TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 224, Number 2, 1976

THE DEGREE OF APPROXIMATION FOR GENERALIZED POLYNOMIALS WITH INTEGRAL COEFFICIENTS

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ABSTRACT. The classical Müntz theorem and the so-called Jackson-Müntz theorems concern uniform approximation on $[0, 1]$ by polynomials whose exponents are taken from an increasing sequence of positive real numbers Λ. Under mild restrictions on the exponents, the degree of approximation for Λ-polynomials with real coefficients is compared with the corresponding degree of approximation when the coefficients are taken from the integers.

Let $C[0, 1]$ be the space of all continuous real valued functions defined on the interval $[0, 1]$ and $\|\cdot\|$ the supremum norm on $[0, 1]$ ($\|f\| = \sup\{|f(x)|: 0 \leqslant x \leqslant 1\}$). It is well known that the ordinary algebraic polynomials with integral coefficients, i.e. integral polynomials, are dense in the subspace

$$C_0[0, 1] = \{f \in C[0, 1]: f(0) = f(1) = 0\}.$$

This seems to be due originally to Kakeya [10], but many other authors have also studied this or related problems: Pál [17], Okada [16], Bernstein [2], Fekete [3]. Finally, Hewitt and Zuckerman [9] obtained necessary and sufficient conditions. With every closed real interval of length less than 4, they associate a certain finite subset J. A continuous real function f on the interval is arbitrarily uniformly approximable by integral polynomials if and only if f is equal to some integral polynomial on the set J.

In 1931, Kantorovič [11] proved that for any positive integer n and any function $f \in C_0[0, 1]$ there exists an integral polynomial $p_n(x) = \Sigma_{k=0}^n b_k x^k$ such that

$$\|f - p_n\| \leqslant 2E_n(f) + O(n^{-1}) \quad \text{for } n \to \infty \tag{1}$$

holds, where

Received by the editors February 19, 1975 and, in revised form, November 4, 1975.

AMS (MOS) subject classifications (1970). Primary 41A25, 41A10.

Key words and phrases. Jackson-Müntz theorem, polynomials with integral coefficients, approximation by polynomials with integral coefficients, degree of approximation.

[1] The Deutsche Forschungsgemeinschaft has sponsored this research under Grant No. Go 270/1.

$$E_n(f) = \inf \left\{ \left\| f(x) - \sum_{k=0}^{n} a_k x^k \right\| : a_k \text{ real} \right\}.$$

Gel′fond [6] and Trigub [18] extended this result to differentiable functions f for the intervals $[0, 1]$ and $[a, b]$, $b - a < 4$, respectively, and obtained analogues of Jackson's and Timan's theorems.

Many theorems, which are well known for ordinary polynomials, are also valid for the so-called Λ-polynomials of the form

$$P_s(x) = \sum_{k=1}^{s} a_k x^{\lambda_k}, \qquad a_k \text{ real},$$

where $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ is a positive increasing sequence of real numbers. Müntz [14] proved that the Λ-polynomials are dense in the subset $\{f \in C[0, 1] : f(0) = 0\}$ of $C[0, 1]$ if and only if $\Sigma_{k=1}^{\infty} 1/\lambda_k = \infty$.

Recently, Le Baron O. Ferguson and von Golitschek [4] showed that for every sequence Λ of distinct positive integers the Λ-polynomials with integral coefficients are dense in $C_0[0, 1]$ if and only if Müntz's condition holds. This result is even valid for every sequence Λ of distinct positive real numbers.

Combining Müntz's and Jackson's theorems, Newman [15], von Golitschek [7], [8], Ganelius and Westlund [5], Leviatan [12], Bak and Newman [1] obtained upper and lower bounds for the degree of approximation when functions f are approximated by the Λ-polynomials with real coefficients.

The purpose of this paper is to find analogous Jackson-Müntz theorems for Λ-polynomials with integral coefficients. More precisely, we shall prove the following two theorems which generalize Kantorovič's result (1).

