specific Haar measure on a certain type of closed subgroup $H \subset G$ and a specific $G$-invariant measure on the quotient space $G / H$.

In §3, the action on $V'$ contragredient to a unipotent action of $G$ on a finite-dimensional vector space $V$ is analyzed by means of the structure matrix (3.6). Theorem 3.1 tells how to parametrize the stability subgroup $G_\gamma$ for almost all $\gamma \in V'$, and describes a $G$-invariant measure on the orbit of $\gamma$ and a Haar measure on $G_\gamma$ which combine to give $m_G$ (formula (3.8)). Theorem 3.2 describes a section for the orbits of $G$ in a nonempty Zariski open subset of $V'$.

Theorem 3.3 gives an explicit formula (3.13) for the disintegration of Lebesgue measure on $V'$ under the contragredient action of $G$. The orbit measures in (3.13) are those in (3.8).

In §4, the results of §§1, 2, and 3 are combined via Kleppner and Lipsman's Plancherel formula for group extensions [15] to obtain a procedure for computing Plancherel measure for idyllic $G$ (Theorem 4.1).

The following groups are known to be idyllic: free nilpotent Lie groups; Heisenberg groups; groups in Kirillov's second example; groups of dimension $\leq 5$; 2-step groups; the nilpotent part of an Iwasawa decomposition of the $\mathbb{R}$-split form of the classical simple groups $G_2$, $A_1$ and $C_1$. Plancherel formulas are listed in Table I.
1. A projective Plancherel measure. Let $H$ be a $q$-dimensional vector space over $R$. Suppose $A : H \times H \rightarrow R$ is bilinear and skew symmetric. Let $\omega : H \times H \rightarrow T$ be the multiplier $\omega(x, y) = e^{iA(x, y)/2}$. $(T = \{z \in C : |z| = 1\})$. Let $\{u_1, \ldots, u_q\}$ be a basis of $H$, and $m_H$ the Haar measure on $H$ defined by

$$m_H(x) = \int_{R^q} f \left( \sum_{i=1}^q x^i u_i \right) dm_{R^q}(x^1, \ldots, x^q).$$

In this section, we compute the measure $\mu$ on the space of equivalence classes of irreducible $\omega$-representations of $H$, denoted $(H, \omega)^\wedge$, such that

$$\int_H |f(x)|^2 dm_{H}(x) = f \ast \omega f^*(0) = \int (H, \omega)^\wedge tr[\sigma(f \ast \omega f^*)] \, d\mu(\sigma),$$

$f \in L^1(H) \cap L^2(H)$. Here, $f \ast \omega f^*(x) = \int_H f(x - y) f^*(y) \omega(y, -x) \, dm_H(y)$, and $f^*(x) = \overline{f(-x)}$.

Suppose $\text{rank}_A = 2l$, and $q = 2l + m$. Then [3, p. 81] there is a $q \times q$ nonsingular matrix $P = (P_{ij})$ such that

$$P(A(u_i, u_j))_{1 \leq i, j \leq q} = \begin{bmatrix} 2l & m \\ 0 & I_l \\ -I_l & 0 \\ 0 & 0 \end{bmatrix}.$$ 

Let $f_i = \sum_{j=1}^q P_{ij} u_j$. Then $\{f_1, \ldots, f_q\}$ is a basis for $H$, and $A(f_i, f_j) = PA(u_i, u_j)^\dagger P$ — that is, $A(f_i, f_{i+l}) = 1 = -A(f_{i+p}, f_i)$, for $1 \leq i \leq l$. The map $\kappa_P : (R^l \times R^l) \times R^m \rightarrow H$ defined by

$$\kappa_P((x, y), z) = \sum_{i=1}^l x_i f_i + \sum_{i=1}^l y_i f_{l+i} + \sum_{i=1}^m z_i f_{2l+i},$$

for $x = (x^1, \ldots, x^l), y = (y^1, \ldots, y^l),$ and $z = (z^1, \ldots, z^m)$, is an isomorphism with the property that

$$\omega(\kappa_P(x_1, y_1, z_1), \kappa_P(x_2, y_2, z_2)) = e^{i(x_1 \cdot y_2 - x_2 \cdot y_1)/2} = \omega_1((x_1, y_1), (x_2, y_2))$$

$$= (\omega_1 \times 1)(((x_1, y_1), z_1), ((x_2, y_2), z_2)),$$

where $\omega_1 : (R^l \times R^l) \times (R^l \times R^l) \rightarrow T$ is the multiplier $\omega_1((x_1, y_1), (x_2, y_2)) = e^{i(x_1 \cdot y_2 - x_2 \cdot y_1)/2}$. Here for $x = (x^1, \ldots, x^l) \in R^l, y = (y^1, \ldots, y^l) \in R^l$, $x \cdot y$ denotes the inner product, $x \cdot y = \sum_{i=1}^l x^i y^i$. Thus, the map $\iota \kappa_P : (H, \omega)^\wedge \rightarrow ((R^l \times R^l) \times R^m, \omega_1 \times 1)$ given by $\iota \kappa_P(\sigma)((x, y), z) = \sigma(\kappa_P((x, y), z))$ for $\sigma \in (H, \omega)^\wedge, ((x, y), z) \in (R^l \times R^l) \times R^m$, is an isomorphism. Hence
\((H, \omega) \simeq ((R^l \times R^l) \times R^m, \omega_1 \times 1)^* = (R^l \times R^l, \omega_1)^* \times (R^m, 1)^* = \{\sigma_1\} \times R^m = \{\sigma_{1,t} = \sigma_1 \cdot \tau_t : t \in R^m\}\),

where \(\sigma_1\) is the unique irreducible \(\omega_1\)-representation of \(R^{2l}\) (see, for example, [17, Example 1, p. 305]), and \(\chi_t\) is a character of \(R^m\). \(\sigma_{1,t}\) can be realized on \(L^2(R^l)\) as follows. If \(h = ((x, y), z) \in (R^l \times R^l) \times R^m\), then

\[
(\sigma_{1,t}(h)F)(\nu) = \chi_t(z)(\sigma_1(x, y)F)(\nu)
\]

\[
e^{i(t \cdot z)} e^{i(y \cdot \nu + (x \cdot y)/2)} F(\nu + x).
\]

From [14, p. 490] the projective Plancherel measure for \((R^{2l}, \omega_1)\) is

\[
\mu_{R^{2l}, \omega_1}(\sigma_1) = 1/(2\pi)^l - \text{that is, }
\]

\[
(2\pi)^l \int_{R^{2l}} |\phi(x, y)|^2 \, dm_{R^{2l}}(x, y) = \frac{1}{(2\pi)^l} \text{tr}(\sigma_1(\phi * \omega_1 \phi^*))
\]

\(\phi \in L^1(R^{2l}) \cap L^2(R^{2l})\). (Here \(m_{R^{2l}}\) is Lebesgue measure \(m_{R^{2l}}\) such that \(m_{R^{2l}}([0, 1]^{2l}) = 1\).

Plancherel measure for \(R^m\) is \(\mu_{R^m} = (2\pi)^{-m} m_{R^m} - \text{i.e.,} \)

\[
\int_{R^m} |f(z)|^2 \, dm_{R^m}(z) = \frac{1}{(2\pi)^m} \int_{R^m} |\chi_t(f)|^2 \, dm_{R^m}(f),
\]

where

\[
\chi_t(f) = \int f(t) = \int_{R^m} f(s) e^{i(s \cdot t)} \, dm_{R^m}(t), \quad f \in L^1(R^m) \cap L^2(R^m).
\]

\(m_{R^m}\) is Lebesgue measure on \(R^m\) such that \(m_{R^m}([0, 1]^m) = 1\).

Let \(\nu_H\) be the image of Lebesgue measure on \((R^l \times R^l) \times R^m\) under the map \(\kappa_P\). Then

\[
\int_H f(h) \, d\nu_H(h) = \int_{(R^l \times R^l) \times R^m} f(\kappa_P((x, y), z))) \, dm_{(R^l \times R^l) \times R^m}((x, y), z)
\]

\[
= \int_{R^{2l+m}} \left( \sum_{i=1}^{2l+m} h^i f_{h_i} \right) \, dm_{R^{2l+m}}(h^1, \ldots, h^{2l+m})
\]

\[
= \int_{R^q} f \left( \sum_{i=1}^{q} \left( \frac{q}{i} h^i f_{h_i} \right) u_j \right) \, dm_{R^q}(h^1, \ldots, h^q)
\]

\[
= |\text{det} P|^{-1} \int_{R^q} f \left( \sum_{j=1}^{q} h^j u_j \right) \, dm_{R^q}(h^1, \ldots, h^q)
\]

\[
= |\text{det} P|^{-1} \int_H f(h) \, dm_H(h),
\]

so that \(m_H = |\text{det} P| \nu_H = |\text{det} P| (\kappa_P(m_{(R^l \times R^l) \times R^m}))\). It follows that

\[
\mu_{(H, \omega)} = |\text{det} P|^{-1} (\kappa_P)^{-1}(\mu_{(R^l \times R^l) \times R^m, \omega_1 \times 1})
\]

\[
= |\text{det} P|^{-1} (\kappa_P)^{-1}(\mu_{R^{2l}, \omega_1} \times \mu_{R^m})
\]

\[
= |\text{det} P|^{-1} (\kappa_P)^{-1}((2\pi)^{-l} \times (2\pi)^{-m} m_{R^m}),
\]
i.e., that \( \mu_{(H, \omega)} \) is the image of the measure \(|\det P|^{-1} (2\pi)^{-(l+m)} m_R \) on \( R^m \) under the map

\[
\psi_P : t \mapsto (t^{\kappa_P})^{-1}(\sigma_{1,t}) : R^m \to (H, \omega)\wedge.
\]

Kleppner and Baggett \([1, \text{Corollary, p. 310}]\) prove that this map is a homeomorphism. To see that

\[
\|f\|_{AH}^2 \, dm_{H}(h) = |\det P| \int_{R^m} \frac{1}{(2\pi)^{l+m}} \int_{R^m} \text{tr} \left[ \left( (t^{\kappa_P})^{-1}(\sigma_{1,t}) (f \ast \omega f^*) \right \|_{R^m}(t),
\right.
\]

we calculate that

\[
(t^{\kappa_P})^{-1}(\sigma_{1,t})(f \ast \omega f^*) = \frac{1}{|\det P|} \sigma_{1,t}((f \circ \kappa_P) \ast_{\omega_1} (f \circ \kappa_P)^*).
\]

Hence,

\[
\frac{|\det P|^{-1}}{(2\pi)^{l+m}} \int_{R^m} \text{tr} \left[ (t^{\kappa_P})^{-1}(\sigma_{1,t}) (f \ast \omega f^*) \right \|_{R^m}(t).
\]

The projective Plancherel measure \( \mu_{(H, \omega)} = (2\pi)^{-(l+m)} m_R \) on \( (H, \omega) \) corresponding to Haar measure \( m_H \) on \( H \) depends on the choice of the matrix \( P \). If \( A \) is nondegenerate, then \( |\det P| = \text{Pfaffian} (A(u_i, u_j))_{1 \leq i, j \leq q} \) \([3, \text{pp. 82–84}]\) is uniquely determined by \( A \). However, if \( A \) is degenerate, then \( P \) is quite arbitrary on the null space of \( (A(u_i, u_j))_{1 \leq i, j \leq q} \), and \( |\det P| \) is not unique.

\[
\psi_P : R^m \to (H, \omega)\wedge \text{ is the following map. Let } Q = (Q_i)_{1 \leq i, j \leq 2l+m} = P^{-1}.
\]

If \( A \) is nondegenerate, then \( M = 0 \); and \( (H, \omega)\wedge \) consists of one point, \( \psi_P = t^{\kappa_P}^{-1}(\sigma_1) \). If \( x = \sum_{i=1}^{2l} x^i u_i \in H \), then

\[
\psi_P(x) = \sigma_1(\kappa_P^{-1}(x)) = \sigma_1(xQ^{(0)}, xQ^{(2n)}),
\]

where

\[
xQ^{(0)} = \left( \sum_{i=1}^{2l} x^i Q^1_i, \ldots, \sum_{i=1}^{2l} x^i Q^l_i \right), \quad xQ^{(2n)} = \left( \sum_{i=1}^{2l} x^i Q^{1+1}_i, \ldots, \sum_{i=1}^{2l} x^i Q^{2l}_i \right).
\]
If \( m > 0 \), then \( \psi_P : \mathbb{R}^m \rightarrow (H, \omega)^p \) is given by
\[
\psi_P(t)(x) = a_1 x_1^t k_{2l}(x),
\]
where \( x = \sum_{i=1}^{2l+m} x_i u_i \in H \), \( t = (t_1, \ldots, t_m) \in \mathbb{R}^m \), and
\[
x^{2l+m} = \left( \sum_{i=1}^{2l+m} x_i^Q q_i, \ldots, \sum_{i=1}^{2l+m} x_i^{Q_2} \right),
\]
\[
x^{Q(2l)} = \left( \sum_{i=1}^{2l+m} x_i^{Q_{l+1}}, \ldots, \sum_{i=1}^{2l+m} x_i^{Q_{2l}} \right),
\]
\[
x^{Q(m)} = \sum_{a=1}^{m} \sum_{i=1}^{2l+m} x_i^{Q_{2l+a}} t_a.
\]
If \( l = 0 \), then \( A = 0; \omega = 1; (H, \omega)^p = \hat{H} \), the character group of \( H; \ m = q \); and \( P \) may be taken as the identity. In this case, \( \psi_P : \mathbb{R}^m \rightarrow \hat{H} \) is given by
\[
\psi_P(t)(x) = e^{i x^T a_1 x^a t_a} = \chi_t(x), \text{ for } x = \sum_{a=1}^{m} x^a u_a \in H, t = (t_1, \ldots, t_m) \in \mathbb{R}^m.
\]

2. Some formulas for Haar measure on \( G \). Let \( G \) be a connected, simply connected nilpotent Lie group over \( \mathbb{R} \) with Lie algebra \( g \). This section is devoted to establishing formulas for Haar measure on \( G \) in terms of certain coordinate systems for \( G \). Suppose \( \dim g = s \). The exponential map, denoted \( \exp \), is a diffeomorphism of \( g \) onto \( G \). Hence the choice of a basis \( \{e_1, \ldots, e_s\} \) in \( g \) determines a coordinate system for \( G \) by the map \( \xi : \mathbb{R}^s \rightarrow G \) given by \( \xi(x^1, \ldots, x^s) = \exp(\sum_{i=1}^{s} x_i^e_i) \). The image of Lebesgue measure on \( \mathbb{R}^s \) under this map is a Haar measure on \( G \), called the measure on \( G \) defined in terms of the basis \( \{e_1, \ldots, e_s\} \) of \( g \).

Let \( B \) be a basis of \( g \). A linear order \( < \) on \( B \) is called a Jordan-Hölder order if, for each \( \nu \) in \( B \), \( [g, \nu] = 0 \) if \( \nu \) is maximal; otherwise, \( [g, \nu] \subseteq \text{span} \{\omega \in B : \nu < \omega\} \). Suppose \( e_1 < \cdots < e_s \) is a basis of \( g \) in Jordan-Hölder order, i.e., \( [g, e_1] = 0, [g, e_i] \subseteq \text{span} \{e_{i+1}, \ldots, e_s\} \) for \( 1 < i < s - 1 \). Let \( m^{B} \) be Lebesgue measure on \( \mathbb{R}^s \) such that \( m^{B}([0, 1]^s) = 1 \). \( m^G \) will denote the Haar measure on \( G \) defined in terms of \( \{e_1, \ldots, e_s\} \); so that
\[
\int_G f(A) \, dm^G(A) = \int_{\mathbb{R}^s} f \left( \exp \left( \sum_{i=1}^{s} x_i^e_i \right) \right) \, dm^{B}(x^1, \ldots, x^s).
\]

Invariance of \( m^G \) under left and right translation follows from the Campbell-Baker-Hausdorff formula, \( \exp x \exp y = \exp(x + y + \frac{1}{2}[x, y] + \cdots) \), and the fact that \( e_1 < \cdots < e_s \) is a Jordan-Hölder basis of \( g \). Then the fact that the measure on \( G \) defined in terms of any basis of \( g \) is a Haar measure follows. Indeed, if \( m \) is the measure on \( G \) defined in terms of the basis \( \{\omega_1, \ldots, \omega_s\} \) of \( g \), and if \( \omega_i = \sum_{j=1}^{s} a_{ij}^i e_j \) for \( 1 < i < s \), then \( m = |\det A|^{-1} m^G \), where \( A = (a_{ij}^i)_{1 < i, j < s} \).
Because \( e_1 < \cdots < e_s \) is a Jordan-Hölder basis of \( \mathfrak{g} \), the measure on \( G \) given in terms of the coordinate system \( \xi(x^1, \ldots, x^s) = \exp(\sum_{i=1}^s x^i e_i) \) is the same as the measure on \( G \) obtained by taking the image of Lebesgue measure on \( \mathbb{R}^s \) under the map \( \eta(x^1, \ldots, x^s) = \exp x_1 e_1 \cdots \exp x^s e_s \). In fact, any sum and any permutation is allowed in the sense of the following lemma.

**Lemma 2.1.** Let \( \{e_1, \ldots, e_s\} \) be a Jordan-Hölder basis of \( \mathfrak{g} \) such that 
\[
[g, e_i] = 0, \quad [g, e_j] \subseteq \text{span}\{e_{i+1}, \ldots, e_s\} \quad \text{for} \quad 1 \leq i \leq s - 1.
\]
Let \( \sigma \) be a permutation of \( \{1, \ldots, s\} \). If \( f \in C_0(G) \) (= continuous functions with compact support), then, for \( 1 \leq m \leq s \),

\[
\int_{\mathbb{R}^s} \left( \exp \left( \sum_{i=1}^s x^i e_i \right) \right) \, dm_{\mathbb{R}^s}(x^1, \ldots, x^s)
= \int_{\mathbb{R}^s} \left[ \exp \left( \sum_{i=m+1}^s x_\sigma(i) e_\sigma(i) \right) \prod_{i=m+1}^s \exp x_\sigma(i) e_\sigma(i) \right] \, dm_{\mathbb{R}^s}(x^1, \ldots, x^s).
\]

**Proof.** For \( x = (x^1, \ldots, x^s) \in \mathbb{R}^s \), put

\[
T(x) = \exp \left( \sum_{i=1}^m x_\sigma(i) e_\sigma(i) \right) \prod_{i=m+1}^s \exp x_\sigma(i) e_\sigma(i).
\]

The Campbell-Baker-Hausdorff formula,

\[
\exp v \exp w = \exp \left( v + w + \frac{1}{2} [v, w] + \frac{1}{12} ([v, [v, w]] - [w, [v, w]]) + \cdots \right),
\]

where \( v, w \in \mathfrak{g} \), shows that \( T(x) = \exp(\sum_{k=1}^s x^k e_k + B(x)) \), where \( B(x) \in \mathfrak{g} \) is a sum of terms of the form

\[
[* \cdots [x^i e_j, [\cdots [x^i e_j, x^i e_j] \cdots] \cdots \cdots \cdots \cdots].
\]

Let \( \phi^k(x) \) denote the \( k \)th component with respect to the basis \( \{e_i\}_{i=1}^s \) of \( \mathfrak{g} \) of \( B(x) \). Since \( e_1 < \cdots < e_s \) is a Jordan-Hölder basis of \( \mathfrak{g} \), \( \phi^k \) is independent of \( (x^k, \ldots, x^s) \). Indeed, if \( j \geq k \), then

\[
[* [x^i e_j, [\cdots [x^i e_j, x^i e_j] \cdots] \cdots \cdots \cdots \cdots] \in \text{span}\{e_{j+1}, \ldots, e_s\} \subseteq \text{span}\{e_{k+1}, \ldots, e_s\}.
\]

Thus the only terms \((*)\) in \( B(x) \) which can have a nonzero component in the direction of \( e_k \) are those brackets involving only \( x^1 e_1, \ldots, x^{k-1} e_{k-1} \). Hence \( \phi^k \) is a function of \( (x^1, \ldots, x^{k-1}) \). Therefore,

\[
T(x) = \exp \left( x^1 e_1 + x^2 e_2 + \sum_{k=3}^s \left( x^k + \phi^k(x^1, \ldots, x^{k-1}) \right) e_k \right).
\]

(2.1) follows from (2.3) by Fubini's Theorem. Considering the right-hand
side of (2.1) as an iterated integral and using (2.3), we make \( s - 2 \) successive substitutions \( x^i \rightarrow x^i - \varphi^{i-1}(x^1, \ldots, x^{i-1}) \) holding \( x^1, \ldots, x^{i-1} \) fixed, for \( i = 0, 1, \ldots, s - 3 \). The result is the left-hand side of (2.1).

