

## STRONG DIFFERENTIABILITY PROPERTIES OF BESSEL POTENTIALS

BY

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**ABSTRACT.** This paper is concerned with the “strong”  $L_p$  differentiability properties of Bessel potentials of order  $\alpha > 0$  of  $L_p$  functions. Thus, for such a function  $f$ , we investigate the size (in the sense of an appropriate capacity) of the set of points  $x$  for which there is a polynomial  $P_x(y)$  of degree  $k < \alpha$  such that

$$\limsup_{\text{diam}(S) \rightarrow 0} (\text{diam } S)^{-k} \left\{ |S|^{-1} \int |f(y) - P_x(y)|^p dy \right\}^{1/p} = 0$$

where, for example,  $S$  is allowed to run through the family of all oriented rectangles containing the origin.

**1. Introduction.** Let  $\mathcal{F}$  be a family of measurable sets in  $\mathbf{R}^n$  which satisfy the following properties:

- (i)  $m(S) > 0$  for  $S \in \mathcal{F}$ ,
- (1) (ii) for every  $\epsilon > 0$ , there is a set  $S \in \mathcal{F}$  such that  $S \subset B(0, \epsilon)$ ,
- (iii)  $\bigcap \{\bar{S} : S \in \mathcal{F}\} = \{0\}$ .

Here  $m$  denotes Lebesgue measure and  $B(x, r)$  denotes the open  $n$ -ball centered at  $x$  with radius  $r$ . In this paper we shall investigate the “strong”  $L_p$  differentiability properties of a certain class of functions with respect to such a family  $\mathcal{F}$ . Specifically, given a function  $f$  we are concerned with the existence of a polynomial  $P_x(y)$  of degree  $k$  such that

$$(2) \quad \limsup_{\delta(S) \rightarrow 0} \delta(S)^{-k} \left\{ m(S)^{-1} \int_{S+x} |f(y) - P_x(y)|^p dy \right\}^{1/p} = 0$$

where it is understood that  $S \in \mathcal{F}$ . The diameter of a set is denoted by  $\delta(S)$  and  $S + x = \{y + x : y \in S\}$ . Whenever (2) is satisfied we shall say that  $f \in t_p^k(x)$  with respect to the family  $\mathcal{F}$ . This concept was introduced in [CZ] for the special case when  $\mathcal{F}$  is the family of balls centered at the origin.

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We were led to the problem of strong  $L_p$  differentiability by certain questions that arose in the study of boundary regularity of solutions of parabolic differential equations [D]. In that study the family  $\mathcal{F}$  under consideration was the class of parabolic rectangles of the form  $B(0, r) \times (0, r^2)$ . This family is “irregular” in the sense that for every  $\epsilon > 0$ , there is a set  $S \in \mathcal{F}$  with  $m(S) \cdot m[B(0, \delta(S))]^{-1} < \epsilon$ . As in [D], we shall be primarily concerned with the size of the set where (2) fails to hold.

We will consider the question of strong  $L_p$  differentiability of functions in the space of Bessel potentials  $\mathcal{L}_p^\alpha = \{G_\alpha * g : g \in L_p(\mathbb{R}^n)\}$ . Here  $G_\alpha$  denotes the Bessel kernel of order  $\alpha > 0$  and  $*$  denotes the usual convolution operation.  $\mathcal{L}_p^\alpha$  is a Banach space under the norm  $\|G_\alpha * f\|_{\mathcal{L}_p^\alpha} = \|g\|_{L_p}$ . When  $p > 1$  and  $\alpha$  is a positive integer,  $\mathcal{L}_p^\alpha$  is isometric to the Sobolev space  $W_p^\alpha$ . For details see [ST].

If  $\mathcal{F} = \mathcal{B}$ , the family of balls centered at the origin, then the classical theorem of Lebesgue can be rephrased to say that  $f \in \mathcal{I}_1^0(x)$  for almost every  $x$ , whenever  $f$  is locally integrable on  $\mathbb{R}^n$ . For this same family  $\mathcal{B}$ , higher order differentiability was first considered by Calderón and Zygmund in [CZ], where they prove that if  $f \in \mathcal{L}_p^k$  for integer  $k$ , then  $f \in \mathcal{I}_p^k(x)$  for almost every  $x \in \mathbb{R}^n$ . This result has recently been improved by considering  $f \in \mathcal{L}_p^\alpha$  for arbitrary  $\alpha$  and by using an appropriate capacity (or Hausdorff measure) to measure the size of the exceptional sets (cf. [FZ], [BZ], [M2]). All of these results continue to hold if  $\mathcal{B}$  is replaced by any family  $\mathcal{F}$  that is “regular”.

One of the first results to deal with an “irregular” family of sets is due to Zygmund [Z1]. In that paper he considers  $\mathcal{F} = \mathcal{R}$ , the family of all oriented rectangles which contain the origin. He proves that  $f \in \mathcal{I}_1^0(x)$  with respect to  $\mathcal{R}$  for almost every  $x$  provided that  $f \in L_p(\mathbb{R}^n)$  for some  $p > 1$ . This assumption can be weakened slightly by requiring only that

$$\int_{\mathbb{R}^n} |f|(\log^+ |f|)^{n-1}$$

be finite. See [JMZ] or [Z2] for a complete discussion. However, some integrability requirement is necessary, for Saks [S] has provided an example of a function  $f \in L_1$  such that  $f \notin \mathcal{I}_1^0(x)$  for each  $x \in \mathbb{R}^n$ .