THEOREM 1. *Let the positive increasing sequence Λ satisfy*

$$\lambda_1 \geq B, \qquad \lambda_{2k} \leq C\lambda_k, \qquad \lambda_k \leq Bk \quad \text{for } k \geq k_0, \tag{2}$$

where k_0 is a positive integer and B and C are positive constants. For any function $f \in C_0[0, 1]$ and any positive integer s there exist integers b_j, $1 \leq j \leq s$, such that

$$\left\| f(x) - \sum_{j=1}^{s} b_j x^{\lambda_j} \right\| \leq 2E_s(f; \Lambda) + O(s^{-1}), \tag{3}$$

where.

$$E_s(f;\Lambda) = \inf \left\{ \left\| f(x) - \sum_{k=1}^{s} a_k x^{\lambda_k} \right\| : a_k \textit{ real} \right\}.$$

THEOREM 2. *Let the positive increasing sequence Λ satisfy*

(4) $$\lambda_{2k} \leqslant C\lambda_k, \quad \lambda_k \geqslant Bk \quad \textit{for } k \geqslant 1.$$

For any function $f \in C_0[0, 1]$ *and any positive integer* s *there exist integers* b_j, $1 \leqslant j \leqslant s$, *such that*

(5) $$\left\| f(x) - \sum_{j=1}^{s} b_j x^{\lambda_j} \right\| \leqslant 2E_s(f; \Lambda) + O(\varphi(s)^{-B}),$$

where $\varphi(s) = \exp(\Sigma_{k=1}^{s} 1/\lambda_k)$.

REMARK 1. The restrictions $\lambda_{2k} \leqslant C\lambda_k$ ($k \geqslant k_0$ or $k \geqslant 1$) in Theorem 1 and Theorem 2 are mild. Indeed, many different sequences have this property, for example

$$\lambda_k = k^\beta \quad (k \geqslant 1), \quad \beta > 0,$$

$$\lambda_k = k \log k \quad (k > 1),$$

and even converging sequences Λ with $\lim_{k\to\infty} \lambda_k = \lambda$, $\lambda > 0$.

REMARK 2. It follows from the theory of width (cf. Lorentz [13, Chapter 9]) that the classes Γ_w of functions,

$$\Gamma_w = \{f \in C[0, 1] : w(f; h) \leqslant w(h) \text{ for } 0 \leqslant h \leqslant 1\},$$

where w is a given modulus of continuity, have the following property. There exists a positive number c not depending on s such that

(6) $$\sup_{f \in \Gamma_w} E_s(f; \Lambda) \geqslant cw(s^{-1}).$$

It is easy to see that the classes $\Gamma_{w0} = \Gamma_w \cap C_0[0, 1]$ satisfy (6). Therefore the summand $O(s^{-1})$ in (3) of Theorem 1 does not change the rate of convergence if we consider the whole class Γ_{w0}.

Combining Theorem 2 and the Jackson-Müntz theorem [8, Theorem 3] for Λ-polynomials with real coefficients, we are led to the following.

COROLLARY. *If* (4) *holds, then for any function* $f \in C_0[0, 1]$ *and any positive integer* s *there exist integers* b_j, $1 \leqslant j \leqslant s$, *such that*

(7) $$\left\| f(x) - \sum_{j=1}^{s} b_j x^{\lambda_j} \right\| = O(w(f; \varphi(s)^{-B^*}))$$

where $B^* = \min\{B; 2\}$ *and* $w(f; h) = \sup\{|f(x + t) - f(x)| : |t| \leqslant h, x, x + t \in [0, 1], 0 \leqslant h \leqslant 1\}$, *denotes the modulus of continuity of* f.

REMARK 3. For $B \geqslant 2$ the rate of convergence in (7) is best possible for classes Γ_{w0}, even if we approximate by Λ-polynomials with real coefficients (cf. Bak and Newman [1]).

Proofs of Theorem 1 and Theorem 2. By [4, Lemma 1] there exists for any positive integers q and s, $q < s$, a Λ-polynomial $Q_{qs}(x) = \sum_{i=q+1}^{s} c_{iqs} x^{\lambda_i}$ such that

$$A_{qs} = \|x^{\lambda_q} - Q_{qs}(x)\| \leq 2 \exp\left(-2\lambda_q \sum_{i=q+1}^{s} 1/\lambda_i\right) \tag{8}$$

and $Q_{qs}(1) = 1$, where the first equality in (8) serves to define A_{qs}.

LEMMA 1. *Let r and s be positive integers such that $r \leq s + 1 - C \log s < s$.* (a) *If the assumptions of Theorem* 1 *hold then*

$$\sum_{q=1}^{r} A_{qs} = O(s^{-1}) \quad \text{for } s \longrightarrow \infty. \tag{9}$$

(b) *If the assumptions of Theorem* 2 *hold then*

$$\sum_{q=1}^{r} A_{qs} = O(\varphi(s)^{-B}) \quad \text{for } s \longrightarrow \infty. \tag{10}$$

PROOF OF LEMMA 1. Let s be so large that $s/2 > C \log s > k_0$.