The following lemma and theorem establish a formula for \( m_G \) in terms of coordinates on a certain type of Lie subgroup \( H \) of \( G \) and on the quotient manifold \( G/H \).

**Lemma 2.2.** Suppose \( \mathfrak{h} \) is a subalgebra of \( \mathfrak{g} \), and \( H = \exp \mathfrak{h} \) is the corresponding Lie subgroup of \( G \). Suppose \( \dim(\mathfrak{g}/\mathfrak{h}) = r \), and \( \mathfrak{h} = \mathfrak{h}_{r+1} \subset \mathfrak{h}_r \subset \cdots \subset \mathfrak{h}_1 = \mathfrak{g} \) is an ascending sequence of subalgebras of \( \mathfrak{g} \) such that

\[
\dim(\mathfrak{h}_i/\mathfrak{h}_{i+1}) = 1 \quad \text{for } 1 \leq i \leq r.
\]

Suppose \( \omega_i \) is in \( \mathfrak{h}_i \), not in \( \mathfrak{h}_{i+1} \), for \( 1 \leq i \leq r \). Then the map \( (t^1, \ldots, t^r) \rightarrow H \exp t^r \omega_r \cdots \exp t^1 \omega_1 \) is a homeomorphism of \( \mathbb{R}^r \) onto \( G/H \). The image of Lebesgue measure on \( \mathbb{R}^r \) under this map is a \( G \)-invariant measure on \( G/H \).

**Proof.** Pukánszky gives a proof in [19, pp. 85, 97].

This measure will be called the measure on \( G/H \) defined in terms of the basis \( \{\omega_1, \ldots, \omega_r\} \) of \( \mathfrak{g}/\mathfrak{h} \).

If \( m_H \) is any Haar measure on \( H \), and \( \nu \) is any \( G \)-invariant measure on \( G/H \), then \( \nu \) and \( m_H \) combine to give a Haar measure on \( G \), i.e.,

\[
\int_G f(x) \, dx = \int_{G/H} \int_H f(hx) \, dm_H(h) \, d\nu(x)
\]

defines a Haar measure on \( G \). For the subgroups of \( G \) which occur in the sequel, the measures \( \nu \) and \( m_H \) can be chosen so that the resulting Haar measure on \( G \) is exactly \( m_G \). The following theorem gives the conditions that will arise and the proof for this type of subgroup \( H \subset G \).

**Theorem 2.1.** Suppose \( \mathfrak{h} \) is a subalgebra of \( \mathfrak{g} \) having a basis \( \{u_1, \ldots, u_q\} \) with the following property. There is a partition \( \{1, \ldots, s\} = \{m_1 < \cdots < m_q\} \cup \{i_1 < \cdots < i_r\} \) such that

\[
u_b = e_{m_b} - \sum_{\{i: m_b < i_r\}} \lambda_i^{m_b} e_{i_r} \quad \text{for } 1 \leq b \leq q.
\]

Let \( H = \exp \mathfrak{h} \), and let \( m_H \) be the Haar measure on \( H \) defined in terms of \( \{u_1, \ldots, u_q\} \).

Then the map \( (t^1, \ldots, t^r) \rightarrow H \exp t^r e_{i_r} \cdots \exp t^1 e_{i_1} \) is a homeomorphism of \( \mathbb{R}^r \) with \( G/H \). The image of Lebesgue measure on \( \mathbb{R}^r \) under this map is a \( G \)-invariant measure, \( \nu \), on \( G/H \). \( m_G, \nu, \) and \( m_H \) satisfy

\[
\int_G f(A) \, dm_G(A) = \int_{G/H} \int_H f(hA) \, dm_H(h) \, d\nu(A),
\]

i.e.,
\[
\int_{R^s} f \left( \exp \left( \sum_{i=1}^{s} x^i e_i \right) \right) \, dm_{R^s}(x^1, \ldots, x^s)
\]

(2.4) \[
= \int_{R^q} \left[ \int_{R^s} f \left( \exp \left( \sum_{i=1}^{s} z^i u_i \right) \exp t^1 e_{i_1} \cdots \exp t^q e_{i_1} \right) \right] \, dm_{R^q}(z^1, \ldots, z^q) \, dm_{R^r}(t^1, \ldots, t^r).
\]

**Proof.** Let \( \mathfrak{h}_{r+1} = \mathfrak{h} \), and \( \mathfrak{h}_k = \mathfrak{h}_{k+1} \oplus (e_{i_k}) \) for \( r \geq k \geq 1 \). Then

\( \mathfrak{h} = \mathfrak{h}_{r+1} \subset \mathfrak{h}_r \subset \mathfrak{h}_s \subset \mathfrak{h}_1 = \mathfrak{g} \) is an increasing sequence of subspaces of \( \mathfrak{g} \) such that \( \text{dim}(\mathfrak{h}_k/\mathfrak{h}_{k+1}) = 1 \); and \( e_{i_k} \) is in \( \mathfrak{h}_k \), not in \( \mathfrak{h}_{k+1} \), for \( 1 \leq k \leq r \). Thus, the fact that the map \( \psi : (t^1, \ldots, t^r) \rightarrow H \exp t^r e_{i_r} \cdots \exp t^1 e_{i_1} \) is a homeomorphism of \( R^r \) onto \( G/H \) and that \( \nu = \psi(m_{R^r}) \) is a \( G \)-invariant measure on \( G/H \) is just Lemma 2.2, once it is shown that each \( \mathfrak{h}_k, r \geq k \geq 1 \), is a subalgebra of \( \mathfrak{g} \).

To prove that each \( \mathfrak{h}_k \) is a subalgebra of \( \mathfrak{g} \), we first prove, by calculating brackets, that \( [\mathfrak{g}, e_{i_k}] \subset \mathfrak{h}_{k+1} \) for \( r \geq k \geq 1 \). Let \( x \in \mathfrak{g} \) and \( 1 \leq k \leq r \). Then

\[
[x, e_{i_k}] = \sum_{n=(i_{k+1})+1}^{s} a_{ik}^n(x)e_n
\]

(by the hypothesis on \( \{u_1, \ldots, u_q\} \)).

Thus \( [x, e_{i_k}] \) is in \( \text{span}(\{u_b : m_b > i_k\} \cup \{e_{i_s} : i_s > i_k\}) \), which is contained in \( \mathfrak{h} \oplus (e_{i_r}) \oplus \cdots \oplus (e_{i_{(k+1)}}) = \mathfrak{h}_{k+1} \).

That each \( \mathfrak{h}_k \) is a subalgebra of \( \mathfrak{g} \) follows by induction. \( \mathfrak{h}_{r+1} = \mathfrak{h} \) is a subalgebra of \( \mathfrak{g} \) by hypothesis. Assume \( \mathfrak{h}_{k+1} \) is a subalgebra of \( \mathfrak{g} \). Then, for \( \mathfrak{h}_k = \mathfrak{h}_{k+1} + (e_{i_k}) \), we have \([\mathfrak{h}_k, \mathfrak{h}_k] = [\mathfrak{h}_{k+1}, \mathfrak{h}_{k+1}] + [\mathfrak{h}_{k+1}, e_{i_k}] \) contained in \( \mathfrak{h}_{k+1} \), since \( [\mathfrak{h}_{k+1}, \mathfrak{h}_{k+1}] \subset \mathfrak{h}_{k+1} \) by inductive hypothesis, and \([\mathfrak{h}_{k+1}, e_{i_k}] \subset [\mathfrak{g}, e_{i_k}] \subset \mathfrak{h}_{k+1} \) by the preceding calculation. Since \( \mathfrak{h}_{k+1} \) is contained in \( \mathfrak{h}_k \), this shows that \( \mathfrak{h}_k \) is a subalgebra of \( \mathfrak{g} \).

The rest of the proof is an application of Lemma 2.1 to show that \( m_G, \nu, \) and \( m_H \) satisfy (2.4). The set \( \{e_{i_1}, \ldots, e_{i_r}\} \cup \{u_1, \ldots, u_q\} \) is a basis of \( \mathfrak{g} \). For \( 1 \leq k \leq s \), let

\[
f_k = e_{i_k} \quad \text{if} \quad k = i_t
\]

\[
= u_b \quad \text{if} \quad k = m_b.
\]
Then $f_1, \ldots, f_s$ is a Jordan-Hölder basis of $g$ such that $[g, f_s] = 0$, $[g, f_k] \subset \text{span}(f_{k+1}, \ldots, f_s)$, for $1 \leq k \leq s - 1$. Indeed, if $k = i$, then

$$[g, f_k] = [g, e_{i_1}] \subset \text{span}([u_b : m_b > i_1] \cup \{e_{i_2} : i_2 > i_1\})$$

by the preceding calculation. Since $\{u_b : m_b > i_1\} = \{f_l : l = m_b > k\}$, and $\{e_{i_2} : i_2 > i_1\} = \{f_l : l = i_2 > k\}$, we have $[g, f_k] \subset \text{span}(f_{k+1}, \ldots, f_s)$. If $k = m_b$, then $[g, f_k] = [g, u_b] = [g, e_{m_b} - \Sigma_{\{s : i_s > m_b\}} \lambda_m e_{i_s}]$, which is contained in $\text{span}\{e_l : l > m_b\}$. Now

$$\text{span}\{e_l : l > m_b\} = \text{span}\{u_a : m_a > m_b\} \cup \{e_{i_2} : i_2 > m_b\}$$

(since $e_{m_a} = u_a + \Sigma_{\{s : i_s > m_a\}} \lambda_m e_{i_s}$). Since $\{u_a : m_a > m_b\} = \{f_l : l = m_a > m_b = k\}$, and $\{e_{i_2} : i_2 > m_b\} = \{f_l : l = i_2 > m_b = k\}$, we have $[g, f_k] \subset \text{span}(f_{k+1}, \ldots, f_s)$.

To apply Lemma 2.1, let $\sigma \in S_s$ be a permutation such that $i_t = \sigma(t - s + 1)$ for $1 \leq t \leq r$, and $m_b = \sigma(b)$ for $1 \leq b \leq q$. Now, taking $f \in C_0(b)$ and using Fubini’s theorem, the right-hand side of (2.4) may be written as

$$\int_{R^r} \left[ \int_{R^q} f \left( \exp \left( \sum_{b=1}^{q} x^m u_b \right) \exp x^r f_{i_1} \cdots \exp x^1 e_{i_1} \right) \right] dm_{R^q}(x^{m_1}, \ldots, x^{m_q}) dm_{R^r}(x^{r_1}, \ldots, x^{r_r})$$

(by Fubini)

$$= \int_{R^s} f \left( \exp \left( \sum_{b=1}^{q} x^m f_{m_b} \right) \exp x^r f_{i_1} \cdots \exp x^1 f_{i_1} \right) dm_{R^s}(x^{1}, \ldots, x^{s})$$

(by definition of $\{f_k : 1 \leq k \leq s\}$)

$$= \int_{R^s} f \left( \exp \left( \sum_{b=1}^{q} x^{\sigma(b)} f_{\sigma(b)} \right) \exp x^{\sigma(q+1)} f_{\sigma(q+1)} \cdots \exp x^{\sigma(s)} f_{\sigma(s)} \right)$$

(by definition of $\sigma$)

$$= \int_{R^s} f \left( \exp \left( \sum_{k=1}^{s} x^k f_k \right) \right) dm_{R^s}(x^{1}, \ldots, x^{s})$$

(by Lemma 2.1).

If $f_k = \Sigma^s_{j=1} a^j e_j$, $1 \leq k \leq s$, then $|\det(a^j)_{1 \leq j \leq k \leq s}| = 1$, since $f_{m_b} = u_b$

$$\equiv e_{m_b} (e_{(m_b)+1}, \ldots, e_s)$,

$1 \leq b \leq q$, and $f_{i_t} = e_{i_t}$, $1 \leq t \leq r$.

Thus the final integral above is equal to
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\[ \int_{\mathbb{R}^s} f \left( \exp \left( \sum_{k=1}^{s} x^k e_k \right) \right) \, dm_{\mathbb{R}^s}(x^1, \ldots, x^s), \]

which is the left-hand side of (2.4).

3. A disintegration theorem. Suppose \( G \) is a connected, simply connected nilpotent Lie group over \( \mathbb{R} \) with Lie algebra \( \mathfrak{g} \); \( V \) a finite-dimensional vector space over \( \mathbb{R} \); and \( G \times V \rightarrow V : (A, v) \rightarrow Av \) a unipotent action of \( G \) on \( V \). This section is devoted to analyzing the contragredient action of \( G \) on the dual space \( V' \) of \( V' \times G \rightarrow V' : (\gamma, A) \rightarrow (v \rightarrow \langle \gamma, Av \rangle) \). After establishing terminology, notation, and preliminary facts about orbits, stability subgroups, and the relation between the action of \( G \) and that of \( \mathfrak{g} \), we develop a technique for (1) computing almost all the stability subgroups for the action of \( G \) on \( V' \), (2) coordinatizing almost all the orbits of \( G \) in \( V' \), and (3) coordinatizing almost all the orbit space \( V'/G \). We establish a formula (3.8) giving Haar measure on \( G \) in terms of Haar measure on the stability subgroup \( G_\gamma \) and a \( G \)-invariant measure on the orbit \( G/G_\gamma \). Lebesgue measure on \( V' \), denoted \( m_{V'} \), is decomposed by \( G \) into a measure on the orbit space \( V'/G \) and measures on the corresponding orbits. We prove an explicit formula (3.13) for this disintegration of \( m_{V'} \), by \( G \), in which the orbit measures are those appearing in (3.8). This coincidence of the orbit measures is necessary for the proof of the Plancherel formula in §4.

Let \( G \) be a connected, simply connected nilpotent Lie group over \( \mathbb{R} \) with Lie algebra \( \mathfrak{g} \). Suppose \( V \) is a \( K \)-dimensional vector space over \( \mathbb{R} \) on which \( G \) acts smoothly as a group of unipotent automorphisms, i.e., the mapping \( G \times V \rightarrow V : (A, v) \rightarrow Av \) is differentiable. Then for each \( \nu \in V \) the map \( F_\nu : G \rightarrow V \) given by \( F_\nu(A) = Av, A \in G \), is differentiable. Its derivative defines an action of \( \mathfrak{g} \) as a nilpotent Lie algebra of endomorphisms of \( V \) by \( \mathfrak{a} \in \mathfrak{g} \), \( v \in V \), then \( (\exp a)(v) = (1 + a + a^2/2! + \cdots + a^k/k!)(v) \).

Let \( V' \) denote the dual space of \( V \). The contragredient action of \( G \) (resp. \( \mathfrak{g} \)) on \( V' \) is given by \( V' \times G \rightarrow V' \) (resp. \( V' \times \mathfrak{g} \rightarrow V' \)) : \((\gamma, A) \rightarrow \gamma A \), where \( \langle \gamma A, v \rangle = \langle \gamma, Av \rangle \) for \( A \in G \) (resp. \( \mathfrak{g} \)), \( \gamma \in V', v \in V \). For \( \gamma \) in \( V' \), let \( F_\gamma : G \rightarrow V' \) be the map \( F_\gamma(A) = \gamma \cdot A \). Let \( O_\gamma = F_\gamma(G) \) denote the orbit of \( \gamma \) in \( V' \); \( G_\gamma = \{ A \in G : \gamma \cdot A = \gamma \} \), the stabilizer of \( \gamma \) in \( G \). \( F_\gamma \) is differentiable. Its derivative at \( A \) in \( G \), denoted \( dF_\gamma(A) \), maps the tangent space to \( G \) at \( A, T_A(G) = dL_A(e)(\mathfrak{g}) \) (where \( L_AB = AB \) for \( A, B \in G \)), into the tangent space to \( V' \) at \( F_\gamma(A) = \gamma \cdot A, T_{\gamma \cdot A}(V') \). If \( x \in \mathfrak{g} = T_e(G) \), then

\[
(3.1) \quad dF_\gamma(e) x = \frac{d}{dt} F_\gamma(e \exp tx) \big|_{t=0} = \frac{d}{dt} (\gamma \cdot \exp tx) \big|_{t=0} = \gamma \cdot x.
\]

Let \( \mathfrak{g}_\gamma = \text{Ker} \, dF_\gamma(e) = \{ x \in \mathfrak{g} : \gamma \cdot x = 0 \} \).
Proposition 3.1.

(i) \( \Omega_\gamma \) is closed in \( V' \).

(ii) \( G_\gamma \) is a Lie subgroup of \( G \), and \( T_e(G_\gamma) = \ker dF_\gamma(e) = \mathfrak{g}_\gamma \).

(iii) \( \Omega_\gamma \) is a submanifold \( (C^\infty) \) of \( V' \); \( h_\gamma : G/G_\gamma \to \Omega_\gamma \); \( G_\gamma x \to \gamma \cdot x \) is a diffeomorphism of the quotient manifold (analytic) \( G/G_\gamma \) onto the manifold \( \Omega_\gamma \); and the tangent space at \( \gamma \) to \( \Omega_\gamma \), \( T_\gamma(\Omega_\gamma) = \text{im} dF_\gamma(e) \).

Proof. (i) is in [2, p. 7]. (ii) and (iii) are in [4, Chapitre 3, Proposition 14, p. 108]. ((i) is necessary for (iii) since one needs \( \Omega_\gamma \) to be a Baire space and \( G \) to be separable to show that \( h_\gamma : G/G_\gamma \to \Omega_\gamma \) is open.)

Proposition 3.1(ii) implies that \( G_\gamma = \exp \mathfrak{g}_\gamma \), since \( \exp : \mathfrak{g} \to G \) is a diffeomorphism, and \( \exp x \in G_\gamma \) implies \( x \in \mathfrak{g}_\gamma \) in this case.

If \( A \in G \), let \( \pi(A) : V' \to V' \) be \( \pi(A)(\gamma) = \gamma \cdot A \). Then for \( \gamma \in V' \),
\[
F_{\gamma \cdot A} = F_\gamma \circ L_A = \pi(A) \circ F_\gamma \circ C_A,
\]
where \( C_A : G \to G : x \mapsto AxA^{-1} \). By the chain rule,
\[
dF_{\gamma \cdot A}(e) = dF_{\gamma}(A)dL_A(e) \\
= d\pi(A)(\gamma)dF_{\gamma}(e)dC_A(e) = d\pi(A)(\gamma)dF_{\gamma}(e)\Ad(A).
\]

(3.2) rank\(_R(dF_{\gamma \cdot A}(e)) = \text{rank}_R(dF_{\gamma}(A)) = \text{rank}_R(dF_{\gamma}(e)).

