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# NONIMMERSION OF LENS SPACES WITH 2-TORSION

BY

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ABSTRACT. From a study of the equivariant unitary $K$-theory of the Stiefel manifold $V_{k+1,2}(\mathbf{C})$, it is shown that the lens space $L^k(n)$, with $n$ a multiple of $2^{2k-1-\alpha(k-1)}$, does not immerse in Euclidean space of dimension $4k - 2\alpha(k) - 2$.

1. **Introduction.** Considerable effort has been devoted to the problem of finding the minimum-dimensional Euclidean space $\mathbf{R}^m$ in which one can immerse the lens space $L^k(n)$ (the quotient of $S^{2k+1}$ by the free action of the cyclic group of order $n$, whose generator $\zeta$ acts as $(z_0, \ldots, z_k) \mapsto (e^{2\pi i/n}z_0, \ldots, e^{2\pi i/n}z_k)$). Although the case where $n$ is odd has met with the most success [9], [10], [12], [13], more attention has been focussed on $n = 2$ [5]. In general, only the cases $\nu_2(n) \leqslant 1$ have previously been discussed (where $\nu_2(n)$ is the exponent of the highest power of 2 dividing $n$). [14, Theorem 5] shows that if $m \geqslant 3k + 2$, then $m$ is a function solely of $k$ and $\nu_2(n)$. We prove here the following. (Let $\alpha(k)$ be the number of nonzero terms in the dyadic expansion of $k$.)

THEOREM. *If $\nu_2(n) \geqslant 2k - 1 - \alpha(k-1)$, then $L^k(n)$ does not immerse in $\mathbf{R}^{4k-2\alpha(k)-2}$.*

In view of the subsequent retraction of the announcement [4], this result may be seen as the nearest approach to date to a strong general nonimmersion result for real projective spaces. It is interesting to compare the above result with [8], which implies the above when $\alpha(k+1) = 1$. Apparently Professors D. M. Davis and M. Mahowald have proved (unpublished) that for $2 \leqslant \alpha(k) \leqslant 8$, $\mathbf{C}P^k$ immerses in $\mathbf{R}^{4k-2\alpha(k)+1+\epsilon_k}$ (so that $L^k(n)$ immerses in $\mathbf{R}^{4k-2\alpha(k)+2+\epsilon_k}$), where $\epsilon_k$ is nonzero only if $k$ is even and $\alpha(k) = 2, 6, 8$ (when $\epsilon_k = 1$) or $\alpha(k) = 3, 7$ (when $\epsilon_k = 2$). In addition, if $k \equiv 3 \pmod 4$, $\alpha(k) = 5$, $k \neq 31$, then $L^k(2)$ immerses in $\mathbf{R}^{4k-12}$, whereas the above theorem implies that $L^k(2^{2k-5})$ does not immerse in $\mathbf{R}^{4k-12}$.

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The chief novelty of the present work is its derivation of nonimmersion results by equivariant $K$-theory. The Gysin sequence of [1] is used in §3 below to derive the $K$-theory of a certain quotient space of the complex Stiefel manifold $V_{k+1,2}(\mathbf{C})$, whose relation to the problem is discussed in §2. Calculations yielding the theorem are performed in the final section.

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2. **The projective tangent bundle.** Let $L$ be the canonical real line bundle over the projective tangent bundle $P\tau M$ of a smooth $r$-dimensional manifold $M$. We begin with a reformulation of [6, Theorem 4.2].

LEMMA 2.1. *There is a function from the set of regular homotopy classes of immersions of $M$ in $\mathbf{R}^m$ to the set of homotopy classes of (never-zero) cross-sections of the real $m$-plane bundle $mL$, which is surjective if $2m \geqslant 3r + 1$, bijective if $2m \geqslant 3r + 2$.*

Here $M = L^k(n)$, $r = 2k + 1$. We pass to a subspace of $P\tau M$ given as follows. Let $G$ denote the direct product of the cyclic group of order $n$, generator $\zeta$, with the group of order 2, generator $\eta$. $\zeta$ acts on $V_{k+1} = V_{k+1,2}(\mathbf{C})$ by sending the ordered pair $(x, y)$, $x \perp y$, to $(e^{2\pi i/n}x, e^{2\pi i/n}y)$, while $\eta$ sends $(x, y)$ to $(x, -y)$. The quotient of $V_{k+1,2}(\mathbf{C})$ by this $G$-action is clearly a subspace of $P\tau L^k(n)$. However, calculations are simplified if a slightly different $G$-action is considered, viz. that where $\eta$ instead maps $(x, y)$ to $(y, x)$. The homeomorphism $(x, y) \mapsto ((x + y)/\sqrt{2}, (y - x)/\sqrt{2})$, from the quotient of $V_{k+1}$ by the latter action to the quotient by the former, pulls back the bundle $L$ above to the canonical line bundle over $V_{k+1}/G$ whose complexification we call $\beta$. (The latter $G$-action is used exclusively henceforth.) So the total space of $\beta$ is given by

$$\beta = \{(x, y, t) \in S^{2k+1} \times S^{2k+1} \times \mathbf{C} \mid x \perp y;$$
$$(x, y, t) = (e^{2\pi i/n}x, e^{2\pi i/n}y, t) = (y, x, -t)\}.$$

The above lemma now has the following consequence (cf. [3, (2.1)c]).