(a) We apply the inequalities (2) and (8). Then, for $1 \leq q \leq C \log s$ and $q_0 = \max\{q + 1; k_0\}$,

$$A_{qs} \leq 2 \exp\left(-2\lambda_q \sum_{i=q_0}^{s} 1/(Bi)\right) \leq 2(q_0/s)^{2\lambda_q/B} \leq 2(q_0/s)^2; \tag{11}$$

for $C \log s < q \leq s/2$,

$$A_{qs} \leq 2 \exp\left(-2\lambda_q \sum_{i=q+1}^{2q} 1/\lambda_i\right) \leq 2e^{-2q/C} \leq 2s^{-2}, \tag{12}$$

and for $s/2 < q \leq s + 1 - C \log s$,

$$A_{qs} \leq 2 \exp(-2\lambda_q(s - q)/\lambda_s) \leq 2e^{2/C}s^{-2}. \tag{13}$$

Combining (11) through (13) we obtain (9).

(b) We apply the inequalities (4) and (8). Let $1 \leq q \leq s/2$. Since

$$B \sum_{i=1}^{q} 1/\lambda_i \leq \sum_{i=1}^{q} 1/i \leq 1 + \log q$$

and

$$\exp\left(-\lambda_q \sum_{i=q+1}^{s} 1/\lambda_i\right) \leq \exp\left(-\lambda_q \sum_{i=q+1}^{2q} 1/\lambda_i\right) \leq e^{-q/C},$$

we obtain

$$A_{qs} \leq 2e^{-q/C} \exp\left(-B \sum_{i=q+1}^{s} 1/\lambda_i\right) \leq 2eqe^{-q/C}\, \varphi(s)^{-B}. \tag{14}$$

Again (13) is valid for $s/2 < q \leqslant s + 1 - C \log s$. This together with (14) completes the proof of (10) if we take into consideration that $\varphi(s)^{-B} \geqslant (es)^{-1}$.

LEMMA 2. *Let n and m be positive integers and α_j, $n \leqslant j \leqslant m$, be real numbers such that*

$$\sum_{j=n}^{m} \alpha_j = 0 \pmod 1. \tag{15}$$

Then there exist integers b_j, $n \leqslant j \leqslant m$, for which

$$\left\| \sum_{j=n}^{m} (\alpha_j - b_j) x^{\lambda_j} \right\| \leqslant 6 \left(\frac{\lambda_m - \lambda_n}{\lambda_n} \right)^2 + \frac{\lambda_m - \lambda_n}{(m-n)\lambda_n}. \tag{16}$$

PROOF OF LEMMA 2. We may assume that $\lambda_{n+1} - \lambda_n \geqslant 4$, because the substitution $x = y^\beta$ does not change the supremum norm in (16) for any positive number β. Applying the method in [4, Lemma 4] we define, recursively, for $j = m, m-1, \ldots, n$, $d_m = \alpha_m - [\alpha_m]$ and

$$d_j = \begin{cases} \alpha_j - [\alpha_j] & \text{if } \sum_{i=j+1}^{m} d_i \leqslant 0 \\ \alpha_j - [\alpha_j] - 1 & \text{if } \sum_{i=j+1}^{m} d_i > 0 \end{cases}, \quad m - 1 \geqslant j \geqslant n,$$

where $[\alpha]$ denotes the largest integer less or equal to α. Then the numbers $\delta_j = \sum_{i=j}^{m} d_i$ have the properties $\delta_n = 0$, $|\delta_j| < 1$ for $n < j \leqslant m$, and

$$p(x) = \sum_{i=n}^{m} d_i x^{\lambda_i} = \sum_{j=n+1}^{m} \delta_j (x^{\lambda_j} - x^{\lambda_{j-1}})$$

where the first equality serves to define p.

Let q be a given integer, $n < q \leqslant m$. Let a^*, β_j, and T be defined by

$$a^* = p'(1)/(\lambda_q - \lambda_{q-1}), \qquad \beta_j = \lambda_j - \lambda_n, \qquad n \leqslant j \leqslant m,$$

and

$$T(x) = x^{-\lambda_n}(p(x) - a^*(x^{\lambda_q} - x^{\lambda_{q-1}}))$$

$$= \sum_{j=n+1}^{m} \gamma_j (x^{\beta_j} - x^{\beta_{j-1}})$$

where p' denotes the first derivative of p. The numbers γ_j are defined by $\gamma_j = \delta_j$ for $j \neq q$ and $\gamma_q = \delta_q - a^*$ and satisfy $|\gamma_j| < 1$ for $j \neq q$ and $|\gamma_q| < 1 + |a^*|$. Since $\beta_n = 0$ and $\beta_1 \geqslant 4$ it is easy to see that

$$\| \beta_j(\beta_j - 1)x^{\beta_j - 2} - \beta_{j-1}(\beta_{j-1} - 1)x^{\beta_{j-1} - 2} \| \leqslant \beta_j^2 - \beta_{j-1}^2,$$