Thus, from Proposition 3.1(iii),
\[
\dim(T_{\gamma \cdot A}(\Omega_\gamma)) = \dim(\text{im} dF_{\gamma \cdot A}(e)) = \dim(\text{im} dF_{\gamma}(e)) = \dim(T_\gamma(\Omega_\gamma)).
\]

Also, by (3.2) \( x \in \mathfrak{g} \) is in \( \ker dF_{\gamma \cdot A}(e) \) if and only if \( dL_A(e)x \) is in \( \ker dF_{\gamma}(A) \) if and only if \( \Ad(A)x \) is in \( \ker dF_{\gamma}(e) \) if and only if \( x \) is in \( \Ad(A)^{-1}(\ker dF_{\gamma}(e)) \).

Hence
\[
(3.5) \mathfrak{g}_{\gamma \cdot A} = \ker dF_{\gamma \cdot A}(e) = \Ad(A^{-1})(\ker dF_{\gamma}(e)) = \Ad(A^{-1})(\mathfrak{g}_\gamma).
\]

To develop computational machinery, we take bases in \( V \) and \( \mathfrak{g} \). Let \( v_1 < \cdots < v_K \) be a basis for \( V \) in Jordan-Hölder order relative to \( \mathfrak{g} \), i.e., \( g\mathfrak{g}_K = 0 \), \( g\mathfrak{g}_i \subseteq \text{span}\{v_{i+1}, \ldots, v_K\} \) for \( 1 \leq i \leq K - 1 \). Let \( \{v^1, \ldots, v^K\} \) be the dual basis of \( V' \), and let \( m_{V'} \) denote the measure on \( V' \) defined in terms of this basis, i.e.,
\[
\int_{V'} f(\gamma) \, dm_{V'}(\gamma) = \int_{\mathbb{R}^K} f\left( \sum_{i=1}^K \gamma_i v^i \right) \, dm_{\mathbb{R}^K}(\gamma_1, \ldots, \gamma_K).
\]

For \( A \in G \), put \( (A(m_{V'}), f) = \int_{V'} f(\gamma) \, dm_{V'}(\gamma) \). Then \( A(m_{V'}) = m_{V'} \), since the determinant of \( (\gamma \mapsto \gamma \cdot A) \) is one for all \( A \) in \( G \). Let \( m_G \) denote the Haar measure on \( G \) defined in terms of the Jordan-Hölder basis \( e_1 < \cdots < e_s \) of \( \mathfrak{g} \) as in §2.
Consider the matrix

\[ M = (e_i u_j)_{1 \leq i < s, 1 \leq j < K}. \]

The entries \( e_i u_j \) are vectors in \( V \), so are elements in the field of fractions of the symmetric algebra of \( V \), denoted \( F_V \). If \( R \) is in \( F_V \), then \( R = P/Q \), for \( P, Q \) in the symmetric algebra, \( S_V \), of \( V \). \( S_V \) is isomorphic to the ring of polynomial functions on \( V' \) by the map \( P \rightarrow (\gamma \rightarrow P(\gamma)) \), where

\[ P(\gamma) = P(\gamma_1, \ldots, \gamma_K) = \sum a_{i_1 \ldots i_K} \gamma_1^{i_1} \cdots \gamma_K^{i_K}, \quad \text{for } \gamma = \sum_{i=1}^{K} \gamma_i v^i V'. \]

If \( R = P/Q \in F_V \), and \( \gamma \in V' \), then define

\[ R(\gamma) = P(\gamma)/Q(\gamma) \text{ whenever } Q(\gamma) \neq 0. \]

The map \( R \rightarrow (\gamma \rightarrow R(\gamma)) \) is an isomorphism of \( F_V \) with the field of rational functions on \( V' \). (As an element in \( F_V \), a vector \( v \in V \) corresponds to the function \( \gamma \rightarrow v(\gamma) = \langle \gamma, v \rangle \) on \( V' \).)

\( M \) is called the structure matrix for the action of \( g \) on \( V \). Since the elements in \( M \) are rational functions on \( V' \), properties of \( M \)—its rank, its independent rows and columns, its minors—are useful in analyzing the contragredient action of \( g \), hence of \( G \), on \( V' \). In fact, all the major formulas in this paper come via \( M \). \( M \) works because \( g \) is nilpotent, and \( \{e_1 < \cdots < e_s\}, \{v_1 < \cdots < v_K\} \) are Jordan-Hölder bases.

For \( \gamma \in V' \), let \( M(\gamma) \) denote the matrix \( (\langle \gamma, e_i u_j \rangle)_{1 \leq i < s, 1 \leq j < K} \). Since

\[ \langle \gamma, e_i u_j \rangle = \langle \gamma e_i, u_j \rangle = \langle dF_{\gamma}(e) v_j, e_i \rangle \text{ by (3.1)}, \]

\( M(\gamma) \) is the matrix for \( dF_{\gamma}(e) : g \rightarrow V' \) in terms of the basis \( \{e_1, \ldots, e_s\} \) of \( g \), and \( \{v_1, \ldots, v_K\} \) of \( V' \). Thus by (3.4)

\[ \text{rank}_R(M(\gamma)) = \text{rank}_R(dF_{\gamma}(e)) = \dim T_{\gamma}(O_\gamma) \]

(3.7)

(3.7) = (the dimension of the orbit of \( \gamma \) under \( G \)).

Suppose \( \text{rank}_{F_V} M = r > 0 \). Let \( d = K - r, q = s - r \). For \( 1 \leq i \leq s, 1 \leq j \leq K \), let \( R_i = (e_i u_1, \ldots, e_i u_K) \) denote the \( i \)th row of \( M \), and

\[ C_j = \begin{pmatrix} e_1 u_j \\ \vdots \\ e_s u_j \end{pmatrix} \]

denote the \( j \)th column of \( M \). Choose indices \( 1 \leq i_1 < \cdots < i_r < s \) (resp. \( 1 \leq l_1 < \cdots < l_r < s \)) as follows: \( i_r \) (resp. \( l_r \)) in the largest integer \( (1 \leq i_r < s) \) such that \( R_{i_r} \neq 0 \) (resp. \( l_r \neq 0 \)). Having chosen \( i_k \) (resp. \( l_k \)), \( i_{k-1} \) (resp. \( l_{k-1} \)) is the largest integer \( (1 \leq i_{k-1} < i_k) \) such that \( R_{i_{k-1}} \) (resp. \( C_{l_{k-1}} \)) is linearly independent in \( (F_V)^K \) (resp. \( (F_V)^d \)) from \( R_{i_k}, \ldots, R_{i_r}, C_{l_k}, \ldots, C_{l_r} \). Next, choose \( 1 \leq m_1 < \cdots < m_q < s \) (resp. \( 1 \leq l_1 < \cdots < l_q = K \)) such that \( \{i_1, \ldots, i_r\}, \{m_1, \ldots, m_q\} \) (resp. \( \{l_1, \ldots, l_r\}, \{j_1, \ldots, j_d\} \)) is a partition of \( \{1, \ldots, s\} \) (resp. \( \{1, \ldots, K\} \)).
In a sense (to be made precise), the dependent columns \( \{C_1, \ldots, C_d\} \) of \( M \) provide a coordinate system for almost all of \( V'/G \); and the independent rows \( \{R_1, \ldots, R_r\} \) of \( M \) provide coordinates for almost all the orbits of \( V' \) under \( G \); while the dependent rows \( \{R_{m_1}, \ldots, R_{m_q}\} \) parametrize almost all the stability subalgebras \( g_\gamma \subset g \).

Let \( M^{(r)} \) denote the \( r \times r \) matrix \((e_1, v_1)_{1 \leq a, b \leq r} \). Since \( \text{rank}_{F,V} M = r \), and \( R_1, \ldots, R_r \) (resp. \( C_1, \ldots, C_r \)) are linearly independent rows (resp. columns) of \( M \),

\[
\text{rank}_{F,V} M^{(r)} = \text{rank}_{F,V} [(e_1, v_1)_{1 \leq a, b \leq r}] = r.
\]

Therefore \( \det M^{(r)} = \sum_{\sigma \in S_r} \text{sign} \sigma (e_1, v_{\sigma(1)}) \cdots (e_r, v_{\sigma(r)}) \) is a nonzero element in \( S_V \), so there is a \( \gamma \in V' \) such that the polynomial

\[
(det M^{(r)}(\gamma) = \sum_{\sigma \in S_r} \text{sign} \sigma \langle \gamma, e_1, v_{\sigma(1)} \rangle \cdots \langle \gamma, e_r, v_{\sigma(r)} \rangle
\]

\[
= \det(M^{(r)}(\gamma)) \neq 0.
\]

Let \( E = \{ \gamma \in V' : \det M^{(r)}(\gamma) \neq 0 \} \). \( E \) is a nonempty Zariski open set in \( V' \).

**Lemma 3.1.** \( E \) is a \( G \)-invariant set containing only maximal dimension orbits.

**Proof.** \( \text{rank}_{F,V} M = r \) implies that every \( (r + 1) \times (r + 1) \) minor of \( M \) is zero. Hence, if \( \gamma \in V' \), then every \( (r + 1) \times (r + 1) \) minor of \( M(\gamma) \) is zero. Thus, \( \text{rank}_{R}(M(\gamma)) \leq r \). If \( \gamma \in E \), then \( \text{rank}_{R}(M(\gamma)) = r \). By (3.7), \( \text{rank}_{R}(M(\gamma)) \) is the dimension of the orbit of \( \gamma \) under \( G \). Thus, if \( \gamma \in E \), then \( O_\gamma \) has maximum possible dimension.

For \( 1 \leq j \leq K \), let \( M_j = (e_1, v_k)_{1 \leq i \leq j, 1 \leq k \leq K} \); \( r_j = \text{rank}_{F,V} M_j \) (then \( 0 = r_K \leq r_{K-1} \leq \cdots \leq r_1 = r \)); \( U_j = \{ \gamma \in V' : \text{rank}_{R}(M(\gamma)) = r_j \} \); and \( U = \bigcap_{j=1}^K U_j \).

Each \( U_j \) is a nonempty Zariski open set in \( V' \). (The set \( B_j \) of all \( r_j \times r_j \) minors of \( M_j \) is a family of polynomial functions on \( V' \), and \( U_j = \{ \gamma \in V' : P(\gamma) \neq 0 \text{ for some } P \in B_j \} \).

To show that \( U_j \) is \( G \)-invariant, we must show that \( \text{rank}_{R} M_j(\gamma \cdot A) = \text{rank}_{R} M(\gamma) \) for all \( A \in G \). Note that \( \{v_1, \ldots, v_K\} \) (the basis of \( V' \) dual to the basis \( \{v_1, \ldots, g \}\) of \( V' \) relative to \( g \)) such that \( v^T \cdot g = 0 \), and \( v^T \cdot g \subset \text{span} \{v_1, \ldots, v^{j-1}\} \) for \( 2 \leq i \leq K \). (For \( x \in g \), the \( (uv)^th \) component of \( u^T \cdot x \) is \( (uv)_{a}(a) = v(a)u(a) \). Since \( x_{a+1}, \ldots, v_K \), \( u^T \cdot x_{a}(a) \) is zero if \( a > i \).) Let \( V_1 = (0) \); \( V_{j+1} = \text{span} \{v_1, \ldots, v^{j-1}\} \) for \( 2 \leq j \leq K + 1 \). Each \( V_j \) is invariant under \( G \), so \( G \) acts on \( V'/V_j \) by \( P(\gamma) \cdot A = P(\gamma \cdot A) \), where \( \gamma \in V' \), \( A \in G \), and \( P(\gamma) : V' \rightarrow V'/V_j \) is the projection. Let \( F(P(\gamma)) : V' \rightarrow V'/V_j \) be the map \( F(P(\gamma))(A) = P(\gamma) \cdot A \). Then for \( \gamma \) \( \gamma \cdot A \).
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\[ (dF_{P_j}(\gamma)(e)(e_i))(v_k) = (P_j(\gamma) \cdot e_i)(v_k) = P_j(\gamma)(e_i) v_k = \sum_{i=1}^{K} \gamma_i v^i(e_i v_k) \]

\[ = \sum_{i=1}^{K} \gamma_i v^i(e_i v_k) = \gamma(e_i v_k). \]

\[ (\Sigma_{i=1}^{j-1} \gamma_i v^i(e_i v_k) = 0 \text{ because } (e_i v_k) \in \text{span}(v_{k+1}, \ldots, v_K) \text{ and } j - 1 < j < k. \]

Thus the matrix for \( dF_{P_j}(\gamma)(e) : \mathfrak{g} \to V'/V_j \) in terms of the basis \( \{e_1, \ldots, e_s\} \) of \( \mathfrak{g} \) and \( \{v', \ldots, v^K\} \) of \( V'/V_j \) is \( M_j(\gamma) \). Hence, if \( A \in G \), we have, by (3.3),

\[ \text{rank}_R(M_j(\gamma)) = \text{rank}_R(dF_{P_j}(\gamma)(e)) = \text{rank}_R(dF_{P_j}(\gamma)(A)(e)) = \text{rank}_R(dF_{P_j(\gamma \cdot A)}(e)) = \text{rank}_R(M_j(\gamma \cdot A)). \]

Since each \( U_i \) is \( G \)-invariant, \( U = \bigcap_{i=1}^{K} U_i \) is \( G \)-invariant.

For \( 1 \leq i \leq s \), let \( N_i = (e_i v_j)_{1 \leq i < s, 1 \leq j \leq K} ; d_i = \text{rank}_R(V'N_i) \) (then \( 0 < d_s < d_{s-1} < \cdots < d_1 = r \) ); \( D_i = \{ \gamma \in V : \text{rank}_R(V'N_i)(\gamma) = d_i \} \); and \( D = \bigcap_{i=1}^{r} D_i \). Each \( D_i \) is a nonempty Zariski open set in \( V' \).

To show \( D_i \) is \( G \)-invariant we must show that \( \text{rank}_R(N_i(\gamma \cdot A) = \text{rank}_R(N_i(\gamma)) \) for all \( A \in G \). Recall that \( \{e_1, \ldots, e_s\} \) is a Jordan-Hölder basis of \( \mathfrak{g} \) such that \( [e_i, \mathfrak{g}] = 0 \), and \( [e_i, e_j] \subset \text{span}(e_{i+1}, \ldots, e_s) \) for \( 1 \leq i < s - 1 \). Therefore \( \mathfrak{h}_i = \text{span}(e_i, \ldots, e_s) \) is an ideal in \( \mathfrak{g} \), and \( H_i = \exp \mathfrak{h}_i \) is a normal Lie subgroup of \( G \). The restriction of the action of \( G \) (resp. \( \mathfrak{g} \)) to \( H_i \) (resp. \( \mathfrak{h}_i \)) defines a smooth action of \( H_i \) (resp. \( \mathfrak{h}_i \)) on \( V' \). Let \( F^t = F_{\gamma}|_{H_i} : H_i \to V' \). Then \( dF^t_\gamma(e) : \mathfrak{h}_i \to V' \), and by (3.1), for \( i < t < s \), \( 1 < j < K \), \( (dF^t_\gamma(e_i))(v_j) = \gamma(e_e)(v_j) = \gamma(e_i, v_j) \) so that the matrix for \( dF^t_\gamma(e) \) in terms of the basis \( \{e_1, \ldots, e_s\} \) of \( \mathfrak{h}_i \) and \( \{v^1, \ldots, v^K\} \) of \( V' \) is \( N_i(\gamma) \). Since \( H_i \) is normal in \( G \), if \( A \in G \), then \( F^t_\gamma \circ A = \pi(A)F^t_\gamma \circ A \); so that (as in (3.3))

\[ \text{rank}(N_i(\gamma \cdot A)) = \text{rank}(dF^t_{\gamma \cdot A}(e)) = \text{rank}(dF^t_\gamma(e)) = \text{rank}(N_i(\gamma)). \]

Since each \( D_i \) is \( G \)-invariant, \( D = \bigcap_{i=1}^{r} D_i \) is \( G \)-invariant. Hence \( U \cap D \) is \( G \)-invariant.

To show that \( U \cap D = E \), let \( \gamma \in V' \). \( \gamma \in E \) if and only if \( \det M^t_\gamma(\gamma) \neq 0 \) if and only if \( R_{i_1}(\gamma), \ldots, R_{i_r}(\gamma) \) are independent rows of \( M(\gamma) \), and \( C_{i_1}(\gamma), \ldots, C_{i_r}(\gamma) \) are independent columns of \( M(\gamma) \) if and only if \( \gamma \in U \cap D \). Indeed, \( \gamma \in D = \bigcap_{i=1}^{r} D_i \) if and only if \( \text{rank}_R(D_i(\gamma)) = d_i \), the maximal possible rank for each \( i = s, s - 1, \ldots, 1 \). From the definition of the indices \( \{i_1, \ldots, i_r\} \), \( i_r \) is the largest integer such that \( d_{i_r} = 1, i_{(k-1)} \) is the largest integer such that \( d_{i_{(k-1)}} = (d_{i_k}) + 1 \) for \( 2 \leq k < r \). Thus \( \gamma \in D \) if and only if \( R_{i_1}(\gamma), \ldots, R_{i_{(k-1)}}(\gamma) \) are linearly independent rows of \( M(\gamma) \). Similarly, \( i_r \) is the largest integer such that \( r_{i_r} = 1, i_{(k-1)} \) is the largest integer such that \( r_{i_{(k-1)}} = (r_{i_k}) + 1 \) for \( 2 \leq k < r \). \( \gamma \in U = \bigcap_{j=1}^{K} U_j \) if and only if \( \text{rank}_R(M_j(\gamma)) = r_j \), the maximum
possible rank for each $j$. Hence $\gamma \in U \iff C_1(\gamma), \ldots, C_{t_1}(\gamma)$ are independent columns of $M(\gamma)$.

In general, the set \( \{\gamma \in V' : \dim O_\gamma \text{ is maximum} \} = \{\gamma \in V' : \rank_R(M(\gamma)) = \alpha \} = U_1 = D_1 \) properly contains $U \cap D = E$.

The following theorem coordinatizes $O_\gamma$ for all $\gamma$ in $E$, and gives a $G$-invariant measure on $O_\gamma$ in terms of these coordinates. The proof shows how to use $M$ to compute all the stability subalgebras $g_\gamma$ for $\gamma \in E$.