LEMMA 2.2. *If $L^k(n)$ immerses in $\mathbf{R}^{2m}$, then the complex $m$-plane bundle $m\beta$ over $V_{k+1}/G$ admits a (never-zero) cross-section.*

3. **Some topology of the complex Stiefel manifold.** In this section the $G$-space $V_{k+1}$ is described in terms of the $G$-spaces $SG_{k+1} = SG_{k+1,2}(\mathbf{C})$, the oriented Grassmannian (of oriented complex 2-planes in $\mathbf{C}^{k+1}$), and $G_{k+1} = G_{k+1,2}(\mathbf{C})$, the (unoriented) Grassmannian. The $G$-action on these latter spaces is such that the projections $f\colon V_{k+1} \longrightarrow SG_{k+1}$ and $g\colon SG_{k+1} \longrightarrow G_{k+1}$ are

$G$-equivariant. In particular, $G$ acts trivially on $G_{k+1}$. Since

$$V_{k+1} = U(k+1)/U(k-1),$$
$$SG_{k+1} = U(k+1)/(U(k-1) \times SU(2)),$$

and

$$G_{k+1} = U(k+1)/(U(k-1) \times U(2)),$$

it follows that $f$ is the projection map of a principal $S^3$-bundle while $g$ is the projection map of a principal $S^1$-bundle. We show now that these bundles are both the sphere bundles of complex $G$-vector bundles which we determine in terms of the canonical complex 2-plane bundle $\gamma$ over $G_{k+1}$.

The unitary representation ring $R(G)$ may be written as

$$R(G) = \mathbf{Z}[\alpha, \beta]/\langle \alpha^n - 1, \beta^2 - 1 \rangle$$

where the representations $\alpha, \beta$ are given by

$$\alpha: \zeta \longmapsto e^{2\pi i/n}, \eta \longmapsto 1;$$
$$\beta: \zeta \longmapsto 1, \eta \longmapsto -1.$$

Then the homomorphism $R(G) \longrightarrow K_G(V_{k+1}) = K(V_{k+1}/G)$, induced from $V_{k+1} \longrightarrow \mathrm{pt}$, sends $\beta$ to its namesake introduced in §2; or rather, to $f^*g^*\beta$ where $R(G)$ is a direct summand of $K_G(G_{k+1}) = K(G_{k+1}) \otimes R(G)$. (We shall be less pedantic when calculations are under way in §4.)

**Lemma 3.1.** *$SG_{k+1}$ is homeomorphic as a G-space to the sphere bundle of the complex G-line bundle*

$$\lambda^2(\gamma) \otimes \alpha^2\beta \in K_G(G_{k+1}).$$

**Proof.** The homeomorphism $h: SG_{k+1} \longrightarrow S(\lambda^2(\gamma) \otimes \alpha^2\beta)$ is given by mapping the oriented plane generated by $(x, y) \in V_{k+1}$ to the pair $([x, y], x \wedge y)$, $[x, y]$ being the unoriented plane which $(x, y)$ generates. Since $x \wedge y$ has isotropy group $SU(2)$ under the $U(2)$-action on the pair $(x, y)$, $h$ is well defined and 1-1. Further, $h$ is clearly onto, and $G$-equivariance is easily checked. □

**Lemma 3.2.** *$V_{k+1}$ is homeomorphic as a G-space to the sphere bundle of the complex G-vector bundle*

$$g^*(\gamma \otimes \alpha) \in K_G(SG_{k+1}).$$

**Proof.**

$$Sg^*(\gamma \otimes \alpha) = \{((x, y), z) \in V_{k+1,2}(\mathbf{C}) \times S^{2k+1} \mid z \in [x, y];$$
$$((x, y), z) = (P(x, y), z)\ \forall P \in SU(2)\},$$

with $G$-action given by

$$\zeta: ((x, y), z) \longmapsto ((e^{2\pi i/n}x, e^{2\pi i/n}y), e^{2\pi i/n}z),$$

$$\eta: ((x, y), z) \longmapsto ((y, x), z).$$

Inverse $G$-homeomorphisms $\varphi: V_{k+1} \longrightarrow Sg^*(\gamma \otimes \alpha)$ and $\psi: Sg^*(\gamma \otimes \alpha) \longrightarrow V_{k+1}$ are given by

$$\varphi: (x, y) \longmapsto ((x, y), (x + y)/\sqrt{2}), \qquad \psi: ((x, y), ax + by) \longmapsto (M_{(a,b)}(x, y)),$$

where, for $(a, b) \in S\mathbf{C}^2$,

$$M_{(a,b)} = \frac{1}{\sqrt{2}} \begin{bmatrix} a + \bar{b} & b - \bar{a} \\ a - \bar{b} & \bar{a} + b \end{bmatrix}.$$

