THE PRIMITIVE LIFTING PROBLEM IN THE EQUIVALENCE PROBLEM FOR TRANSITIVE PSEUDOGROUP STRUCTURES: A COUNTEREXAMPLE

BY

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ABSTRACT. A transitive Lie pseudogroup \( \Gamma_M \) on \( M \) is a primitive extension of \( \Gamma_N \) if \( \Gamma_N \) is the quotient of \( \Gamma_M \) by an invariant fibration \( \pi: M \rightarrow N \) and if the pseudogroup induced by \( \Gamma_M \) on the fiber of \( \pi \) is primitive. In the present paper an example of this situation is given with the following property (counterexample to the primitive lifting property): the equivalence theorem is true for almost-\( \Gamma_N \)-structures but false for almost-\( \Gamma_M \)-structures.

1. We shall consider the following situation: Let \( \pi: M \rightarrow N \) be a fibration of smooth manifolds, \( \Gamma_M \) a transitive Lie pseudogroup on \( M \) respecting the fibration \( \pi \). It is not always true that a quotient pseudogroup on \( N \) can be defined, but by projecting the equations of \( \Gamma_M \) (that is to say the associated structures) we obtain the equations of a transitive Lie pseudogroup \( \Gamma_N \) on \( N \). \( \Gamma_N \) will be referred to as the pseudogroup defined by passing to the quotient. In the study of the equivalence problem for almost-\( \Gamma_M \)-structures (see [1], [3], [4]), it is a standard method to use such quotients. If the equivalence theorem is true for almost-\( \Gamma_N \)-structures, any almost-\( \Gamma_M \)-structure defines a quotient almost-\( \Gamma_N \)-structure, and the given equivalence problem reduces to a “lifting problem” from an equivalence for the quotient structure to an equivalence for the given structure.

Let \( K \) be a fiber of \( \pi \), \( \Gamma_K \) the family of restrictions to \( K \) of those transformations in \( \Gamma_M \) which map \( K \) into \( K' \). We assume here that \( \Gamma_K \) is a flat irreducible transitive Lie pseudogroup on \( K \). Then \( \Gamma_K \) is either an affine pseudogroup (\( \Gamma_M \) is an “affine extension” of \( \Gamma_N \)) or a primitive pseudogroup (\( \Gamma_M \) is a “primitive extension” of \( \Gamma_N \)).

The purpose of this paper is to give an example where \( \Gamma_M \) is a primitive extension of \( \Gamma_N \) and the equivalence theorem is true for \( \Gamma_N \) but false for \( \Gamma_M \). In the terminology of A. Pollack (see [4]), the “primitive lifting theorem for \( \Gamma_M \rightarrow \Gamma_N \)” is not true.

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In the paper just mentioned, A. Pollack asserts that, for $\Gamma_M$ and $\Gamma_N$ flat, "the primitive lifting theorem can be established with little difficulty", while "the affine lifting theorem requires deep results from partial differential equation theory" (essentially Ehrenpreis-Malgrange's theorem on partial differential equations with constant coefficients).

As a matter of fact, if $\Gamma_K$ is a primitive simple pseudogroup, the lifting theorem can be easily proved by using the results of V. Guillemin (see [2]).

If $\Gamma_K$ is primitive but not simple, the lifting theorem is true in the flat case ($\Gamma_M$ and $\Gamma_N$ flat) but the demonstration requires, as in the affine case, Ehrenpreis-Malgrange's results. The method has been indicated in [1].

2. Construction of the counterexample. The counterexample is constructed by a refinement of the Lewy-Guillemin-Sternberg counterexample to the equivalence problem for $G$-structures [3].

(a) First, recall the data of this counterexample: We consider on $\mathbb{R}^3$ a transitive algebra of vector fields isomorphic to $g = SO(3, \mathbb{R})$. $\{X_1, X_2, X_3\}$ is the standard basis of this algebra, that is to say, $X_1, X_2, X_3$ are globally defined vector fields on $\mathbb{R}^3$ satisfying the following relations:

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = X_1, \quad [X_2, X_3] = -X_2.$$ 

Let $\{Y_1, Y_2\}$ be the standard basis of vector fields on $\mathbb{R}^2$. $E$ is the $G$-structure on $\mathbb{R}^2 \times \mathbb{R}^3$ defined by the moving frame $\{Y_1, Y_2, X_1, X_2, X_3\}$, where $\hat{G}$ is the group of matrices of the form

$$\begin{bmatrix}
1 & 0 & a & b & c \\
0 & 1 & d & e & f \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}.$$ 

$E$ is the $G$-structure on $\mathbb{R}^2 \times \mathbb{R}^3$ defined by the same moving frame, where $G$ is the group of matrices of the form

$$\begin{bmatrix}
1 & 0 & a & b & c \\
0 & 1 & -b & a & d \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}.$$
We denote by \( \hat{\gamma} \) (resp. \( \gamma \)) the pseudogroup of automorphisms of \( \hat{E} \) (resp. \( E \)) and by \( \hat{\mathcal{I}} \) (resp. \( \mathcal{I} \)) the associated Lie algebra sheaf, in the terminology of [5].

\( \hat{\mathcal{I}} \) is a Lie algebra sheaf of vector fields of the following type:

\[
X = \xi + \phi_1(x_1, x_2, x_3)Y_1 + \phi_2(x_1, x_2, x_3)Y_2
\]

where \( \xi \) is in the algebra of (local) right-invariant vectorfields on \( R^3 \subset SO(3, R) \).