**Theorem 3.1.** (a) If $\gamma \in E$, then the mapping $t = (t^1, \ldots, t^r) \mapsto G_\gamma \cdot \exp t^1 e_{i_1} \cdots \exp t^r e_{i_r}$ is a homeomorphism of $\mathbb{R}^r$ onto $G/G_\gamma$. Let $\nu_\gamma$ be the measure on $G/G_\gamma$ defined by

\[
\langle \nu_\gamma, f \rangle = \int_{G/G_\gamma} f(G_\gamma x) \, d\nu_\gamma(G_\gamma x)
\]

\[= \int_{\mathbb{R}^r} f(G_\gamma \exp t^1 e_{i_1} \cdots \exp t^r e_{i_r}) \, dm_{\mathbb{R}^r}(t^1, \ldots, t^r).
\]

There is a basis \( \{u_1(\gamma), \ldots, u_q(\gamma)\} \) of $g_\gamma$ such that if Haar measure $m_{G_\gamma}$ on $G_\gamma$ is taken as

\[
\langle m_{G_\gamma}, f \rangle = \int_{\mathbb{R}^q} f \left( \exp \sum_{b=1}^q z^b u_b(\gamma) \right) \, dm_{\mathbb{R}^q}(z^1, \ldots, z^q),
\]

then, for $f \in C_0(G)$,

\[
(3.8) \quad \int_G f(x) \, dm_G(x) = \int_{G/G_\gamma} \int_{G_\gamma} f(zx) \, dm_{G_\gamma}(z) \, d\nu_\gamma(G\gamma x).
\]

(b) If $\gamma \in E$, then the mapping $t = (t^1, \ldots, t^r) \mapsto G_\gamma \cdot \exp t^1 e_{i_1} \cdots \exp t^r e_{i_r}$ is a homeomorphism of $\mathbb{R}^r$ onto $O_\gamma$. The measure on $O_\gamma$ given by $\langle \nu_\gamma, f \rangle = \int_{\mathbb{R}^r} f(\gamma \cdot \exp t^1 e_{i_1} \cdots \exp t^r e_{i_r}) \, dm_{\mathbb{R}^r}(t)$ is $G$-invariant.

**Proof.** (b) follows from (a) by Proposition 3.1. The map $h_\gamma : G/G_\gamma \to O_\gamma : G_\gamma x \mapsto \gamma \cdot x$ carries coordinates and measures on $G/G_\gamma$ to $O_\gamma$.

The proof of (a) consists in showing that if $\gamma \in E$, then $g_\gamma$ has a basis

\[
\{u_1(\gamma), \ldots, u_q(\gamma)\}
\]

satisfying the requirement of Theorem 2.1 with respect to the indices $i_1 < \cdots < i_r$ of the independent rows of $M$ and $m_1 < \cdots < m_q$ of the dependent rows of $M$. In other words, there are scalars $\lambda_{mb}^{i_s}(\gamma)$, $1 \leq b \leq q$, $1 \leq s \leq r$, with $\lambda_{mb}^{i_s}(\gamma) = 0$ if $i_s < m_b$, such that the vectors $u_b(\gamma) = e_{mb} - \sum_{s=1}^r \lambda_{mb}^{i_s}(\gamma) e_{is}$, $1 \leq b \leq q$, form a basis of $g_\gamma$.

By definition, $1 \leq m_1 < \cdots < m_q < s$ are indices such that $\{1, \ldots, s\} = \{m_1, \ldots, m_q\} \cup \{i_1, \ldots, i_r\}$. By definition of $\{i_1, \ldots, i_r\}$, for $1 \leq b \leq q$, $R_{mb} = \sum_{s: i_s > m_b} \lambda_{mb}^{i_s} R_{is}$, with
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(3.9) \[
\chi_{mb}^{is} = \begin{bmatrix}
    e_{i1} v_{l1} & \cdots & e_{i1} v_{lr} \\
    \vdots & & \vdots \\
    e_{i(s-1)} v_{l1} & \cdots & e_{i(s-1)} v_{lr} \\
    e_{mb} v_{l1} & \cdots & e_{mb} v_{lr} \\
    e_{i(s+1)} v_{l1} & \cdots & e_{i(s+1)} v_{lr} \\
    \vdots & & \vdots \\
    e_{ir} v_{l1} & \cdots & e_{ir} v_{lr}
\end{bmatrix}, \quad 1 \leq s \leq r.
\]

By definition of \(\{i_1, \ldots, i_r\}, \chi_{mb}^{is} = 0\) if \(i_s < m_b\). Hence \(e_{mb} v_a = \sum_{s=1}^{r} \chi_{mb}^{is} (\gamma) e_{is}, 1 \leq a \leq K\). If \(\gamma \in E\), let

(3.10) \[
u_b = u_b(\gamma) = e_{mb} - \sum_{s=1}^{r} \chi_{mb}^{is} (\gamma) e_{is}, \quad 1 \leq b \leq q.
\]

Then, for \(1 \leq a \leq K\), \(\gamma u_b v_a = e_{mb} v_a - \sum_{s=1}^{r} \chi_{mb}^{is} (\gamma) e_{is} v_a = 0\). Hence \(u_b \in g_\gamma\) for \(1 \leq b \leq q\). Since \(\dim g_\gamma = \dim g - \dim O_\gamma = s - r = q\), and \(u_1, \ldots, u_q\) are linearly independent, \(\{u_1, \ldots, u_q\}\) is a basis of \(g_\gamma\).

Since \(E\) is a nonempty Zariski open set in \(V'\), \(E\) is \(m_{\nu'}\)-conull. Thus, to obtain a disintegration formula for \(m_{\nu'}\), we may restrict consideration to the \(G\)-invariant space \(E\) and the orbit space \(E/G\). \(V'\) has dimension \(K\), and \(m_{\nu'}\) is essentially \(m_{R^K}\), Lebesgue measure on \(R^K\). The orbits in \(E\) are \(r\)-dimensional manifolds, and each carries a \(G\)-invariant measure \(\nu_\gamma\) (Theorem 3.1) which is essentially \(m_{R^r}\), Lebesgue measure on \(R^r\). One would expect the measure on the orbit space \(V'/G\) in the disintegration of \(m_{\nu'}\) by \(G\) to be essentially \(m_{R^d}\), where \(d = K - r\) is the codimension of a maximal dimension orbit. To get the precise form of the measure on the orbit space, we need coordinates on \(V'/G\). The advantage of \(E\) is that we can use \(M\) to compute coordinates on \(E/G\) and the measure in terms of these coordinates. The following theorem gives a coordinate system for the orbit space \(E/G\).

Theorem 3.2. Let \(p : V' \rightarrow V'/G\) be the projection. Let \(s : R^d \rightarrow V'\) be the map \(s(y) = s(y_1, \ldots, y_d) = \sum_{k=1}^{d} y_k v_k\), where \(\{i_1, \ldots, i_d\}\) are the indices previously defined for the dependent columns of \(M\). Let \(W = \{y \in R^d : s(y) \in E\}\).

Then \(W\) is a nonempty Zariski open set in \(R^d\), and the map \((y_1, \ldots, y_d) \rightarrow p(\sum_{k=1}^{d} y_k v_k) : W \rightarrow E/G\) is a homeomorphism.

Proof. By definition of \(E\), \(W = \{y \in R^d : \det M^{(r)}(s(y)) \neq 0\}\) is a Zariski open set in \(R^d\). To show that \(W\) is not empty, and that \(p \circ s|_W\) is a bijection of
Lemma 3.2. If $\gamma \in E$, then the map $\pi_r |_{O_\gamma}: O_\gamma \rightarrow \mathbb{R}^r$ given by $\pi_r(\beta) = (\beta(u_1), \ldots, \beta(u_r))$ is bijective. (Here, $\{l_1, \ldots, l_r\}$ are the indices previously defined for the independent columns of $M$.)

Proof. The proof of Lemma 3.2 follows that of Pukánszky's orbit parametrization theorem [19, Theorem, pp. 50–54]. To show $\pi_r |_{O_\gamma}$ is bijective, we need suitable coordinates on $G/G_\gamma$. Recall from the proof of Lemma 3.1 that $M_j(\gamma)$ is the matrix for the mapping $dF_{P_j(\gamma)}(e): g \rightarrow V'/V_j$ in terms of the basis $\{e_1, \ldots, e_s\}$ of $g$ and $\{v'_1, \ldots, v'_K\}$ of $V'/V_j$. $Ker M_j(\gamma)$ is the stability subalgebra

$$Ker M_j(\gamma) = \{x: xu_j = xv_j + x = \cdots = xv_K = 0\}.$$ 

For $l_k < j \leq l_{(k+1)}$, $\text{rank } M_j(\gamma) = \text{rank } M_{l(k+1)+1}(\gamma) = \text{rank } M_{l_k}(\gamma) - 1$, $1 < l < r$ ($M_j = 0$ if $j > l_r$). Thus,

$$\dim Ker M_j(\gamma) = s - \text{rank } M_j(\gamma) = s - \text{rank } M_{l(k+1)+1}(\gamma)$$

$$= s - (\text{rank } M_{l_k}(\gamma)) + 1 = (\dim Ker M_{l_k}(\gamma)) + 1.$$ 

Since $Ker M_{l_k}(\gamma) \subseteq Ker M_j(\gamma)$ whenever $j > l_k$, if $w_k \in Ker M_{l(k)+1}(\gamma)$, $w_k \notin Ker M_j(\gamma)$, then $(Ker M_{l_k}(\gamma)) \oplus (w_k) = Ker M_j(\gamma)$ for $(l_k) + 1 \leq j \leq l_{(k+1)}$.

For $1 < k < r$, choose $w_k = w_k(\gamma) \in Ker M_{l(k)+1}(\gamma)$, $w_k \notin Ker M_{l_k}(\gamma)$, such that $(\gamma \cdot w_k)(v_k) = 1$. Then setting $n_0 = Ker M_{l_1}(\gamma)$, $n_k = n_{k-1} \oplus (w_k)$ for $1 \leq k < r$, we have an ascending sequence of subalgebras $g_{\gamma} = n_0 \subseteq n_1 \subseteq \cdots \subseteq n_r = g$ such that $n_k/n_{k-1} \cong (w_k)$. Let $Q: \mathbb{R}^r \rightarrow G$ be the map $Q(t) = Q(t^1, \ldots, t^r) = \exp t^1 w_1 \cdots \exp t^r w_r$. By Lemma 2.2, the map $t \rightarrow G_\gamma \cdot Q(t): \mathbb{R}^r \rightarrow G/G_\gamma$ is a homeomorphism. Thus, by Proposition 3.1(iii), the map $t \rightarrow \gamma \cdot Q(t): \mathbb{R}^r \rightarrow O_\gamma$ is a homeomorphism. The components of $\beta = \gamma \cdot Q(t)$ with respect to the basis $\{v^1, \ldots, v^K\}$ of $V'$, $\beta_a = \gamma \cdot Q(t)(v_a)$, $1 \leq a \leq K$, have the following form:

$$\beta_{t_r} = \gamma_{t_r} + t^r,$$

$$\beta_{t_k} = \gamma_{t_k} + t^k + \psi_k(t^{k+1}, \ldots, t^r; \gamma), \quad 1 \leq k \leq r - 1;$$

$$\beta_j = \gamma_j + F_j(t^k, \ldots, t^r; \gamma),$$ 

$k$ the largest integer such that $j > l_{k-1}$ (setting $l_0 = 0$).

Hence $t^r$, $\ldots$, $t^1$ may be recursively determined from

$$t^1 = \beta_{t_1} = \gamma_{t_1} - \psi_k(t^2, \ldots, t^r; \gamma).$$
Thus, given $z = (z_1, \ldots, z_r) \in \mathbb{R}^r$, there is one and only one $t = (t^1, \ldots, t^r)$ such that $\gamma \cdot Q(t)(v_{i_k}) = z_k$, $1 \leq k \leq r$. This says there is one and only one point $\beta \in O_\gamma$ such that $\pi_\gamma(\beta) = z$. Hence $\pi_\gamma$ is a bijection of $O_\gamma$ onto $\mathbb{R}^r$.

To show that $W$ is not empty, choose $\gamma \in E$. Then (Lemma 3.1) $O_\gamma \subset E$. By Lemma 3.2, there is a point $\beta \in O_\gamma$ such that $\pi_\gamma(\beta) = 0$. Since $\{l_1, \ldots, l_r\}$, $\{j_1, \ldots, j_d\}$ is a partition of $\{1, \ldots, K\}$, $\beta = \sum_{k=1}^d \beta_k v_{i_k} = s(\beta_{j_1}, \ldots, \beta_{j_d}) \in E$, so that $(\beta_{j_1}, \ldots, \beta_{j_d}) \in W$. Since $\gamma \in E$ was arbitrary, this also shows that $ps(W) = E/G$ ($\beta \in s(W)$ and $p\beta = p\gamma$).

If $y, z \in W$, and if $ps(y) = ps(z)$, then $O_{s(y)} = O_{s(z)} \subset E$. By Lemma 3.2, $\pi_\gamma \circ O_{s(y)}$ is injective. $\pi_\gamma(s(y)) = 0 = \pi_\gamma(s(z)) \implies s(y) = s(z) \implies y = z$. Thus $p \cdot s|_W : W \rightarrow E/G$ is bijective.

$s(W) = (\text{span}\{u^1, \ldots, u^d\}) \cap E$ intersects each orbit in $E$ in exactly one point, so that $\psi : E/G \rightarrow V'$ defined by $\psi(p\gamma) = p^{-1} p\gamma \cap s(W)$ is a cross-section for $E/G$ in $V'$.

$p \circ s|_W : W \rightarrow E/G$ is continuous since both $p$ and $s$ are continuous. To show that $p \circ s|_W$ is open, we introduce the following map, which is also used in the proof of the disintegration formula. For $t = (t^1, \ldots, t^r) \in \mathbb{R}^r$, let $g(t) = \exp \sum_{i=1}^r t^i e_{i_t} \in G$, where $i_1, \ldots, i_r$ are the indices previously defined for the independent rows of $M$. Let $H : \mathbb{R}^d \times \mathbb{R}^r \rightarrow V'$ be the map $H(y, t) = s(y) \cdot g(t)$. $H(y, t)$ is linear in $(y_1, \ldots, y_d)$ and a polynomial in $(t^1, \ldots, t^r)$, so $H$ is an analytic mapping of $\mathbb{R}^d \times \mathbb{R}^r$ into $V'$.

For $(y, t) \in \mathbb{R}^d \times \mathbb{R}^r$, let $J(y, t)$ be the absolute value of the determinant of the $K \times K$ matrix

$$
\begin{vmatrix}
\frac{\partial H_{i_1}}{\partial y_1} & \cdots & \frac{\partial H_{i_1}}{\partial y_d} & \frac{\partial H_{i_1}}{\partial t^1} & \cdots & \frac{\partial H_{i_1}}{\partial t^r} \\
\vdots & & \vdots & \vdots & & \vdots \\
\frac{\partial H_{i_d}}{\partial y_1} & \cdots & \frac{\partial H_{i_d}}{\partial y_d} & \frac{\partial H_{i_d}}{\partial t^1} & \cdots & \frac{\partial H_{i_d}}{\partial t^r} \\
\frac{\partial H_{i_1}}{\partial y_1} & \cdots & \frac{\partial H_{i_1}}{\partial y_d} & \frac{\partial H_{i_1}}{\partial t^1} & \cdots & \frac{\partial H_{i_1}}{\partial t^r} \\
\vdots & & \vdots & \vdots & & \vdots \\
\frac{\partial H_{i_r}}{\partial y_1} & \cdots & \frac{\partial H_{i_r}}{\partial y_d} & \frac{\partial H_{i_r}}{\partial t^1} & \cdots & \frac{\partial H_{i_r}}{\partial t^r}
\end{vmatrix}
$$

evaluated at $(y, t)$, where $H_a(y, t) = H(y, t)(v_a)$, $1 \leq a \leq K$. Then $J(y, t) = |\det dH(y, t)|$, where $dH(y, t) : \mathbb{R}^d \times \mathbb{R}^r \rightarrow V'$ is the derivative of $H$ at $(y, t)$.

Since each $H_a$ is a polynomial in $y$ and $t$, the partials are polynomials in $y$ and $t$. Hence $\det dH(y, t)$ is a polynomial in $y$ and $t$.

By calculation,
\[
\frac{\partial H_{ik}}{\partial y_m}(y, 0) = \lim_{h \to 0} \frac{1}{h} [H_{ik}(y_1, \ldots, y_m + h, \ldots, y_d; 0) - H_{ik}(y_1, \ldots, y_m, \ldots, y_d; 0)]
\]
\[
= \lim_{h \to 0} \frac{1}{h} \left[ s(y_1, \ldots, y_m + h, \ldots, y_d) - s(y_1, \ldots, y_m, \ldots, y_d) \right] (v_{ik})
\]
\[
= v_m(v_{ik}) = \delta^m_k, \quad 1 \leq k \leq d, 1 \leq m \leq d.
\]

\[
H_{ik}(y, 0) = s(y)(v_{ik}) = 0, \quad 1 \leq k \leq r, \forall y \in \mathbb{R}^d.
\]

Therefore,

\[
\frac{\partial H_{ik}}{\partial y_m}(y, 0) = 0, \quad 1 \leq k \leq r, 1 \leq m \leq d.
\]

\[
\frac{\partial H_a}{\partial t^k}(y, 0) = \lim_{h \to 0} \frac{1}{h} [H_a(y; 0 \cdot \cdot \cdot 0, h, 0 \cdot \cdot \cdot 0) - H_a(y; 0)]
\]
\[
= \lim_{h \to 0} \frac{1}{h} \left[ s(y) \cdot \exp he_{ik} - s(y) \right] (v_a)
\]
\[
= \lim_{h \to 0} \frac{1}{h} \left[ \left( s(y) + s(y) \cdot he_{ik} + \frac{h^2}{2!} e_{ik}^2 + \cdots \right) - s(y) \right] (v_a)
\]
\[
= s(y) \cdot e_{ik}(v_a), \quad 1 \leq a \leq K, 1 \leq k \leq r.
\]

Therefore,

\[
(3.12) \quad J(y, 0) = \det \begin{bmatrix}
1 & 0 \cdots 0 \\
0 & 1 \cdots 0 \\
\vdots & \vdots \\
0 & \cdots 1 \\
\end{bmatrix}
\begin{bmatrix}
s(y)e_{i_1}v_{i_1} & \cdots & s(y)e_{i_r}v_{i_1} \\
0 & \vdots \\
s(y)e_{i_1}v_{i_r} & \cdots & s(y)e_{i_r}v_{i_r} \\
\end{bmatrix}
\]
\[
= |\det M^{(r)}(s(y))|.
\]

If \( y \in W \), then \( J(y, 0) \neq 0 \). By the inverse function theorem [20, p. 35] there is an \( \mathbb{R}^d \times \mathbb{R}^r \) open neighborhood \( A \times B \) of \( (y, 0) \) and a \( V' \)-open neighborhood \( C \) of \( H(y, 0) \) such that \( H|_{A \times B} : A \times B \to C \) is a diffeomorphism of \( A \times B \) onto \( C \).

Now, to show \( p \circ s|_U \) is open, let \( U \subseteq W \) be open, and \( y \in p^{-1}(s(U)) = s(U) \cdot G \). Then \( y = s(y_0) \cdot g_0 \) for some \( y_0 \in U, g_0 \in G \). Since \( y_0 \) is in \( U \subseteq W \), by the preceding paragraph, there is an \( \mathbb{R}^d \)-open neighborhood \( A \) of \( y_0 \) (by taking \( A \cap U \), we may assume \( A \subseteq U \)), an open neighborhood \( B \) of \( 0 \) in \( \mathbb{R}^r \), and an open neighborhood
C of \( H(y_0, 0) = s(y_0) \) in \( V' \) such that \( H(A \times B) = C \). Since \( C \) is a \( V' \)-neighborhood of \( s(y_0) \) and \( g_0 \in G \) is a homeomorphism of \( V' \), \( C \cdot g_0 \) is a \( V' \)-neighborhood of \( \gamma = s(y_0) \cdot g_0 \). If \( \beta \in C \cdot g_0 \), then \( \beta = H(y, t) \cdot g_0 = s(y) \cdot g(t) \cdot g_0 \) for some \((y, t) \in A \times B \subset U \times B\). Therefore, \( \gamma \in s(U) \cdot G \). Thus \( \gamma \) is an interior point of \( s(U) \cdot G \). Therefore, \( s(U) \cdot G = p^{-1}(s(U)) \) is open in \( V' \), so \( p \circ s(U) \) is open in \( E/G \).