The map $\psi$ is well defined, since if $P \in SU(2)$ then $M_{P(a,b)} = M_{(a,b)}P^T$. $M_{(1/\sqrt{2},1/\sqrt{2})} = I$ implies that $\psi \circ \varphi = \mathrm{id}$, while $M^T_{(a,b)}(1/\sqrt{2}, 1/\sqrt{2}) = (a, b)$ implies that $\varphi \circ \psi = \mathrm{id}$. Again, $G$-equivariance may be readily checked. □

4. **Calculations in $K$-theory.** It is well known (e.g. [1]) that if $m\beta$ admits a section over $V_{k+1}/G$, then the "Thom class" $\lambda_{-1}[m\beta] = \Sigma_{i=0}^m(-1)^i\lambda^i(m\beta)$ vanishes in $K_G(V_{k+1})$. Conversely, if $\lambda_{-1}[m\beta] = 0$ then it follows from [2] that $K_G^*(S\beta)$ and $K_G^*(V_{k+1} \times S^{2m-1})$ are isomorphic as $K_G^*(V_{k+1})$-algebras. We therefore determine this obstruction.

Hoggar [7] has adapted [1] to present $K(G_{k+1})$ in a form amenable to calculation. Letting $\gamma$ be the canonical bundle as in §3, write

$$x = 2 - \gamma, \qquad y = \gamma - \lambda^2(\gamma) - 1 \in K(G_{k+1}),$$

and also in this ring set

$$v_0 = 1, \qquad v_1 = x, \qquad v_t = xv_{t-1} + yv_{t-2} = \sum_{s \geq 0} \binom{t-s}{s} x^{t-2s}y^s, \qquad 2 \leq t \leq k + 1.$$

If a weight function on monomials $x^ry^s$ be defined by $\mathrm{wt}(x^ry^s) = r + 2s$, then all the monomials in the sum $v_t$ have the same weight, viz. $t$; so $v_t$ is homogeneous.

LEMMA 4.1 [7].

$$K(G_{k+1}) = \mathbf{Z}[x, y]/\langle v_k, v_{k+1}\rangle$$
$$= \mathbf{Z}[x, y]/\langle y^{k-t}v_t, 0 \leq t \leq k\rangle,$$

*a torsion-free ring in which all monomials of weight exceeding* $2k - 2$ *vanish.*

Now for two purely combinatorial preliminaries to the proof of the theorem.

LEMMA 4.2. *The ring* $\mathbf{Z}[x, y]/\langle v_k, v_{k+1}, x-2\rangle$ *has torsion* $2^{2k-\alpha(k)}$.

PROOF. By (4.1), $y^{k-1} \neq 0$ whereas $2y^{k-1} = 0$. We show by induction on $r$ that

$$2^{2r-\alpha(r)-1}y^{k-r} = y^{k-1}, \qquad 1 \leqslant r \leqslant k.$$

Accordingly suppose that the result holds for $1 \leqslant r \leqslant t-1 \leqslant k-1$, the case $r = 1$ being trivial.

Observe that

$$\begin{aligned}\nu_2\left(\binom{t-i}{i}\right) &+ 2(t-i) - \alpha(t) - 1 \\ &= \alpha(i) + \alpha(t-2i) - \alpha(t-i) - \alpha(t) + 2(t-i) - 1 \\ &= \nu_2\left(\binom{t}{2i}\right) + (2(t-i) - \alpha(t-i) - 1).\end{aligned}$$

When this is combined with the induction hypotheses and inserted in the expansion of $2^{t-\alpha(t)-1}y^{k-t}v_t = 0$ (4.1), the following occurs. (Since $2y^{k-1} = 0$, the calculations are effectively modulo 2.)

$$\begin{aligned}2^{2t-\alpha(t)-1}y^{k-t} &= \sum_{i\geqslant 1}\binom{t-i}{i}2^{2(t-i)-\alpha(t)-1}y^{k-t+i} \\ &= \sum_{i\geqslant 1}\binom{t}{2i}2^{2(t-i)-\alpha(t-i)-1}y^{k-(t-i)} \\ &= \sum_{i\geqslant 1}\binom{t}{2i}y^{k-1} \\ &= (2^{t-1}-1)y^{k-1} \\ &= y^{k-1},\end{aligned}$$

since $t \geqslant 2$. □

LEMMA 4.3. *If* $\nu_2(r) \geqslant 2k - 2 - \alpha(k-1)$, *then the ring*

$$\mathbf{Z}[x, y]/\langle v_k, v_{k+1}, x-2, (1+y)^r - 1\rangle$$

*has torsion* $2^{2k-\alpha(k)}$.