$n+1 \leqslant j \leqslant m$. Therefore we obtain for the second derivative T'' of T,

$$\|T''\| \leqslant \sum_{j=n+1}^{m} |\gamma_j| (\beta_j^2 - \beta_{j-1}^2) \leqslant \beta_m^2 + |a^*| (\beta_q^2 - \beta_{q-1}^2).$$

Since $|p'(1)| \leqslant \beta_m$ it follows that $|a^*| \leqslant \beta_m/(\lambda_q - \lambda_{q-1})$ and

$$\|T''\| \leqslant 3\beta_m^2. \tag{17}$$

We notice that $T(1) = T'(1) = 0$. Thus, by the mean value theorem,

$$|T(x)| \leqslant (x-1)^2 \|T''\|/2, \qquad 0 \leqslant x \leqslant 1,$$

and by the inequality $\|x^{\lambda_n}(x-1)^2\| \leqslant 4\lambda_n^{-2}$, we find

$$|p(x) - a^*(x^{\lambda_q} - x^{\lambda_{q-1}})| = x^{\lambda_n}|T(x)| \leqslant 6(\lambda_m - \lambda_n)^2\lambda_n^{-2}. \tag{18}$$

We define the integers b_j, $n \leqslant j \leqslant m$, by

$$b_j = \begin{cases} \alpha_j + d_j, & j \neq q, \ \ j \neq q-1, \\ \alpha_q + d_q - [a^*], & j = q, \\ \alpha_{q-1} + d_{q-1} + [a^*], & j = q-1. \end{cases}$$

It is evident that

$$\begin{aligned} \left\| \sum_{j=n}^{m} (\alpha_j - b_j)x^{\lambda_j} \right\| \leqslant {} & \|p(x) - a^*(x^{\lambda_q} - x^{\lambda_{q-1}})\| \\ & + (a^* - [a^*])\|x^{\lambda_q} - x^{\lambda_{q-1}}\|. \end{aligned} \tag{19}$$

We choose the integer q such that $\lambda_q - \lambda_{q-1} = \inf\{\lambda_j - \lambda_{j-1} : n+1 \leqslant j \leqslant m\}$ and therefore

$$\lambda_q - \lambda_{q-1} \leqslant (\lambda_m - \lambda_n)/(m-n). \tag{20}$$

Applying the inequalities $\|x^{\lambda_q} - x^{\lambda_{q-1}}\| \leqslant (\lambda_q - \lambda_{q-1})/\lambda_q$ and (18) through (20) we obtain (16) and thus conclude the proof of Lemma 2.

REMARK 4. In the proof of [4, Lemma 4] we have constructed the Λ-polynomial p and have proved in [4, Lemma 2] that

$$\|p\| \leqslant (\lambda_m - \lambda_n)/\lambda_n. \tag{21}$$

Setting $a^* = 0$ and applying (21) we find integers b_j, $n \leqslant j \leqslant m$, such that

$$\left\| \sum_{j=n}^{m} (\alpha_j - b_j)x^{\lambda_j} \right\| \leqslant (\lambda_m - \lambda_n)/\lambda_n. \tag{22}$$

However, the upper bound in (22) is larger than in (16) and not small enough to prove the results of our Theorems 1 and 2. Thus the more complicated construction of our Lemma 2 is necessary.

Let $f \in C_0[0, 1]$ and s be sufficiently large. We denote $t = [(s + 1)/2]$ and consider the set of all pairs of integers $\Phi = \{(u, v)\colon v = u + [C \log s], t \leqslant u < v \leqslant s\}$. We choose $(r, m) \in \Phi$ such that $\lambda_m - \lambda_r = \min\{\lambda_v - \lambda_u\colon (u, v) \in \Phi\}$. Then it follows that

$$\frac{\lambda_m - \lambda_r}{\lambda_r} \leqslant \frac{(\lambda_s - \lambda_t)C \log s}{\lambda_r(s - t - C \log s)} \leqslant \frac{C(C - 1) \log s}{s - t - C \log s} \leqslant 3C(C - 1)s^{-1} \log s. \tag{23}$$