We have shown that the orbit space \( E/G \) is homeomorphic to a Zariski-open set in \( \mathbb{R}^d \) (Theorem 3.2) and that each orbit in \( E \) is homeomorphic to \( \mathbb{R}^r \) (Theorem 3.1). The following theorem uses the coordinate system \( y \to ps(y) : \mathcal{W} \to E/G \) just established for \( E/G \) and the coordinate system \( y, y \to s(y) \cdot g(t) = s(y) \cdot \exp t^r e_{i_r} \cdots \exp t^1 e_{i_1} \) for the orbits in \( E \) (Theorem 3.1) to decompose \( m_{V'} \), relative to the action of \( G \) in \( V' \).

Let \( x_T \) denote the characteristic function of the set \( T \).

**Theorem 3.3.** The formula

\[
\int_{V'} f(\gamma) \, dm_{V'}(\gamma) = \int_E f(\gamma) \, dm_V(\gamma)
\]

is a disintegration of \( m_{V'} \) by \( G \), that is, \( m_{V'}(p^{-1}(V'/G - E/G)) = m_{V'}(V' - E) = 0 \).

The image of the measure \( |\det M^{(r)}(s)| x_w m_{R^d} \) under the homeomorphism \( p \circ s : \mathcal{W} \to E/G \) is a measure on \( E/G \) which is a pseudo-image of \( x_E m_{V'} \), by \( p |_{\mathcal{E}} \), the projection of \( E \) onto \( E/G \). If, for \( y \in \mathcal{W} \), \( \nu_{s(y)} \) is the measure on \( E \) given by

\[
\langle \nu_{s(y)}, f \rangle = \int_{R^r} f(s(y) \cdot g(t)) \, dm_{R^r}(t),
\]

then \( y \to \nu_{s(y)} : \mathcal{W} \to M_+(E) \) (positive measures on \( E \)) has the following properties:

(i) \( \nu_{s(y)} \neq 0 \) for \( y \in \mathcal{W} \);

(ii) \( \nu_{s(y)} \) is concentrated in \( O_s(y) \) for all \( y \in \mathcal{W} \);

(iii) if \( f \in L^1(x_E m_{V'}) \), then \( y \to \langle \nu_{s(y)}, f \rangle \in L^1(|\det M^{(r)}(s)| x_w m_{R^d}) \),

and

\[
\langle x_E m_{V'}, f \rangle = \int_{\mathcal{W}} \langle \nu_{s(y)}, f \rangle |\det M^{(r)}(s(y))| \, dm_{R^d}(y).
\]

**Proof.** By Lemma 3.1, \( E \) is a nonempty, \( G \)-invariant Zariski open set in \( V' \). Therefore, \( p^{-1}(V'/G - E/G) = V' - E \) is \( m_{V'} \)-null. That \( \nu_{s(y)} \in M_+(E) \) and properties (i) and (ii) follow from Theorem 3.1 and the fact that \( G \) orbits are closed in \( V' \) (Proposition 3.1(a)). The proof of (iii) and formula (3.13) consists in (1) showing that \( p \circ s(x_w m_{R^d}) \) is a pseudo-image of \( x_E m_{V'} \) by \( p \); (2) using Bourbaki's theorem [6, Chapitre 6, Théorème 2, p. 64] on the disintegration of
a measure relative to a pseudo-image to get a disintegration of $x_E m_{V'}$, relative to $p \circ s(x_w m_{R^d})$; and (3) showing that the orbit measures provided by Bourbaki's theorem are $|\det M(y)| v_s(y)$.

The following three lemmas show that the measure $p \circ s(x_w m_{R^d})$ on $E/G$ is a pseudo-image of the measure $x_E m_{V'}$ on $E$. Equation (3.14) in Lemma 3.3 would be the disintegration formula (3.13) if we knew that $|\det dH(y, t)| = J(y, t) = J(y, 0)m_{R^d \times R'}$ a.a. $(y, t)$. This is proved in Lemma 3.7.

**Lemma 3.3.** If $f: V' \to R$ is $m_{V'}$-integrable, then

$$\int_{V'} f(y) dm_{V'}(y) = \int_{R^d \times R'} f(H(y, t)) |\det dH(y, t)| dm_{R^d \times R'}(y, t).$$

**Proof.** Let $A = \{(y, t) \in R^d \times R' : \det dH(y, t) \neq 0\}$. $A$ is a Zariski open set in $R^d \times R'$. $A \supset W \times \{0\}$, so $A$ is nonempty. Suppose $H(y_1, t_1) = H(y_2, t_2)$ for $(y_1, t_1) \in W \times R'$. Then $s(y_2) \in O_s(y_1) \subset E$. By Lemma 3.2,

$$\pi_t(s(y_2)) = 0 = \pi_t(s(y_1)) \iff s(y_2) = s(y_1) \iff y_2 = y_1.$$  By Theorem 3.1(b),

$$s(y_1) \cdot g(t_1) = s(y_1) \cdot g(t_2) \iff t_1 = t_2.$$  Therefore $H|_{A \cap (W \times R')} : A \cap (W \times R') \to V'$ is a 1-1, continuously differentiable function such that $\det dH(y, t) \neq 0$ for all $(y, t) \in A \cap (W \times R')$. By the change of variable theorem for integrals on $R^d$ [20, p. 67], if $f: H(A \cap (W \times R')) \to R$ is integrable, then

$$\int_{H(A \cap (W \times R'))} f(y) dm_{V'}(y) = \int_A \chi_{(W \times R')} f \circ H(y, t) |\det dH(y, t)| dm_{R^d \times R'}(y, t).$$

Since $A \cap (W \times R')$ is a nonempty Zariski open set in $R^d \times R'$, it is conull. Hence the integral on the right-hand side of (3.15) is

$$\int_{R^d \times R'} f \circ H(y, t) J(y, t) dm_{R^d \times R'}(y, t).$$

Let $B = \{(y, t) \in R^d \times R' : \det dH(y, t) = 0\}$. By Sard's theorem [20, p. 72], $H(B)$ is an $m_{V'}$-null set in $V'$. Since $H$ is 1-1 on $W \times R'$, $H(W \times R')$ is the disjoint union of $H(A \cap (W \times R'))$ and $H(B \cap (W \times R')) \subset H(B)$. Hence, the integral on the left-hand side of (3.15) is $\int_{H(W \times R')} f(y) dm_{V'}(y)$. By Theorem 3.2, $H(W \times R') = E$, which is $m_{V'}$-conull. This proves (3.14).

**Corollary.** $f: V' \to R$ is $m_{V'}$-measurable $\iff f \circ H: R^d \times R' \to R$ is $m_{R^d \times R'}$-measurable.

**Proof.** Lemma 3.3 says that

$$m_{V'} = \int_{R^d \times R'} e_{H(y, t)} J(y, t) dm_{R^d \times R'}(y, t)$$

(where $(e_{H(y, t)}, f) = f(H(y, t))$). By [5, Chapitre 5, Proposition 3, p. 39],
\[ f : V' \to R \text{ is } m_{V'}\text{-measurable } \iff (f \circ H) \circ J \text{ is } m_{R^d \times R^r}\text{-measurable} \]

\[ \iff \left( (f \circ H) \right) \upharpoonright_A \text{ is } m_{R^d \times R^r}\text{-measurable} \]

(where \( A = \{(y, t) \in R^d \times R^r : J(y, t) \neq 0 \} \)). Since \( A \) is conull in \( R^d \times R^r \), \( f : V' \to R \) is \( m_{V'}\text{-measurable } \iff f \circ H : R^d \times R^r \to R \) is \( m_{R^d \times R^r}\text{-measurable} \).

**Lemma 3.4.** Suppose \( f : V'/G \to R \) is nonnegative. Then \( f \circ p : V' \to R \)

\[ \text{is } m_{V'}\text{-measurable } \iff f \circ p \circ s : R^d \to R \text{ is } m_{R^d}\text{-measurable.} \]

**Proof.** By the above corollary, \( f \circ p : V' \to R \) is \( m_{V'}\text{-measurable } \iff f \circ p \circ H : R^d \times R^r \to R \) is \( m_{R^d \times R^r}\text{-measurable.} \)

Suppose \( f \circ p \circ s : R^d \to R \) is \( m_{R^d}\text{-measurable}. \) Then \( f \circ p \circ H(y, t) = f \circ p(s(y) \cdot g(t)) = f(p(s(y))) \) for all \( (y, t) \in R^d \times R^r \Rightarrow f \circ p \circ H \) is \( m_{R^d \times R^r}\text{-measurable.} \) \((\{(y, t) : f \circ p \circ H(y, t) > a\} = \{y : f \circ p \circ s(y) > a\} \times R^r\) \)

Suppose \( f \circ p : V' \to R \) is \( m_{V'}\text{-measurable.} \) Let \( \beta \approx m_{R^r} \) be a finite measure on \( R^r \). By Tonneli's theorem, \( y \to \int_{R^d} f \circ p \circ H(y, t) \, d\beta(t) : R^d \to R \) is \( m_{R^d}\text{-measurable.} \) \((f \circ p \circ H) \in (m_{R^d \times R^r} = m_{R^d} \times m_{R^r})\text{-measurable } \iff f \circ p \circ H \in (m_{R^d \times \beta})\text{-measurable.} \) Since \( f \circ p \circ H(y, t) = f(p(s(y))) \), this implies \( y \to f(p(s(y))) \beta(R^r) : R^d \to R \) is \( m_{R^d}\text{-measurable, so } f \circ p \circ s \) is \( m_{R^d}\text{-measurable.} \)

Let \( \Omega = \{U \subset V'/G : p^{-1}(U) \) is \( m_{V'}\text{-measurable} \}. \) Lemma 3.4 shows that \( \Omega = \{U \subset V'/G : (p \circ s)^{-1}(U) \) is \( m_{R^d}\text{-measurable} \}. \) (Take \( f = x_U \), the characteristic function of \( U \).)

**Lemma 3.5.** Let \( N \subset V'/G, N \in \Omega. \) Then \( m_{V'}(p^{-1}(N)) = 0 \iff m_{R^d}((p \circ s)^{-1}(N)) = 0. \)

**Proof.** \( m_{V'}(p^{-1}(N)) = 0 \iff x_N \circ p = 0 \text{ m}_{V^1} \text{ a.e. } \iff (x_N \circ p \circ H) \cdot J = 0 \text{ m}_{R^d \times R^r} \text{ a.e. } \) (by Lemma 3.3) \iff x_N \circ p \circ H = 0 \text{ m}_{R^d \times R^r} \text{ a.e. } \) (since \( A \) is conull).

Suppose \( x_N \circ p \circ H = 0 \text{ m}_{R^d \times R^r} \text{ a.e.} \) By Fubini's theorem, for \( m_{R^d} \) almost all \( y, x_N \circ p \circ H(y, t) = x_N(p(s(y))) = 0 \) for \( m_{R^r} \text{ a.a. } t. \) Hence \( m_{R^d}((p \circ s)^{-1}(N)) = 0. \)

Conversely, suppose \( x_N \circ p \circ s = 0, \) m_{R^d} a.e. Then by Tonneli's theorem \( (x_N \circ p \circ H) \) is \( m_{R^d \times R^r}\text{-measurable by the corollary to Lemma 3.3}, \)

\[
\int_{R^d \times R^r} x_N \circ p \circ H(y, t) \, dm_{R^d \times R^r}(y, t) = \int_{R^r} \left( \int_{R^d} x_N(p(H(y, t))) \, dm_{R^d}(y) \right) \, dm_{R^r}(t) = \int_{R^r} \left( \int_{R^d} x_N(p(s(y))) \, dm_{R^d}(y) \right) \, dm_{R^r}(t) = \int_{R^r} 0 \, dm_{R^r}(t) = 0.
\]

Thus \( x_N \circ p \circ H = 0 \) m_{R^d \times R^r} a.e.
The following argument uses Bourbaki's theorem on the disintegration of a measure relative to a pseudo-image [6, Chapitre 6, Théorème 2, p. 64] to get a disintegration of \( x_{E}m_{V} \), relative to \( (p \circ s)(x_{w}m_{Rd}) \). The rest of the proof of Theorem 3.3 consists of showing that the orbit measures \( \lambda_{b} \) \((b = psy \in E/G)\) from [6, Chapitre 6, Théorème 2, p. 64] are equal to \( |\det M^o(s(y))|\nu_{s(y)} \).

Since \( E \) is an open set in \( V' \), \( E \) is a locally compact topological space with a countable basis. By Theorem 3.2, \( p \circ s \mid W \) is a homeomorphism of the Zariski open set \( W \subset R^d \) onto \( E/G \). Therefore \( E/G \) is a locally compact space with a countable basis. Since \( W \) is \( m_{Rd} \)-null, and \( E \) is \( m_{V} \)-null, Lemma 3.5 shows that the measure on \( E/G \), \( (p \circ s)(x_{w}m_{Rd}) \), is a pseudo-image of \( x_{E}m_{V} \), by \( p \mid E \), i.e., \( N \subset E/G \) is \( (p \circ s)(x_{w}m_{Rd}) \)-null \( \iff \) \( p^{-1}(N) \) is \( (x_{E}m_{V}) \)-null. By [6, Chapitre 6, Théorème 2, p. 64] there exists a \((p \circ s)(x_{w}m_{Rd})\)-adequate family \([5, Chapitre 5, Définition 1, p. 19]\) \( b \to \lambda_{b} \) \((b \in E/G)\) of positive measures on \( E \) having the following properties:

(a) \( \lambda_{b} \neq 0 \) for \( b \in p(E) = E/G \);
(b) \( \lambda_{b} \) is concentrated in \( p^{-1}(b) \) for all \( b \in E/G \);
(c) \( x_{E}m_{V} = \int_{E/G} \lambda_{b} d(p \circ s)(x_{w}m_{Rd})(b) \).

Thus, if \( f : E \to R \) is \((x_{E}m_{V})\)-integrable \((f \text{ is } (x_{E}m_{V})\)-measurable, and \( \int_{E} |f(\gamma)| dm_{V}(\gamma) < \infty \), then \( b \to \langle \lambda_{b}, f \rangle = \int_{p^{-1}(b)} f(\gamma) d\lambda_{b}(\gamma) : E/G \to R \) is \((p \circ s)(x_{w}m_{Rd})\)-integrable; \( y \to \langle \lambda_{s(t)(y)}, f \rangle = \int_{p^{-1}(s(t)(y))} f(\gamma) d\lambda_{s(t)(y)}(\gamma) : W \to R \) is \((x_{w}m_{Rd})\)-integrable; and

\[
\int_{E} f(\gamma) dm_{V}(\gamma) = \int_{E/G} \left( \int_{p^{-1}(b)} f(\gamma) d\lambda_{b}(\gamma) \right) d(p \circ s)(x_{w}m_{Rd})(b) \\
= \int_{W} \left( \int_{p^{-1}(s(t))} f(\gamma) d\lambda_{s(t)}(\gamma) \right) dm_{Rd}(y).
\]

(3.16)

To complete the proof of Theorem 3.3, we show that for \((x_{w}m_{Rd})\) a.a. \( y \), \( \lambda_{s(t)}(y) = |\det M^{o}(s(y))|\nu_{s(y)} \).

Since \( x_{E}m_{V} \) is \( G \)-invariant, \((x_{w}m_{Rd})\) almost all the \( \lambda_{s(t)}(y) \) are \( G \)-invariant [16, Lemma 11.5, p. 126]. Let \( N \subset W \) be a null set such that \( y \in W - N \implies \lambda_{s(t)}(y) \) is \( G \)-invariant. Then \( \lambda_{s(t)}(y) \) and \( \nu_{s(y)} \) are both \( G \)-invariant measures on \( O_{s(y)} \approx G/G_{s(y)} \). Therefore, if \( y \in W - N \), there is a positive number \( c(y) \) such that

(3.17)

\[ \lambda_{s(t)}(y) = c(y)\nu_{s(y)} \]

Put \( c(y) = 1 \) if \( y \in N \cup (R^d - W) \).

\textbf{Lemm}\textsc{a} 3.6. \( c : R^{d} \to R \) is \( m_{R^{d}} \)-measurable.

\textbf{Proof.} Let \( f : V' \to R \) be an everywhere positive, continuous, \( m_{V'} \)-integrable function. By the corollary to Lemma 3.3, \( f \circ H \) is \( m_{R^{d} \times R^{d}} \)-measurable, nonnegative. By Tonelli's theorem \( y \to f_{R^{d}}(f(H(y, t)))dm_{R^{d}}(t) \) is \( m_{R^{d}} \)-measurable.
If \( y \in W \), then by Theorem 3.1(b), \( \langle \omega_{s(y)} f, f \rangle = \int_{R} f(s(y) \cdot g(t)) m_{R}(t) \) is an everywhere positive, \( (x_{w} m_{R}) \)-measurable function. Hence \( y \rightarrow \frac{1}{\langle \omega_{s(y)} f, f \rangle} : W \rightarrow R \) is \( m_{R}d \)-measurable. Since \( f \) is \( m_{V} \)-integrable, \( y \rightarrow \langle \lambda_{ps(y)} f, f \rangle = \langle c(y), \langle \nu_{s(y)} f, f \rangle \rangle \) a.e. is \( (x_{w} m_{R}) \)-integrable, hence measurable. Therefore \( y \rightarrow \langle \lambda_{ps(y)} f, f \rangle / \langle \nu_{s(y)} f, f \rangle = c(y) \) is \( m_{R}d \)-measurable on \( W - N \). Hence \( y \rightarrow c(y) \) is \( m_{R}d \)-measurable on \( W \), hence on \( R^{d} \).