PROOF. By (4.2) it suffices to show that in the expansion

$$(1+y)^r - 1 = \sum_{s=1}^{k-1}\binom{r}{s}y^s,$$

$\nu_2(\binom{r}{s}) \geqslant 2(k-s) - \alpha(k-s)$ for each $s$. We first evaluate the left-hand side of this inequality. From the expansion

$$\binom{r}{s} = r(r-1)\cdots(r-s+1)/s(s-1)\cdots 1,$$

$$\nu_2\left(\binom{r}{s}\right) = \nu_2(r) - \nu_2(s) + \sum_{t=1}^{s-1} (\nu_2(r-t) - \nu_2(t)).$$

However, $t < s < k$ implies $\nu_2(t) < \nu_2(r)$ and thence $\nu_2(r-t) = \nu_2(t)$. So $\nu_2(\binom{r}{s}) = \nu_2(r) - \nu_2(s)$ and it remains to show that

$$(2k - 2 - \alpha(k-1)) - \nu_2(s) \geqslant 2(k-s) - \alpha(k-s).$$

But the difference between these two expressions is just

$$\begin{aligned} &2s - 2 + \alpha(k-s) - \alpha(k-1) - \nu_2(s) \\ &= 2s - 2 + \alpha(k-s) - \alpha(k-1) - \alpha(s-1) + \alpha(s) - 1 \\ &= 2[(s-1) - \alpha(s-1)] + \nu_2\left(\binom{k-1}{s-1}\right) + (\alpha(s) - 1), \end{aligned}$$

a sum of three nonnegative terms. □

Proof of the theorem. We show that if $m \leqslant 2k - \alpha(k) - 1$, then $\lambda_{-1}[m\beta] = 2^{m-1}(1-\beta)$ is nonzero in $f^*g^*K_G(G_{k+1})$. It follows from the exact Gysin sequences

$$K_G(G_{k+1}) \xrightarrow{.\lambda_{-1}} K_G(G_{k+1}) \xrightarrow{g^*} K_G(SG_{k+1}),$$

$$K_G(SG_{k+1}) \xrightarrow{.\lambda_{-1}} K_G(SG_{k+1}) \xrightarrow{f^*} K_G(V_{k+1}),$$

that the ring $f^*g^*K_G(G_{k+1})$ may be obtained by factoring out

$$K_G(G_{k+1}) = \mathbf{Z}[x, y]/\langle v_k, v_{k+1}\rangle \otimes \mathbf{Z}[\alpha, \beta]/\langle \alpha^n - 1, \beta^2 - 1\rangle$$

by the principal ideals generated by

$$\lambda_{-1}[\lambda^2(\gamma) \otimes \alpha^2\beta] = 1 - (1 - x - y)\alpha^2\beta \quad (3.1),$$

and

$$\lambda_{-1}[\gamma \otimes \alpha] = 1 - (2 - x)\alpha + (1 - x - y)\alpha^2 \quad (3.2).$$

However, since our interest lies in the torsion of $(1-\beta)$, we factor out further by the ideal generated by $(1+\beta)$. It suffices to show that $2^m \neq 0$ in this last ring, namely

$$\mathbf{Z}[x, y, \alpha]/\langle v_k, v_{k+1}, \alpha^n - 1, x - 2, 1 - (1+y)\alpha^2\rangle$$

$$= \mathbf{Z}[x, y]/\langle v_k, v_{k+1}, x - 2, (1+y)^{n/2} - 1\rangle.$$

(4.3) now clinches the proof. □

It can be shown that $\lambda_{-1}[(2k - \alpha(k))\beta] = 0$ in $f^*g^*K_G(G_{k+1})$, although the proof is too long to present here.

By applying (2.1) to complex projective space $CP^k$ immersed in $\mathbf{R}^m$ and arguing as for (2.2), one infers the existence of a section to the canonical bundle over $V_{k+1}/H$ ($H$ being a semidirect product of $S^1 \times S^1$ by $\mathbf{Z}_2$) whose pull-back to $V_{k+1}/G$ is $m\beta$, and thence a section to $m\beta$ itself. So our theorem has the following (known) consequence [11].

COROLLARY 4.4. *$CP^k$ does not immerse in* $\mathbf{R}^{4k-2\alpha(k)-2}$.

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