\( X \) is in \( \mathcal{I} \) if and only if \( \phi_1, \phi_2 \) satisfy a system \( (\Sigma) \) of linear partial differential equations.

(b) Let us consider now, for \( p \geq 2 \), the product \( C^p \times R^3 \) endowed with the \( \hat{H} \)-structure \( \tilde{E} \) obtained from the moving frame \( \{ \partial/\partial z_1, \ldots, \partial/\partial z_p, X_1, X_2, X_3 \} \) by linear transformations in the group \( \hat{H} \) of matrices of the following type:

\[
\begin{bmatrix}
\alpha_{ij} & xx & \cdots & xx \\
0 & 1 & 0 & 0 \\
\cdots & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad \text{with } [\alpha_{ij}] \in GL(p, C).
\]

Let \( \tilde{\gamma} \) be the pseudogroup of automorphisms of this structure, \( \tilde{\mathcal{L}} \) the associated \( LAS \). We consider the projection \( \tilde{\pi} : \tilde{E} \rightarrow R^2 \times R^3 \) defined by:

\[
\tilde{\pi}(\tilde{z}(z, x)) = (\det(\alpha_{ij}), x) \quad \text{where } \tilde{z} = \left\{ \sum_i \alpha_{ij} \frac{\partial}{\partial z_i}, \tilde{X}_1(x), \tilde{X}_2(x), \tilde{X}_3(x) \right\}.
\]

If \( \tilde{Y}_1 = \sum_{k=1}^{p} z_k \partial/\partial z_k \) and \( \tilde{Y}_2 = \sum_{k=1}^{p} i z_k \partial/\partial z_k \), let \( \hat{\mathcal{L}} \) be the \( LAS \) of vector fields on \( C^p \times R^3 \) of the following type:

\[
(2) \quad \hat{X} = \sum_{k=1}^{p} X_k(x, z) \frac{\partial}{\partial z_k} + \varphi_1(x_1, x_2, x_3)\tilde{Y}_1 + \varphi_2(x_1, x_2, x_3)\tilde{Y}_2 + \xi
\]

where \( X_k \) is holomorphic with respect to \( z \), with the sole condition \( \Sigma_k \partial X_k/\partial z_k = 0 \).

\( \hat{\mathcal{L}} \) defines a transitive Lie pseudogroup \( \hat{\Gamma} \) on \( C^p \times R^3 \), with \( \hat{\Gamma} \subset \tilde{\Gamma} \). By the projection \( \tilde{\pi} \), \( \hat{\Gamma} \) acting on \( \tilde{E} \) projects onto \( \hat{\gamma} \). It is in this case a true quotient!

The vector field (2) projects onto (1).

Let \( \Gamma \) be the preimage of \( \gamma \) by this projection \( \hat{\Gamma} \rightarrow \hat{\gamma} \). The associated \( LAS \ \mathcal{L} \) is the \( LAS \) of vector fields of the type (2) with the condition \( (\Sigma) \) on \( \varphi_1, \varphi_2 \).

The fibration \( \pi : C^p \times R^3 \rightarrow R^3 \) allows us to consider \( \Gamma \) as a primitive extension of \( SO(3, R) \). The counterexample follows now from the following proposition:
Proposition. The equivalence theorem is not true for the pseudogroup \( \Gamma \).

Proof. Let \( \hat{E}^1 \) be the second order structure associated to \( \hat{T} \). We have a natural projection \( \hat{\pi}^1 : \hat{E}^1 \to \hat{E} \). The second order structure \( E^1 \) associated to \( \Gamma \) is the preimage of \( E \) by \( \hat{\pi}^1 \).

If \( E' \) is an arbitrary \( G \)-subbundle of \( \hat{E} \), let \( E^{1'} = [\hat{\pi}^1]^{-1}(E') \). By [3], \( E^{1'} \) defines an almost-\( \gamma \)-structure on \( \mathbb{R}^2 \times \mathbb{R}^3 \). Then \( E^{1'} \) will define an almost-\( \Gamma \)-structure on \( \mathbb{C}^p \times \mathbb{R}^3 \). If \( E' \) is not a \( \gamma \)-structure, \( E^{1'} \) does not define a \( \Gamma \)-structure. Q. E. D.

3. The primitive lifting theorem in the flat case. Recall the principle of the demonstration in the flat case (following the method indicated in [1]): If \( \Gamma_K \) is not simple, we consider the subpseudogroup \( \Gamma_{K'} \), the LAS of which is the derived algebra of the LAS of \( \Gamma_K \). Then we can obtain a \( \Gamma_K \)-extension \( \Gamma'_M \) of \( \Gamma_N \), with \( \Gamma_M' \subset \Gamma_M \).

In the first order structure associated to \( \Gamma_M' \), we have an invariant foliation such that the first order structure associated to \( \Gamma_{M'} \) is foliated. If \( \pi^1 \) is the (local) projection onto a (local) quotient, we resolve the equivalence problem for the pseudogroup defined by passing to the quotient. This allows us to obtain for any almost-\( \Gamma_M \)-structure a subordinate almost-\( \Gamma_M' \)-structure. Now, the primitive lifting theorem reduces to the simple case, which is elementary.

But the equivalence problem for the pseudogroup obtained by passing to the quotient by \( \pi^1 \) requires Ehrenpreis-Malgrange’s theorem as an essential tool: in fact it is equivalent to the equivalence problem for a flat abelian pseudogroup.

BIBLIOGRAPHY


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