**Lemma 3.7.** For \( m_{R}d \) almost all \( y \in R^{d} \),

(3.18) \[ c(y) = | \det M^{(y)}(s(y)) |. \]

**Proof.** We substitute \( c(y) \nu_{s(y)} \) for \( X_{p}(y) \) in (3.16), write \( \nu_{s(y)} \) in terms of the coordinates \( t = (t^{1}, \ldots, t^{r}) \rightarrow s(y) \cdot g(t) = H(y, t) \), and compare the resulting equation with (3.14). The result is

\[
\int_{W} \left( \int_{R} f(H(y, t)) \, dm_{R}(t) \right) c(y) \, dm_{Rd}(y) = \int_{W \times R} f(H(y, t)) J(y, t) \, dm_{Rd \times R}(y, t), \quad f \in L^{1}(m_{V}).
\]

Suppose \( f \in L^{1}(m_{V}) \) is nonnegative. By the corollary to Lemma 3.3, \( f \circ H : R^{d} \times R^{r} \rightarrow R \) is \( m_{Rd \times R} \)-measurable. By Lemma 3.6, \( c : R^{d} \rightarrow R \) is \( m_{Rd} \)-measurable. Hence \( (f \circ H) \circ c \) is \( m_{Rd \times R} \)-measurable, nonnegative. By Tonelli’s theorem, the left-hand side of (3.19) is equal to

\[
\int_{W \times R} f(H(y, t), c(y)) \, dm_{Rd \times R}(y, t).
\]

Therefore, whenever \( f \geq 0 \) is \( m_{V} \)-integrable,

(3.20) \[ 0 = \int_{W \times R} f(H(y, t), J(y, t) - c(y)) \, dm_{Rd \times R}(y, t). \]

Let \( D = \{(y, t) \in W \times R^{r} : J(y, t) > c(y)\} \). \( x_{D} = (x_{D} \circ H^{-1}) \circ H \) is \( m_{Rd \times R} \)-measurable so (by the corollary to Lemma 3.3) \( x_{D} \circ H^{-1} \) is \( m_{V} \)-measurable. Let \( f : V' \rightarrow R \) be an everywhere positive, integrable function. \( x_{D} \circ H^{-1} \circ f \leq f \), so \( x_{D} \circ H^{-1} \circ f \) is \( m_{V} \)-integrable, nonnegative. By (3.20),

\[ 0 = \int_{W \times R} x_{D}(y, t) f(H(y, t), J(y, t) - c(y)) \, dm_{Rd \times R}(y, t). \]

Hence \( x_{D}(y, t) (J(y, t) - c(y)) = 0 \) for \( m_{Rd \times R} \) a.a. \( (y, t) \). Since \( J(y, t) - c(y) > 0 \) on \( D \), \( m_{Rd \times R}(D) = 0 \). Similarly,

\[ m_{Rd \times R}(\{(y, t) \in W \times R^{r} : J(y, t) < c(y)\}) = 0. \]

Therefore, \( J(y, t) = c(y) \) for \( m_{Rd \times R} \) a.a. \( (y, t) \) in \( W \times R^{r} \), hence for \( m_{Rd \times R} \) a.a. \( (y, t) \). By Fubini’s theorem, for almost all \( y \in R^{d} \), \( J(y, t) = c(y) \) for almost all \( t \in R^{r} \). Since \( t \rightarrow J(y, t) \) is continuous on \( R^{r} \), \( J(y, t) = c(y) \) for all \( t \in R^{r} \).
Hence, \( c(y) = J(y, 0) \) for almost all \( y \in \mathbb{R}^d \). By (3.12), \( J(y, 0) = |\det M^{(r)}(s(y))| \) for \( y \in \mathcal{W} \). Thus \( c(y) = |\det M^{(r)}(s(y))| \) for almost all \( y \in \mathbb{R}^d \).

Substituting \( c(y) = |\det M^{(r)}(s(y))| \) for \( \lambda_{p_0} \) in (3.16), we obtain (3.13). This completes the proof of Theorem 3.3. The above proof also gives the following fact.

**Theorem 3.4.** \( H: \mathcal{W} \times \mathbb{R}^r \to E: (y, t) \to s(y) \cdot g(t) \) is a diffeomorphism.

**Proof.** \( H \) is a polynomial in \( y \) and \( t \) so it is differentiable. The proof of Lemma 3.7 shows that the continuous function \( (y, t) \to J(y, t) - |\det A'(s(y))| \) is zero for \( m_{\mathbb{R}^d \times \mathbb{R}^r} \) almost all \( (y, t) \). Hence \( J(y, t) = |\det dH(y, t)| = |\det M^{(r)}(s(y))| \) for all \( (y, t) \in \mathbb{R}^d \times \mathbb{R}^r \). Thus \( \{(y, t) \in \mathbb{R}^d \times \mathbb{R}^r : |\det dH(y, t)| \neq 0\} = \mathcal{W} \times \mathbb{R}^r \). From the proof of Lemma 3.3, \( H \) is a bijection of \( \mathcal{W} \times \mathbb{R}^r \) onto \( E \). Therefore, the inverse function theorem shows \( H \) is a diffeomorphism.

4. A Plancherel formula for idyllic nilpotent Lie groups. In §4 we bring together the results of §§1-3 to obtain a procedure for computing Plancherel measure for the following class of nilpotent Lie groups.

Suppose \( G \) is a connected, simply connected nilpotent Lie group with Lie algebra \( g \). \( g \) will be called "idyllic" if \( g \) has an abelian ideal \( n \) such that for Lebesgue almost all \( \gamma \) in \( n' \), \( g_\gamma/n \) is abelian, where \( g_\gamma = \{x \in g : [\gamma, x, n] = 0 \ \forall n \in n\} \). Such an ideal \( n \) will be called an "idyll" of \( g \). \( G \) is called idyllic if its Lie algebra \( g \) is idyllic. If \( n \) is an idyll of \( g \), then \( N = \exp n \) is called an idyll of \( G \).

To compute Plancherel measure for idyllic \( G \) with idyll \( N \), we combine the projective Plancherel formula from §1 with the disintegration theorem of §3 (Theorem 3.3) via Kleppner and Lipsman's Plancherel formula for group extensions [15, Theorem 23, p. 108]

\[
(4.1) \quad \int_G |f(x)|^2 \, dm_G(x) = \int_{\hat{N}/G} \int_{(G\gamma/N, \omega_\gamma)} \lim_{\tau \to \gamma} \tau \pi_{\tau, \alpha} (f \ast f^*) \, d\mu_\gamma(\alpha) \, d\overline{\mu}_N(\gamma),
\]

which expresses Plancherel measure on \( \hat{G} \) corresponding to a given Haar measure \( m_G \) on \( G \) as a fibered measure with base \( \hat{N}/G \) and fibers \( (G\gamma/N, \omega_\gamma) \), where \( G_\gamma \) is the stability subgroup at \( \gamma \in \hat{N} \). \( \mu_N \) is Plancherel measure on \( \hat{N} \) corresponding to a given Haar measure \( m_N \) on \( N \). \( \overline{\mu}_N \) is a pseudo-image of \( \mu_N \) by the projection \( p: \hat{N} \to \hat{N}/G \). Since \( \hat{N}/G \) is countably separated, there are orbit measures \( \nu_\gamma \) which provide a disintegration of Plancherel measure \( \mu_N \) on \( \hat{N}/G \) relative to the pseudo-image \( \overline{\mu}_N \) on \( \hat{N}/G \), i.e.,

\[
(4.2) \quad \mu_N = \int_{\hat{N}/G} \nu_\gamma d\overline{\mu}_N(\gamma),
\]

\( \nu_\gamma \) concentrated on \( \gamma \cdot G \cong G/G_\gamma \). The projective Plancherel measure \( \mu_\gamma \) on \( (G_\gamma/N, \omega_\gamma) \) corresponds to the Haar measure \( m_{G\gamma/N} \) on \( G_\gamma/N \) which satisfies
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\[(4.3) \int_G f(x) \, dm_G(x) = \int_{G/G_N} \int_{G_N/N} \int_N f(nx) \, dm_N(n) \, dm_{G/N}(Nz) \, dv_N(x). \]

For \( \gamma \in \hat{N} \), \( \pi_{\gamma, \sigma} = \text{ind}_{G_N}^{G} \gamma' \otimes \sigma' \) is an irreducible representation of \( G \). \( \gamma' \) is the extension of \( \gamma \) to an \( \omega_{\gamma} \)-representation of \( G_N \), where \( \omega_{\gamma} \) is a multiplier on \( G_N/N \). \( \sigma \) is an irreducible \( \omega_{\gamma} \)-representation of \( G_N/N \), and \( \sigma'' \) denotes the lift of \( \sigma \) to \( G_N \).

If \( \mu_{\gamma} \) is the projective Plancherel measure on \((G_N/N, \omega_{\gamma})^*\) corresponding to \( m_{G/N} \) satisfying \( 4.3 \), then \([15, (2.10), p. 109]\), for \( f \in C_0(G) \) (continuous functions with compact support),

\[
\int_{(G_N/N, \omega_{\gamma})^*} \text{tr} [\pi_{\gamma, \sigma}(f \star f^\ast)] \, d\mu_{\gamma}(\sigma) = \int_{G/G_N} \text{tr} [\gamma \cdot A(f \star f^\ast |_N)] \, dv_N(A),
\]

so that

\[
\int_{\hat{N}/G} \int_{(G_N/N, \omega_{\gamma})^*} \text{tr} [\pi_{\gamma, \sigma}(f \star f^\ast)] \, d\mu_{\gamma}(\sigma) \, d\mu_N(\gamma) = \int_{\hat{N}/G} \int_{G/G_N} \text{tr} [\gamma \cdot A(f \star f^\ast |_N)] \, dv_N(A) \, d\mu_N(\gamma)
\]

\[
= \int_{\hat{N}/G} \int_{G/G_N} \text{tr} [\gamma \cdot A(f \star f^\ast |_N)] \, dv_N(A) \, d\mu_N(\gamma)
\]

\[
= \int_{\hat{N}/G} \text{tr} [\gamma(f \star f^\ast |_N)] \, d\mu_N(\gamma) = f \star f^\ast(e) = \int_G |f(x)|^2 \, dm_G(x).
\]

This implies the validity of \( 4.1 \) for \( f \in L^1(G) \cap L^2(G) \) since \( C_0(G) \) is dense in the \( C^* \)-algebra of \( G \).

The Plancherel measure for idyllic \( G = \exp \gamma \) with idyll \( N = \exp \gamma \) computed via \( 4.1 \) is given in terms of coordinates on \( \hat{N}/G \) and on the fibers \((G_N/N, \omega_{\gamma})^*\). We start by making an explicit choice of Haar measures \( m_G \) and \( m_N \) in terms of coordinates on \( G \) and \( N \), respectively. Then we compute Plancherel measure \( \mu_N \), in terms of coordinates on \( \hat{N} \) corresponding to \( m_N \). Next, we use Theorem 3.3 to obtain a disintegration of \( \mu_N \) by \( G \),

\[
\mu_N = \int_{\hat{N}/G} \nu_{\gamma} \, d\mu_N(\gamma),
\]

in which the pseudo-image \( \mu_N \) is given in terms of coordinates on almost all of \( \hat{N}/G \), and the orbit measures \( \nu_{\gamma} \) are expressed in terms of coordinates on the orbit of \( \gamma \). Then we use Theorem 3.1 to find the Haar measure \( m_{G/N} \) on \( G_N/N \) which satisfies \( 4.3 \). Then we use \( \S 1 \) to compute the projective Plancherel measure \( \mu_{\gamma} \) corresponding to \( m_{G/N} \) in terms of coordinates on \((G_N/N, \omega_{\gamma})^*\). Finally, we combine \( \mu_N \) and the \( \mu_{\gamma} \) to obtain a Plancherel formula for \( G \). The steps involved in the computational process and the resulting Plancherel formula are described in the following theorem.

**Theorem 4.1.** A Plancherel-measure-computing procedure for idyllic \( G = \exp \gamma \) with idyll \( N = \exp \gamma \) consists of the following steps:

1. Take a basis \( \{v_1 < \cdots < v_K\} \) of \( \gamma \) in Jordan-Hölder order relative to
the adjoint action of $g$ on $n$, and a Jordan-Hölder basis $\{e_1 < \cdots < e_s\}$ of $g/n$.

Let $\{v^1, \ldots, v^K\}$ be the basis of $n'$ such that $\langle v^j, v^l \rangle = \delta^j_l$.

(2) Compute $M = (e_i v_j)_{1 < i < s, 1 < j < K}$, where $(e_i v_j) = [e_i, v_j]$.

(3) Find the partitions defined in §3 (p. 13)

\[
\{1, \ldots, K\} = \{i_1, \ldots, i_p\} \cup \{j_1, \ldots, j_q\},
\]

\[
\{1, \ldots, s\} = \{i_1, \ldots, i_p\} \cup \{m_1, \ldots, m_q\},
\]

i.e., determine the independent columns of $M$ from the right and the independent rows of $M$ from below.

(4) For $y = (y_1, \ldots, y_d) \in \mathbb{R}^d$, let $sy = \sum_{k=1}^d y_k v^k$, and compute

\[
\det M^{(r)}(sy) = |sy(e_i v_j)_{1 < a, b < r}|.
\]

(5) For $y \in W = \{y \in \mathbb{R}^d : \det M^{(r)}(sy) \neq 0\}$, compute, for $1 < b < q$,

\[
u_b(sy) = e_m - \sum_{s=1}^r \lambda_{mb}^{(s)}(sy)e_s;
\]

(6) For $y \in W$, compute the matrix $(sy, [u_i(sy), u_j(sy)])_{1 < i, j < q}$.

(7) For $y \in W, \{y \in W : ([sy, [u_i(sy), u_j(sy)])_{1 < i, j < q} has maximal rank, 2l\}$, find a nonsingular $q \times q$ matrix $P_{sy}$ such that

\[
P_{sy}([sy, [u_i(sy), u_j(sy)])_{1 < i, j < q} = \begin{bmatrix}
0 & I_l & 0 \\
-I_l & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

Let $m = q - 2l$, and let

(4.4) \[\mu_{sy} = |\det P_{sy}|^{-1} \frac{1}{(2 \pi)^{1+m}} \psi_{P_{sy}(m_{R^n})},\]

where $\psi_{P_{sy}}$ is defined in §1.

Then
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\[ (4.5) \quad \mu_G = \frac{1}{(2\pi)^K} \int_{W_1} \mu_{sy} \left| \det M^G(sy) \right| dm_{R^G(\gamma)}(y) \]

is Plancherel measure on \( \hat{G} \) corresponding to \( m_G \), Haar measure on \( G \) defined in terms of the basis \( \{ e_1, \ldots, e_s, v_1, \ldots, v_K \} \) of \( \mathfrak{g} \).

The Plancherel formula is

\[ (4.6) \quad \int_G |f(x)|^2 \, dm_G(x) = \frac{1}{(2\pi)^{K+1+m}} \int_{W_1} \int_{R^m} \text{tr} \pi_{s}(f \ast f^*) \, dm_{R^m}(t) \]

\[ \left| \det P_{s} \right|^{-1} \left| \det M^r(\gamma) \right| dm_{R^G(\gamma)}(y). \]

For \((y, t) \in W_1 \times R^m, \pi_{s}(t) = \text{ind}_{G_{sy}}^{G} (\chi_{sy})' \otimes (\psi_{P_{sy}}(t))''\) is an irreducible representation of \( G \), where \( G_{sy} \) is the stability subgroup at sy for the coadjoint representation of \( G \) in \( n' \). \( \chi_{sy} \) is the character of \( N = \exp n \) defined by

\[ (X_{sy})'(\exp n) = e^{i\langle sy, n \rangle}, \quad n \in n. \]

(\( (\chi_{sy})' \) is the extension of \( \chi_{sy} \) to an \( \omega_{sy} \)-representation of \( G_{sy} \), where

\[ \omega_{sy}(\exp x, \exp z) = e^{-i/2\langle sy, [x, z] \rangle}, \quad x, z \in \mathfrak{g}_{sy}, \]

the stability subalgebra at sy for the coadjoint representation of \( \mathfrak{g} \) in \( n' \). \( \psi_{P_{sy}}(t) \) is an irreducible \( \tilde{\omega}_{sy} \)-representation of \( G_{sy}/N \), and \( (\psi_{P_{sy}}(t))'' \) denotes the lift of \( \psi_{P_{sy}}(t) \) to \( G_{sy} \).

**Proof.** To prove Theorem 4.1, we relate steps (1)–(7) to \( \tilde{N}, \mu_N \) (step (1)); the disintegration of \( \mu_N \) by \( G \) (steps (2)–(4)); equation (4.3) (step (5)); and \((G_{s}/N, \tilde{\omega}_{s})', \mu_{s} \) (steps (6) and (7)). Then we use (4.1).

Since \( n \) is abelian, \( \exp : n \rightarrow N \) is an isomorphism \( \exp(x + y) = \exp x \exp y \), and may be used to identify \( \tilde{N} \) with \( n' \). If \( \gamma \in n' \), let \( \chi_{\gamma} \) be the character of \( N \) defined by

\[ (\chi_{\gamma})(\exp x) = e^{i\langle \gamma, x \rangle}, \quad x \in n. \]

The map \( \gamma \rightarrow \chi_{\gamma} : n' \rightarrow \tilde{N} \) is an isomorphism. Let \( m_n \) be the Haar measure on \( N \) defined in terms of the basis \( \{ v_1, \ldots, v_K \} \) of \( n \). Let \( m_n' \) be the measure on \( n' \) defined by

\[ \langle m_{n'}, f \rangle = \int_{R^K} f \left( \sum_{j=1}^{K} \gamma_j v_j \right) \, dm_{R^K}(\gamma_1, \ldots, \gamma_K). \]

**Lemma 4.1.** Plancherel measure \( \mu_N \) on \( \hat{N} \) corresponding to \( m_n \) is the image of \((2\pi)^{-K} m_n \) under the map \( \gamma \rightarrow \chi_{\gamma} : n' \rightarrow \tilde{N} \).

**Proof.** If \( f \in C_0(N) \), let \( f_1 \in C_0(R^K) \) be

\[ f_1(x^1, \ldots, x^K) = f \left( \exp \sum_{j=1}^{K} x^j v_j \right), \quad (x^1, \ldots, x^K) \in R^K. \]
Then, for $\gamma = \sum_{j=1}^{K} \gamma_j u^j \in n'$,

$$
\chi_{\gamma}(f) = \int_{N} f(n) \chi_{\gamma}(n) \, dm_N(n)
$$

$$
= \int_{R^K} f \left( \exp \sum_{j=1}^{K} x^j u^j \right) \chi_{\gamma} \left( \exp \sum_{j=1}^{K} x^j u^j \right) \, dm_{R^K}(x^1, \ldots, x^K)
$$

$$
= \int_{R^K} f_1(x^1, \ldots, x^K) e^{i \langle \gamma, x^1 \rangle} \, dm_{R^K}(x^1, \ldots, x^K)
$$

$$
= \int_{R^K} f_1(x^1, \ldots, x^K) e^{i \sum_{j=1}^{K} \gamma_j x^j} \, dm_{R^K}(x^1, \ldots, x^K)
$$

$$
= \hat{f}_1(\gamma_1, \ldots, \gamma_K).
$$

Hence

$$
\int_{N} \left| \chi(f) \right|^2 \, d\mu_N(x) = (2\pi)^{-\frac{K}{2}} \int_{n'} \left| \chi_{\gamma}(f) \right|^2 \, dm_{n'}(\gamma)
$$

$$
= (2\pi)^{-\frac{K}{2}} \int_{R^K} \hat{f}_1(\gamma_1, \ldots, \gamma_K) \, dm_{R^K}(\gamma_1, \ldots, \gamma_K)
$$

$$
= \int_{R^K} \left| \chi_{\gamma}(x^1, \ldots, x^K) \right|^2 \, dm_{R^K}(x^1, \ldots, x^K)
$$

by the Plancherel formula for $R^K$. By definition of $f_1$, the latter integral is

$$
\int_{R^K} \left| f \left( \exp \sum_{j=1}^{K} x^j u^j \right) \right|^2 \, dm_{R^K}(x^1, \ldots, x^K) = \int_{N} \left| f(n) \right|^2 \, dm_N(n),
$$

by definition of $m_N$.

The action of $G$ on $\hat{N}$ corresponds to the coadjoint action of $G$ on $g'$ restricted to $n'$. If $\gamma \in n'$, $A \in G$ and $x \in n$, then

$$
(\chi_{\gamma} \cdot A)(\exp x) = \chi_{\gamma}(A \exp x A^{-1})
$$

$$
= \chi_{\gamma}(\exp \text{Ad} A(x)) = e^{i \langle \gamma, \text{Ad} A(x) \rangle} = e^{i \langle \gamma \cdot A, x \rangle} = \chi_{\gamma \cdot A}(\exp x).
$$

Hence the map $\gamma \rightarrow \chi_{\gamma} : n'/G \rightarrow \hat{N}/G$ identifies $\hat{N}/G$ with $n'/G$. We apply §3 to the adjoint action of $G$ on $n : G \times n \rightarrow n : (A, x) \mapsto A \cdot x$, where $A \cdot x = \text{Ad} A(x) = (d/dt)A \exp t x A^{-1} \big|_{t=0}$. $A \in G$, $x \in n$. The contragredient action of $G$ on $n' : n' \times G \rightarrow n' : (\gamma, A) \mapsto \gamma \cdot A$, where $\langle \gamma \cdot A, x \rangle = \langle \gamma, A \cdot x \rangle$, $\gamma \in n'$, $A \in G$, $x \in n$, is the coadjoint action of $G$ on $g'$ restricted to $n'$.