For r and s, we proceed as in the proof of [4, Lemma 4]. There exist real coefficients a_{j0}, $1 \leqslant j \leqslant s$, such that

$$\left\| f(x) - \sum_{j=1}^{s} a_{j0} x^{\lambda_j} \right\| \leqslant 2E_s(f; \Lambda) \quad \text{and} \quad \sum_{j=1}^{s} a_{j0} = 0. \tag{24}$$

We define the integers b_j, $1 \leqslant j \leqslant r$, and the real coefficients a_{jq}, $1 \leqslant q \leqslant r$ and $q + 1 \leqslant j \leqslant s$, by the induction on q which has been described in the proof of [4, Lemma 4]. Thus we obtain real coefficients a_{jr}, $r + 1 \leqslant j \leqslant s$, for which the inequalities

$$\left\| f(x) - \sum_{j=1}^{r} b_j x^{\lambda_j} - \sum_{j=r+1}^{s} a_{jr} x^{\lambda_j} \right\| \leqslant 2E_s(f; \Lambda) + \sum_{q=1}^{r} A_{qs} \tag{25}$$

and

$$\sum_{j=1}^{r} b_j + \sum_{j=r+1}^{s} a_{jr} = 0 \tag{26}$$

are satisfied.

If $m < s$ we define the integers b_j, $m + 1 \leqslant j \leqslant s$, by $b_j = [a_{jr}]$. There exist Λ-polynomials $Q_q(x) = \sum_{i=r+1}^{m} c_{iq} x^{\lambda_i}$, $m + 1 \leqslant q \leqslant s$, such that

$$\| x^{\lambda_q} - Q_q(x) \| \leqslant 2 \prod_{i=r+1}^{m} \frac{\lambda_q - \lambda_i}{\lambda_q + \lambda_i} \quad \text{and} \quad Q_q(1) = 1. \tag{27}$$

(Cf. [7, Lemma 2] and [4, Lemma 1].) We apply the inequality $(1 - x)/(1 + x) \leqslant e^{-2x}$ factorwise for $x = \lambda_i/\lambda_q$ and obtain from (27),

$$\| x^{\lambda_q} - Q_q(x) \| \leqslant 2 \exp\left(-2 \sum_{i=r+1}^{m} \lambda_i/\lambda_q \right).$$

Since $\lambda_q \leqslant C\lambda_r$ and $m - r \geqslant -1 + C \log s$ we get

$$\| x^{\lambda_q} - Q_q(x) \| \leqslant 2 \exp(-2(m - r)\lambda_r/\lambda_q) \leqslant 2e^{2/C} s^{-2}. \tag{28}$$

In the next step we define the real coefficients α_j, $r + 1 \leqslant j \leqslant m$, by

$$\alpha_j = a_{jr} + \sum_{q=m+1}^{s} (a_{qr} - b_q) c_{jq} \tag{29}$$

and obtain from (25) and (28)

$$\left\| f(x) - \sum_{j=1}^{r} b_j x^{\lambda_j} - \sum_{j=m+1}^{s} b_j x^{\lambda_j} - \sum_{j=r+1}^{m} \alpha_j x^{\lambda_j} \right\| \leqslant 2E_s(f; \Lambda) + \sum_{q=1}^{r} A_{qs} + e^{2/C} s^{-1}. \tag{30}$$

Since $Q_q(1) = 1$ for $m + 1 \leqslant q \leqslant s$ it follows by (26) and (29) that

$$\sum_{j=r+1}^{m} \alpha_j = 0 \pmod 1. \tag{31}$$

If $m = s$ we define the real numbers α_j by $\alpha_j = a_{jr}$, $r + 1 \leqslant j \leqslant m$. Then, by (26), the equality (31) is also valid. Now we can apply Lemma 2 for m and $n = r + 1$. We find integers b_j, $r + 1 \leqslant j \leqslant m$, for which (16) is satisfied. Hence by (30) (if $m < s$) or (25) (if $m = s$), (16), (23) and the inequality $m - r \geqslant C \log s - 1$ we are led to

$$\left\| f(x) - \sum_{j=1}^{s} b_j x^{\lambda_j} \right\| \leqslant 2E_s(f;\Lambda) + \sum_{q=1}^{r} A_{qs} + O(s^{-1}). \tag{32}$$

Finally we combine (32) and Lemma 1. This concludes the proofs of Theorem 1 and Theorem 2. For the latter we apply that $\varphi(s)^{-B} \geqslant (es)^{-1}$ if (4) holds.

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