The derivative of the adjoint action of $G$ on $n$ is the adjoint action of $g$ on $n : g \times n \rightarrow n : (x, n) \mapsto x \cdot n = [x, n]$. The contragredient action of $g$ on $n'$ is the coadjoint action of $g$ on $n' : n' \times g \rightarrow n' : (\gamma, x) \mapsto \gamma \cdot x$, where $\langle \gamma \cdot x, n \rangle = \langle \gamma, [x, n] \rangle$, $\gamma \in n'$, $x \in g$, $n \in n$.

Since $\{v_1 < \cdots < v_K\}$ is a basis of $n$ in Jordan-Hölder order relative to $g$, and $\{\bar{e}_1 < \cdots < \bar{e}_s\}$ is a Jordan-Hölder basis of $g/n$, $\{e_1 < \cdots < e_s < v_1 < $
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... is a Jordan-Hölder basis of \( \mathfrak{g} \). We take Haar measure on \( G \) to be the measure \( m_G \) defined in terms of this basis.

Define \( e_{s+j} = v_j, 1 \leq j \leq K \). Since \( \mathfrak{n} \) is abelian, \([e_{s+j}, v_k] = 0, 1 \leq j, k \leq K\). Thus, the matrix \( M = (e_i v_j)_{1 \leq i \leq s+K, 1 \leq j \leq K} \) defined in §3 has the form

\[
M = \begin{bmatrix}
e_1 v_1 & \cdots & e_1 v_K \\
e_s v_1 & \cdots & e_s v_K \\
0 & \cdots & 0 \\
0 & \cdots & 0
\end{bmatrix}
\]

Disregarding the last \( K \) rows, we have \( M = (e_i v_j)_{1 \leq i \leq s, 1 \leq j \leq K} \) as in step (2). As in §3, \( E = \{ \gamma \in \mathfrak{n}' : \det M^{(r)}(\gamma) \neq 0 \} \).

By Theorem 3.2, for \( sy = \Sigma_{k=1}^d y_k v^k, W = \{ y \in \mathbb{R}^d : sy \in E \} \), and \( p : \mathfrak{n}' \to \mathfrak{n}'/G \) the projection \( p \circ s \mid W : W \to E/G \) is a homeomorphism. By Theorem 3.3,

\[
m_{n'} = \int_{W} \nu_{sy} | det M^{(r)}(sy) | dm_{\mathbb{R}^d}(y)
\]

is a disintegration of \( m_{n'} \) by \( G \). By Lemma 4.1, \( \mu_N = (2\pi)^{-K} m_{n'} \). Since \( \hat{N}/G = \mathfrak{n}'/G \),

\[
(4.7) \quad \mu_N = (2\pi)^{-K} \int_{W} \nu_{sy} | det M^{(r)}(sy) | dm_{\mathbb{R}^d}(y)
\]

is a disintegration of \( \mu_N \) by \( G \), in which the pseudo-image \( \tilde{\mu}_N \) is given in terms of coordinates on \( E/G \).

By Theorem 3.1, if \( u_b(sy), 1 \leq b \leq q \), are computed as in step (5), then \( \{u_1(sy), \ldots, u_q(sy)\} \) is a basis of \( \mathfrak{g}_{sy}/\mathfrak{n} \), and Haar measure \( m_{\mathfrak{g}_{sy}/\mathfrak{n}} \) on \( G_{sy}/N \) defined in terms of this basis satisfies (4.3) relative to the orbit measure \( \nu_{sy} \) and \( m_N \).

As stated, Theorem 3.1 gives a basis of the stability subalgebra \( \mathfrak{g}_{sy} \) such that Haar measure \( m_{\mathfrak{g}_{sy}} \) on \( G_{sy} = \exp \mathfrak{g}_{sy} \) computed in terms of this basis satisfies

\[
\int_{G} f(x) \, dm_G(x) = \int_{G/\mathfrak{g}_{sy}} \int_{\mathfrak{g}_{sy}} f(zx) \, dm_{\mathfrak{g}_{sy}}(z) \, \nu_{sy}(\bar{x})
\]

In the present situation, \( \mathfrak{g}_{sy} = \text{span} \{u_1(sy), \ldots, u_q(sy)\} \oplus \mathfrak{n} \), and the basis of \( \mathfrak{g}_{sy} \) computed in Theorem 3.1 is \( \{u_1(sy), \ldots, u_q(sy), u_1, \ldots, u_K\} \). By definition of \( m_{\mathfrak{g}_{sy}} \) (§2),

\[
\int_{G_{sy}} f(z) \, dm_{\mathfrak{g}_{sy}}(z) = \int_{\mathbb{R}^q \times \mathbb{R}^K} f \left( \exp \left( \sum_{i=1}^q z_i u_i(sy) + \sum_{i=1}^K n_i v_i \right) \right) \\
(dm_{\mathbb{R}^q \times \mathbb{R}^K})(z^1, \ldots, z^q, n^1, \ldots, n^K).
\]

By Lemma 2.1 applied to the Jordan-Hölder basis \( \{u_1(sy)\} < \cdots < u_q(sy) < v_1 < \cdots < v_K \).
\[
\int_{R} \left( \int_{R^{K}} f \left( \exp \left( \sum_{i=1}^{K} n' r_{i(j)} \right) \cdot \exp \left( \sum_{i=1}^{q} z' r_{i(j)} \right) \right) \right) \, dm_{R}^{n} dm_{R}^{z} \]

by definition of \( m_{K} \) and \( m_{G_{n}/N} \) (§3).

Steps (6) and (7) are the projective Plancherel measure parts of the procedure. Using the Campbell-Baker-Hausdorff formula, we write, for \( x, y \in g \),

\[
\exp x \exp y = \exp (x + y + B(x, y)),
\]

where

\[
B(x, y) = (1/2)[x, y] + (1/12)([x, [x, y]] - [y, [x, y]]) + \text{(terms of the form } [x, [\ldots, [x, y] \cdot \cdot \cdot ]]} \]

and \([y, [\ldots, [x, y] \cdot \cdot \cdot ]}).

Since \( g \) is nilpotent, \( B(x, y) \) has only finitely many terms.

**Lemma 4.2.** Suppose \( G = \exp g \) is a nilpotent Lie group. If \( f \in g' \), let

\[
\omega_{f}(\exp x, \exp y) = e^{-i(f, B(x, y))}.
\]

Then \( \omega_{f} \) is a normalized, trivial multiplier on \( G \).

**Proof.** Since \((\exp x)^{-1} = \exp(-x), B(x, -x) = 0, so \omega_{f}(\exp x, (\exp x)^{-1}) = 1. The cocycle identity follows from associativity of multiplication on \( G \).

\[
(\exp x \exp y) \exp z = \exp (x + y + B(x, y)) \exp z
\]

\[
= \exp((x + y + B(x, y)) + z + B(x + y + B(x, y), z))
\]

\[
= \exp x(\exp y \exp z) = \exp x \exp (y + z + B(y, z))
\]

\[
= \exp(x + (y + z + B(y, z))) + B(x, y + z + B(y, z)).
\]

Since \( \exp \) is injective,

\[
B(x, y) + B(x + y + B(x, y), z) = B(y, z) + B(x, y + z + B(y, z)).
\]

Thus,

\[
\omega_{f}(\exp x, \exp y) \omega_{f}(\exp x \exp y, \exp z)
\]

\[
= e^{-i(f, B(x, y))} e^{-i(f, B(x + y + B(x, y), z))}
\]

\[
= e^{-i(f, B(y, z))} e^{-i(f, B(x, y + z + B(y, z)))}
\]

\[
= \omega_{f}(\exp y, \exp z) \omega_{f}(\exp x, \exp y \exp z).
\]
To see that $\omega_f$ is trivial, let $\chi_f : G \to T$ be defined by $\chi_f(\exp x) = e^{i\langle f, x \rangle}$, $x \in g$. Then

$$\chi_f(\exp x \exp y) = \chi_f(\exp(x + y + B(x, y))) = e^{i\langle f, x + y + B(x, y) \rangle} = \chi_f(\exp x)\chi_f(\exp y)\omega_f(\exp x, \exp y),$$

so that

$$\omega_f(\exp x, \exp y) = \frac{\chi_f(\exp x)\chi_f(\exp y)}{\chi_f(\exp x \exp y)}.$$

The above proof shows that if $\gamma \in \mathfrak{n}'$, then $\chi_{\gamma}$ may be extended to a multiplier representation of $G$ as follows. Let $\gamma'$ in $g'$ be any extension of $\gamma$ to $g$. Then $\omega_{\gamma'} \big|_{G \cdot \gamma' \times G \cdot \gamma'}$ is a multiplier on $G_{\gamma'/N}$ because, if $x \in g_{\gamma'}$, then $\langle \gamma, [x, n] \rangle = 0$. This implies that $\langle \gamma', B(x + n + B(n, x), y) \rangle = \langle \gamma', B(x, y) \rangle$ for $x, y \in g_{\gamma'}$, $n \in \mathbb{R}$, which says that $\omega_{\gamma'}(\exp n \exp x, \exp y) = \omega_{\gamma'}(\exp x, \exp y)$. Although $\omega_{\gamma'}$ is a trivial multiplier on $G_{\gamma'}$, it is not, in general, trivial on $G_{\gamma'/N}$ (unless $\gamma = 0$), because $\chi_{\gamma'}(\exp n) = e^{i\langle \gamma, n \rangle}$ is not one on $N$.

Now suppose $G_{\gamma'/N}$ is abelian. Then $[g_{\gamma'}, g_{\gamma'}] \subset n$. If $x, y \in g_{\gamma'}$, $[x, y] \in n$, $n$ is an ideal, so

$$B(x, y) = \frac{1}{2} [x, y] + \text{terms of the form } [x, \text{an element of } n] \text{ or } [y, \text{an element of } n].$$

Since $\langle \gamma, [x, n] \rangle = \langle \gamma, [y, n] \rangle = 0$, $\langle \gamma, B(x, y) \rangle = \frac{1}{2} \langle \gamma, [x, y] \rangle$. Therefore $\omega_{\gamma}(\exp x, \exp y) = e^{i\beta_{\gamma}(\langle x, y \rangle)}$. Since $g_{\gamma}/n$ is abelian, $\exp : g_{\gamma}/n \to G_{\gamma'/N}$ is an isomorphism. Define $A_{\gamma} : g_{\gamma}/n \times g_{\gamma}/n \to \mathbb{R}$ by $A_{\gamma}(x, y) = \langle \gamma, [x, y] \rangle$. Then $A_{\gamma}$ is bilinear and skew symmetric, and $\omega_{\gamma}$ has the form of the multiplier in §1, $\omega_{\gamma}(x, y) = e^{i\beta_{\gamma}(A(x, y))}$, $x, y \in G_{\gamma'/N}$ (identified with $g_{\gamma}/n$).

By definition of idyllic, $g_{\gamma}/n$ is abelian for $m_n$, almost all $\gamma$ in $\mathfrak{n}'$. The following lemma shows that $g_{\gamma}/n$ is abelian for all $\gamma$ in $E$.

**Lemma 4.3.** If there is a $\gamma$ in $E$ such that $g_{\gamma}/n$ is not abelian, then $g_{\gamma}/n$ is not abelian for all $\gamma$ in a nonempty Zariski open subset of $E$.

**Proof.** Let $\gamma$ be in $E$, and $\{u_a(\gamma) : 1 \leq a \leq q\}$ be the basis of $g_{\gamma}/n$ defined in step (5). Then

$$[g_{\gamma}, g_{\gamma}] \subset n \iff [u_a(\gamma), u_b(\gamma)] \in n,$$

for $1 \leq a, b \leq q$. This requirement, when written out in terms of the definition of $u_a(\gamma)$, determines a family of rational functions of the form

$$R_{ab}^l(\gamma) = \Gamma_{m_a m_b}^l + \frac{P_{ab}^l(\gamma)}{\det M^r(\gamma)} + \frac{Q_{ab}^l(\gamma)}{(\det M^r(\gamma))^2}$$
(where $\Gamma^I_{mabp} \in \mathbb{R}$, and $P_{ab}^I$, $Q_{ab}^I$ are polynomials in $\gamma_1, \ldots, \gamma_K$), which must vanish for $1 \leq l \leq s$, $1 \leq a, b \leq q$. Each $R_{ab}^I(\gamma) = 0$ is the family of polynomials $F_{ab}^I(\gamma) = (\det M^{(r)}(\gamma))^2 R_{ab}^I(\gamma) = 0$ for $1 \leq l \leq s$, $1 \leq a, b \leq q$. Therefore, $\{\gamma \in E : [\gamma, \gamma] \subseteq \mathbb{C} \} = \{\gamma \in E : F_{ab}^I(\gamma) = 0, 1 \leq l \leq s, 1 \leq a, b \leq q\} = F$, a Zariski closed set in $E$.

The projective Plancherel measure determined in §1 for the multiplier on a vector space $H$ arising from a bilinear skew-symmetric mapping $A : H \times H \rightarrow \mathbb{R}$ depends on the rank of the form $A$, where rank $A$ is the rank of the matrix $(A(u_p u_q))_{1 \leq i, j \leq \dim H}$, for any basis $\{u_i\}$ of $H$. The following lemma shows that the rank of the form $A_\gamma : g_\gamma/\mathfrak{n} \times g_\gamma/\mathfrak{n} \rightarrow \mathbb{R}$, $A_\gamma(x, y) = (\gamma, [x, y])$, is constant on a nonempty, $G$-invariant Zariski open set $E_1$ of $E$. By passing to $E_1$, we obtain a Plancherel measure for $G$ in which the dimension of the coordinate space of the fibers $(G_\gamma/\mathfrak{n}, \mathfrak{g}_\gamma)^*$ is constant.

**Lemma 4.4.** There is an integer $l$, $0 \leq l \leq q/2$, such that rank $A_\gamma = 2l$ for all $\gamma$ in a nonempty, $G$-invariant Zariski open set $E_1 \subseteq E$.

**Proof.** Let $\gamma \in E$. For $0 \leq k \leq q$, let $T_k(\gamma)$ be the set of all $k \times k$ minors of the matrix $(A(u_p u_q))_{1 \leq i, j \leq \dim H}$, for any basis $\{u_i\}$ of $H$. The following lemma shows that the rank of the form $A_\gamma : g_\gamma/\mathfrak{n} \times g_\gamma/\mathfrak{n} \rightarrow \mathbb{R}$, $A_\gamma(x, y) = (\gamma, [x, y])$, is constant on a nonempty, $G$-invariant Zariski open set $E_1$ of $E$. By passing to $E_1$, we obtain a Plancherel measure for $G$ in which the dimension of the coordinate space of the fibers $(G_\gamma/\mathfrak{n}, \mathfrak{g}_\gamma)^*$ is constant.

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(4.8) \[ \mu_N = (2\pi)^{-K} \int_{W_1} \nu_{sy} |\det M(r)(sy)| \, dm_Rd(y). \]

By §1 for \( y \in W_1 \), the map \( \psi_{Psy} : R^m \to (G_{sy}/N, \bar{\omega}_{sy}) \) is a homeomorphism, where \( m = q - 2l \); and (4.4) in step (7),

\[ \mu_{sy} = |\det P_{sy}|^{-1} (2\pi)^{-(t+m)} \psi_{Psy}(m_{R^m}), \]

is the projective Plancherel measure on \( (G_{sy}/N, \bar{\omega}_{sy}) \) corresponding to the Haar measures \( m_{G_{sy}/N} \) on \( G_{sy}/N \).

Since \( m_{G_{sy}/N} \) satisfies (4.3) with respect to the orbit measure \( \nu_{sy} \) in the disintegration formula (4.8), Kleppner and Lipsman’s Plancherel formula for group extensions (4.1) [15, Theorem 2.3, p. 108] says that (4.5),

\[ \mu_G = (2\pi)^{-K} \int_{W_1} \mu_{sy} |\det M(r)(sy)| \, dm_Rd(y), \]

is Plancherel measure on \( \hat{G} \) corresponding to Haar measure \( m_G \) on \( G \), and that formula (4.6) is a Plancherel formula for \( G \).

**Table I: Plancherel formulas.** Plancherel formulas computed in [23] are summarized here. For each group \( G = \exp g \), data are listed in the following order

1. A Jordan-Hölder basis \( B = \{ e_i : 1 \leq i \leq \dim g \} \). (The basis of \( g' \) dual to \( B \) is denoted \( \{ e^i : 1 \leq i \leq \dim g \} \).)
2. Nonzero vectors in the set \( \{ [x, y] : x, y \in B \} \).
3. A basis of \( n \), the idyll of \( g \) used to compute \( \mu_G \). \( (N = \exp n < G) \).
4. A basis of \( g_\gamma/n \), where \( g_\gamma = \{ x \in g : \langle \gamma, [x, n] \rangle = 0 \ \forall n \in n \} \) for \( \gamma \in E \)
   \( (E = \{ \gamma \in n' : \det M(r)(\gamma) \neq 0 \} \) as in §3 and Theorem 4.1.)
5. The Plancherel formula,

\[ \int_G |f|^2 = \int_{W_1} \int_{R^m} \text{tr}[\pi_{sy,t}(f \ast f^*)] \, dm_Rm(t)R(y) \, dm_Rd(y), \]

\( f \in L^1(G) \cap L^2(G) \).

In each case, \( \int_G |f|^2 \) denotes the \( \int_G |f(x)|^2 \, dm_G(x) \), where \( m_G \) is the Haar measure on \( G \) defined in terms of the basis \( B \) of \( g \) (as in §2). \( R(y) \) is the rational function of \( y \) defined in Theorem 4.1. \( d \) is the codimension of a maximal dimension orbit in \( n' \) under the coadjoint representation of \( G \) in \( n' \). \( s : R^d \to n' \) is the section for the orbits of \( G \) in \( n' \) used to compute \( \mu_G \). \( W = \{ y \in R^d : \det M(r)(sy) \neq 0 \} \). For \( y \in W_1 \subset W \), \( \pi_{sy,t} = \text{ind}_{g_{sy}}^{G} (x_{sy})^t \otimes (\psi_{Psy}(t))^t \) (Theorem 4.1) is an irreducible representation of \( g \) for \( t \in R^m \).

The following procedure gives most of the idylls listed below. Let \( \delta_1 \subset \cdots \subset \delta_n = g \) be the ascending central series of \( g \). Let \( n_1 = \delta_1 \). Having chosen \( n_i \), let \( n_{i+1} \) be a maximal dimensional abelian subalgebra of \( \delta_{i+1} \) containing \( n_i \).
Then $n = n_n$. It is a conjecture that if $g$ is idyllic, then the maximal abelian ideal $n$ of $g$ obtained in this way is an idyll.

A. HEISENBERG GROUPS, $H_n$

(1) $\{e_1, \ldots, e_{2n}, e_{2n+1}\}$
(2) $[e_i, e_{n+i}] = -[e_{n+i}, e_i] = e_{2n+1}, 1 \leq i \leq n$
(3) $\{e_{n+1}, \ldots, e_{2n}, e_{2n+1}\}$
(4) $\{0\}$
(5) $\int_G |f|^2 = (2\pi)^{-n} \int_{W} \text{tr} [\pi_{sy}(f * f^*)] |y| n \, dm_R(y)$
$s : R \rightarrow n', sy = ye^{2n+1}$
$W = R - \{0\}$
$\pi_{sy} = \text{ind}_N^G \chi_{sy}$
$\chi_{sy}(\exp(\sum_{i=n+1}^{2n+1} x^i e_i)) = e^{y x^{2n+1}}$

B. KIRILLOV'S SECOND EXAMPLE [12, p. 102]

(1) $\{e_0, \ldots, e_n\}$
(2) $[e_0, e_i] = -[e_i, e_0] = e_{i+1}, 1 \leq i \leq n - 1$
(3) $\{e_1, \ldots, e_n\}$
(4) $\{0\}$
(5) $\int_G |f|^2 = (2\pi)^{-n} \int_{W} \text{tr} [\pi_{sy}(f * f^*)] |y| n \, dm_R(y)$
$s : R^{n-1} \rightarrow n'$,
$s(y_1, \ldots, y_{n-1}) = y_1 e^1 + \cdots + y_{n-2} e^{n-2} + y_{n-1} e^n$
$W = \{y = (y_1, \ldots, y_{n-1}) : y_{n-1} \neq 0\}$
$\pi_{sy} = \text{ind}_N^G \chi_{sy}$
$\chi_{sy}(\exp(\sum_{i=1}^{n-1} x^i e_i)) = e^{(y_1 x^1 + \cdots + y_{n-2} x^{n-2} + y_{n-1} x^n)}$

C. GROUPS OF DIMENSION $\leq 5$

These are the groups $\Gamma = \exp g$, where $g$ is one of the algebras listed by Dixmier [9, Proposition 1, p. 323]. The Plancherel formula is given here for those groups which are not products.

$\Gamma_1 = R$

$$\int_{\Gamma_1} |f|^2 = \frac{1}{2\pi} \int_R \chi_y(f * f^*) \, dm_R(y).$$

$\chi_y(x) = e^{yx \cdot x}, y \in R.$

$\Gamma_3 = H_1.$

(1) $\{e_1, e_2, e_3\}$
(2) $[e_1, e_2] = -[e_2, e_1] = e_3$
(3) $\{e_2, e_3\}$
(4) $\{0\}$
(5) $\int_{\Gamma_3} |f|^2 = (2\pi)^{-2} \int_{W} \text{tr} [\pi_{sy}(f * f^*)] |y| \, dm_R(y)$
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\[ s : \mathbb{R} \rightarrow \mathbb{R}', s y = y e^3 \]
\[ \mathcal{W} = \mathbb{R} - \{0\} \]
\[ \pi sy = \text{ind}_\Gamma^\mathbb{R} \chi_{sy} \]
\[ \chi_{sy}(\exp(x^2e_2 + x^3e_3)) = e^{iy \cdot x^3} \]

Dimension 4: \( \Gamma_4 \)

1. \{e_1, e_2, e_3, e_4\}
2. \([e_1, e_2] = -[e_2, e_1] = e_3 \]
   \([e_1, e_3] = -[e_3, e_1] = e_4 \]
3. \{e_2, e_3, e_4\}
4. \{0\}
5. \( \int_{\Gamma_4} |f|^2 = (2\pi)^{-3} \int_{\mathcal{W}} \text{tr}[\pi_{sy}(f \ast f^*)] \, dy_2 \, dm_{\mathbb{R}^2}(y_2, y_4) \)
   \[ s : \mathbb{R}^2 \rightarrow \mathbb{R}', sy = y e^2 + y_4 e^4 \]
   \[ \mathcal{W} = \{ y = (y_2, y_4) : |y_4| \neq 0 \} \]
   \[ \pi_{sy} = \text{ind}_\Gamma^\mathbb{R} \chi_{sy} \]
   \[ \chi_{sy}(\exp(x^2e_2 + x^3e_3 + x^4e_4)) = e^{iy_2x^2 + y_4x^4} \]

\[ \Gamma_{5,1} \]

1. \{e_1, e_2, e_3, e_4, e_5\}
2. \([e_1, e_2] = -[e_2, e_1] = e_5 \]
   \([e_3, e_4] = -[e_4, e_3] = e_5 \]
3. \{e_2, e_4, e_5\}
4. \{0\}
5. \( \int_{\Gamma_{5,1}} |f|^2 = (2\pi)^{-4} \int_{\mathcal{W}} \text{tr}[\pi_{sy}(f \ast f^*)] \, dy_2 \, dm_{\mathbb{R}}(y) \)
   \[ s : \mathbb{R} \rightarrow \mathbb{R}', sy = ye^5 \]
   \[ \mathcal{W} = \mathbb{R} - \{0\} \]
   \[ \pi_{sy} = \text{ind}_\Gamma^\mathbb{R} \chi_{sy} \]
   \[ \chi_{sy}(\exp(x^2e_2 + x^4e_4 + x^5e_5)) = e^{iy \cdot x^5} \]

\[ \Gamma_{5,2} \]

1. \{e_1, e_2, e_3, e_4, e_5\}
2. \([e_1, e_2] = -[e_2, e_1] = e_4 \]
   \([e_1, e_3] = -[e_3, e_1] = e_5 \]
3. \{e_2, e_3, e_4, e_5\}
4. \{0\}
5. \( \int_{\Gamma_{5,2}} |f|^2 = (2\pi)^{-4} \int_{\mathcal{W}} \text{tr}[\pi_{sy}(f \ast f^*)] \, dy_5 \, dm_{\mathbb{R}^3}(y_2, y_4, y_5) \)
   \[ s : \mathbb{R}^3 \rightarrow \mathbb{R}', s(y_2, y_4, y_5) = y_2 e^2 + y_4 e^4 + y_5 e^5 \]
   \[ \mathcal{W} = \{ y = (y_2, y_4, y_5) : y_5 \neq 0 \} \]
   \[ \pi_{sy} = \text{ind}_\Gamma^\mathbb{R} \chi_{sy} \]
   \[ \chi_{sy}(\exp(x^2e_2 + x^3e_3 + x^4e_4 + x^5e_5)) = e^{iy_2x^2 + y_4x^4 + y_5x^5} \]
\[ \Gamma_{5,3} \]

(1) \{e_1, e_2, e_3, e_4, e_5\}

(2) \[[e_1, e_2] = - [e_2, e_1] = e_3\]
\[ [e_1, e_3] = - [e_3, e_1] = e_4\]
\[ [e_2, e_3] = - [e_3, e_2] = e_5\]

(3) \{e_3, e_4, e_5\}

(4) \{0\}

(5) \int_{\Gamma_{5,3}} |f|^2 = (2\pi)^{-3} \int_{W} \text{tr}[\pi_{xy}(f \ast f^*)] |y|^2 \, dm_R(y)
\[ s : \mathbb{R} \rightarrow n', s(y) = ye_5 \]
\[ W = \mathbb{R} - \{0\} \]
\[ \pi_{xy} = \text{ind}_N^G \chi_{xy} \]
\[ \chi_{xy} = \exp(x^3e_3 + x^4e_4 + x^5e_5) = e^{ly \ast x^5} \]

\[ \Gamma_{5,4} \]

(1) \{e_1, e_2, e_3, e_4, e_3\}

(2) \[ [e_1, e_2] = - [e_2, e_1] = e_3\]
\[ [e_1, e_3] = - [e_3, e_1] = e_4\]
\[ [e_2, e_3] = - [e_3, e_2] = e_5\]

(3) \{e_3, e_4, e_5\}

(4) \text{span}_R \{e_1 - \langle \gamma, e_4 \rangle / \langle \gamma, e_5 \rangle e_2\}

(5) \int_{\Gamma_{5,4}} |f|^2 = (2\pi)^{-4} \int_{W} \text{tr}[\pi_{xy}(f \ast f^*)] \, dt \, dy \, dm_R(y_4, y_5)
\[ s : \mathbb{R}^2 \rightarrow n', s(y_4, y_5) = y_4e^4 + y_5e^5 \]
\[ W = \{y = (y_4, y_5) : y_5 \neq 0\} \]
\[ \pi_{xy} = \text{ind}_N^G \chi_{xy} \otimes (\chi_y)^n \]
\[ \chi_{xy} = \exp(x^3e_3 + x^4e_4 + x^5e_5) = e^{ly_4x^4 + y_5x^5} \]
\[ \chi_t = \exp(\alpha(e_1 - (y_4/y_5)e_2)) = e^{t \ast \alpha}, \alpha, t \in \mathbb{R} \]

\[ \Gamma_{5,5} \]

(1) \{e_1, e_2, e_3, e_4, e_5\}

(2) \[ [e_1, e_2] = - [e_2, e_1] = e_3\]
\[ [e_1, e_3] = - [e_3, e_1] = e_4\]
\[ [e_2, e_3] = - [e_3, e_2] = e_5\]

(3) \{e_2, e_3, e_4, e_5\}

(4) \{0\}

(5) \int_{\Gamma_{5,5}} |f|^2 = (2\pi)^{-4} \int_{W} \text{tr}[\pi_{xy}(f \ast f^*)] \, dy \, dm_R(y_2, y_3, y_5)
\[ s : \mathbb{R}^3 \rightarrow n', s(y_2, y_3, y_5) = y_2e^2 + y_3e^3 + y_5e^5 \]
\[ W = \{y = (y_2, y_3, y_5) : y_5 \neq 0\} \]
\[ \pi_{xy} = \text{ind}_N^G \chi_{xy} \]
\[ \chi_{xy} = \exp(x^3e_2 + x^3e_3 + x^4e_4 + x^5e_5) = e^{(y_2x^2y_3x^3 + y_5x^5)} \]
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\[ \Gamma_{5,6} \]

(1) \( \{e_1, e_2, e_3, e_4, e_5\} \)

(2) \( [e_1, e_2] = - [e_2, e_1] = e_3 \)
\( [e_1, e_3] = - [e_3, e_1] = e_4 \)
\( [e_1, e_4] = - [e_4, e_1] = e_5 \)
\( [e_2, e_3] = - [e_3, e_2] = e_5 \)

(3) \( \{e_3, e_4, e_5\} \)

(4) \( \{0\} \)

(5) \( \int_{\Gamma_{5,6}} |f|^2 = (2\pi)^{-3} \int_W \text{tr}[\pi_{s_y}(f * f^*)] |\nu|^2 \, dm_R(\nu) \)
\( s : R \rightarrow n', s_y = ye^x \)
\( W = R - \{0\} \)
\( \pi_{s_y} = \text{ind}_N^G X_{s_y} \)
\( X_{s_y}(\exp(x^3 e_3 + x^4 e_4 + x^5 e_5)) = e^{\nu'x^5} \)

D. TWO-STEP GROUPS

(1) \( \{e_1, \ldots, e_S\} \cup \{v_1, \ldots, v_K\} \)

(2) \( [e_i, e_j] = - [e_j, e_i] \subset \text{span}\{v_1, \ldots, v_K\}, 1 \leq i, j \leq S \)

(3) \( \{v_1, \ldots, v_K\} = \text{center of } g \)

(4) \( \{e_i, \ldots, e_S\} \)

(5) \( \int_G |f|^2 = (2\pi)^{-l} \int_{W_1} \int_{R_m} \text{tr}[\pi_{s_y^f}(f * f^*)] \)
\( \det P_{s_y} |^{-1} dm_{R_K}(\nu). \)

For \( y \in W_1, P_{s_y} \) is a nonsingular \( S \times S \) matrix such that

\[
P_{s_y}(\langle s_y, [e_p e_j] \rangle)_{1 \leq i, j \leq S} P_{s_y}^T = \begin{bmatrix}
0 & I_l \\
-I_l & 0 \\
0 & 0 \\
\end{bmatrix}
\]

\( 2l \times m \)

For \( (\nu, t) \in W_1 \times R^m, \)

\( \pi_{s_y, t} = (\chi_{s_y})' \otimes (\psi_{s_y^f}(t))^\nu; \quad \chi_{s_y}(\exp(\sum_{j=1}^K u_j v_j)) = e^{i \sum_{j=1}^K y_j m_j}; \)
\[
\psi_{P_{sy}}(t) \left( \exp \left( \sum_{i=1}^{S} x^{i} e_{i} \right) \right) \\
= a_{1} \left( \left( \sum_{i=1}^{S} x^{i} Q_{i}^{1}, \ldots, \sum_{i=1}^{S} x^{i} Q_{i}^{1} \right) \right) \left( \sum_{i=1}^{S} x^{i} Q_{i}^{1+1}, \ldots, \sum_{i=1}^{S} x^{i} Q_{i}^{1} \right) e^{\sum_{a=1}^{m} \sum_{i=1}^{S} x^{i} Q_{i}^{1+a} e_{a}},
\]
where \((Q_{i})_{1 \leq i \leq S} = P_{sy}^{-1}\).

E1. Nilpotent Part of \(G_{2l}\) (see [10, 11, 21])

(1) \{e_{1}, e_{2}, e_{3}, e_{4}, e_{5}, e_{6}\}
(2) \[e_{1}, e_{2}\] = \(-[e_{2}, e_{1}] = e_{3}\)
[\[e_{1}, e_{3}\] = \(-[e_{2}, e_{1}] = e_{4}\)
[\[e_{1}, e_{4}\] = \(-[e_{2}, e_{1}] = e_{5}\)
[\[e_{1}, e_{5}\] = \(-[e_{2}, e_{1}] = e_{6}\)
[\[e_{2}, e_{3}\] = \(-[e_{2}, e_{1}] = e_{6}\)
(3) \{e_{4}, e_{5}, e_{6}\}
(4) \{e_{1} + \langle y, e_{5}\rangle, \langle y, e_{6}\rangle\} e_{3}\}
(5) \int_{G} |f|^{2} = (2\pi)^{-a} \int_{W} \text{tr} \left[ \pi_{sy}(f \ast f^{*}) \right] \text{dm}_{R}(t) |y|^{2} \text{dm}_{R}(y)
\]
\(s : R \rightarrow n', sy = ye^{6}\)
\(W = R - \{0\}\)
\(\pi_{sy} = \text{ind}_{G}^{G}(x_{sy}\rangle) \otimes (x_{y})^{n}\)
\(x_{sy}(\exp(x^{4} e_{4} + x^{5} e_{5} + x^{6} e_{6})) = e^{yx_{6}}\)
\(X_{y}(\exp \lambda e_{4}) = e^{\lambda x_{4}}, \lambda \in R\)

E2a. Nilpotent Part of \(A_{2l}, l + 1 = 2m\)

(1) \{e_{ij} : 1 \leq i < j \leq 2m\}
(2) \[e_{ir}, e_{rj}\] = \(-[e_{rj}, e_{ir}] = e_{ij}, 1 \leq i < r < j \leq 2m\)
(3) \{e_{ij} : 1 \leq i \leq m, m + 1 \leq j \leq 2m\}
(4) \{0\}
(5) \int_{G} |f|^{2} = (2\pi)^{-m} \int_{W} \text{tr} \left[ \pi_{sy}(f \ast f^{*}) \right] \Pi_{k=1}^{m-1} \left| y_{k, 2m+1-k} \right|^{2(m-k)}
\text{dm}_{R}(y_{1,2m}, \ldots, y_{m-1,m+2}, y_{m,m+1})
\text{W} = \{y = (y_{1,2m}, \ldots, y_{m,m+1}) : \Pi_{k=1}^{m-1} \left| y_{k, 2m+1-k} \right| \neq 0\}
\pi_{sy} = \text{ind}_{G}^{G}(x_{sy})
\chi_{sy}(\exp(\sum_{1 \leq i \leq m, m+1 \leq j \leq 2m} x^{ij} e_{ij})) = e^{\sum_{k=1}^{m} y_{k, 2m+1-k} x_{k, 2m+1-k}}\)
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E2b. Nilpotent part of \( A_l, l + 1 = 2m - 1 \)

1. \( \{ e_{ij} : 1 \leq i < j \leq 2m - 1 \} \)
2. \( \{ e_{ir}, e_{rj} \} = - \{ e_{ij} \} \quad 1 \leq i < r < j \leq 2m - 1 \)
3. \( \{ e_{ij} : 1 \leq i \leq m, m + 1 \leq j \leq 2m - 1 \} \)
4. \( \{ 0 \} \)
5. \( \int_G |f|^2 = (2\pi)^{-m(m-1)} \int_{W} \text{tr} [\pi_{sy}(f * f^*)] \prod_{k=1}^{m-1} |y_{k,2m-k}|^{2(m-k)-1} \)

\( s : R^{m-1} \rightarrow n', s(y_{1,2m-1}, \ldots, y_{m-1,m+1}) = \sum_{k=1}^{m-1} y_{k,2m-k}e^{k,2m-k} \)

\( W = \{ y = (y_{1,2m-1}, \ldots, y_{m-1,m+1}) : \Pi_{k=1}^{m-1} |y_{k,2m-k}| \neq 0 \} \)

\( \pi_{sy} = \text{ind}_N^G \chi_{sy}, \)

\( \chi_{sy}(\exp (\Sigma_{1<i<m; m+1<j<2m-1} x_{ij}^k e_{ij})) = e^{i \sum_{k=1}^{m-1} y_{k,2m-k}e^{k,2m-k}} \)

E3. Nilpotent part of \( G_l \)

1. \( \{ a_{ij} : 1 \leq i < j \leq l \} \cup \{ b_{ij} : 1 \leq i \leq l, 2l + 1 - i \leq j \leq 2l \} \)
2. \( \{ a_{ij}, a_{jk} \} = - \{ a_{kj}, a_{ij} \} = a_{ik}, 1 \leq i < j < k \leq l \)

For \( 1 \leq i < j < l, 1 \leq t < l, 2l + 1 - t \leq s \leq 2l, \)

\[ [a_{ij}, b_{ts}] = - [b_{ts}, a_{ij}] = \begin{cases} 
2b_{t,2l+1-t} & \text{if } t = j = 2l + 1 - s \\
b_{2l+1-s,2l+1-t} & \text{if } t = j > 2l + 1 - s \\
b_{ts} & \text{if } t > j > 2l + 1 - s \\
b_{t,2l+1-t} & \text{if } t > j > 2l + 1 - s 
\end{cases} \]

3. \( \{ b_{ij} : 1 \leq i \leq l, 2l + 1 - i \leq j \leq 2l \} \)
4. \( \{ 0 \} \)
5. \( \int_G |f|^2 = (2\pi)^{-l(l+1)/2} \int_{W} \text{tr} [\pi_{sy}(f * f^*)] \prod_{k=1}^{l-1} |y_{k,2l+1-k}|^{l-k} \)

\( s : R^l \rightarrow n', s(y_{1,2l}, \ldots, y_{l,l+1}) = \sum_{k=1}^{l} y_{k,2l+1-k}e^{k,2l+1-k} \)

\( W = \{ y = (y_{1,2l}, \ldots, y_{l,l+1}) : \Pi_{k=1}^{l-1} |y_{k,2l+1-k}| \neq 0 \} \)

\( \pi_{sy} = \text{ind}_N^G \chi_{sy}, \)

\( \chi_{sy}(\exp (\Sigma_{1<i<l; 2l+1-i<j<2l} x_{ij}^k e_{ij})) = e^{i \sum_{k=1}^{l} y_{k,2l+1-k}e^{k,2l+1-k}} \)

BIBLIOGRAPHY


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