A final result worth mentioning is due to Riviere [R]. He considers a Vitali family  $\mathcal{V} = \{U_\alpha\}_{\alpha>0}$  of bounded open sets with the following properties:

1.  $\alpha < \beta \Rightarrow U_\alpha \subset U_\beta$  (nestedness),
2.  $\bigcap_{\alpha>0} \bar{U}_\alpha = \{0\}$ ,
3.  $m(U_\alpha - U_\alpha) \leq A m(U_\alpha)$  for some constant  $A$ , independent of  $\alpha$ ,
4.  $m(U_\alpha)$  is left continuous as a function of  $\alpha$ .

Here  $U_\alpha - U_\alpha = \{x - y : x, y \in U_\alpha\}$ . He then proves that for any locally integrable function  $f$ ,  $f \in \mathcal{I}_1^0(x)$  with respect to  $\mathcal{V}$  for almost every  $x \in \mathbb{R}^n$ .

In this paper we shall first generalize the results of Zygmund and Riviere to

the function classes  $\mathcal{L}_p^\alpha$ , and prove that the exceptional set is null with respect to an appropriate capacity. We then consider the question of higher order differentiability. Theorem 2 will show, for example, that the results of [BZ] and [M2] will hold for the family  $\mathfrak{R}$  as well as  $\mathfrak{B}$ , provided we assume that  $p > 1$ .

**2. First order differentiability.** The Bessel potential space  $\mathcal{L}_p^\alpha$  was defined above. The Bessel capacity associated with this function space is defined as follows. For any set  $E \subset \mathbf{R}^n$ , define

$$B_{\alpha,p}(E) = \text{Inf} \{ \|g\|_{L_p}^p : g \in L_p^+, G_\alpha * g \geq 1 \text{ on } E \}.$$

This set function is an outer measure on  $\mathbf{R}^n$  and moreover  $B_{\alpha,p}(E) = \text{Inf} \{ B_{\alpha,p}(G) : G \supset E \text{ and } G \text{ is open} \}$ . If  $\alpha = 0$  we set  $\mathcal{L}_p^0 = L_p$  and  $B_{0,p} = m$ . If  $p > 1$  and  $\alpha p < n$ , then  $B_{\alpha,p}$  is related to Hausdorff measure of dimension  $n - \alpha p$  by the following relations:

1.  $H^{n-\alpha p}(E) = 0 \Rightarrow B_{\alpha,p}(E) = 0$ ,
2.  $B_{\alpha,p}(E) = 0 \Rightarrow H^{n-\alpha p+\varepsilon}(E) = 0$  for any  $\varepsilon > 0$ .

See Meyers [M1] for further details.

For any family  $\mathfrak{F}$  defined as in §1, we introduce the maximal operator

$$M_{\mathfrak{F}}f(x) = \text{Sup} \left\{ \frac{1}{M(S)} \int_{S+x} f(y) dy : S \in \mathfrak{F} \right\}$$

defined for every  $f \in L_1(\mathbf{R}^n)$ . Often  $\mathfrak{F}$  may be chosen to contain only bounded sets, in which case  $f$  need be only locally integrable. To prove that functions from a Lebesgue class  $L_p(\mathbf{R}^n)$  are in  $L_p^0(x)$  almost everywhere with respect to  $\mathfrak{F}$ , it suffices to prove a weak type (1, 1) estimate on the maximal operator, i.e.

$$(3) \quad m\{M_{\mathfrak{F}}f(x) > t\} \leq C \|f\|_{L_1}/t$$

for some constant  $C$  independent of  $f \in L_1(\mathbf{R}^n)$ . Given (3), we choose functions  $f_n \in C^\infty(\mathbf{R}^n)$  such that  $\|f - f_n\|_{L_p} \rightarrow 0$  and observe that

$$\begin{aligned} \frac{1}{m(S)} \int_{S+x} |f(y) - f(x)|^p dy &\leq \frac{C_p}{m(S)} \int_{S+x} |f(y) - f_n(y)|^p dy \\ &\quad + \frac{C_p}{m(S)} \int_{S+x} |f_n(y) - f_n(x)|^p dy + C_p |f_n(x) - f(x)|^p. \end{aligned}$$

The first term on the right is dominated by  $C_p M_{\mathfrak{F}}(|f - f_n|^p)$ , and since  $f_n$  is continuous we have

$$\overline{\text{Lim}}_{\delta(S) \rightarrow 0} \frac{1}{m(S)} \int_{S+x} |f(y) - f(x)|^p dy \leq C_p M_{\mathfrak{F}}(|f - f_n|^p) + C_p |f_n(x) - f(x)|^p.$$

In view of (3) and the fact that  $f_n \rightarrow f$  in measure, we can choose  $n$  sufficiently

large so that the right side is less than  $\varepsilon$  except on a set of measure less than  $\varepsilon$ , hence the left side is zero almost everywhere.

For functions in  $\mathcal{L}_p^\alpha$  we will prove that in fact the same conclusion holds  $B_{\alpha,p}$  almost everywhere, by replacing estimate (3) by a capacity weak type inequality. This can be done provided the maximal operator is a bounded operator on  $L_p$  (see Adams [A]). The following result was first proved in [D] for the case  $\mathcal{F} = \mathcal{R}$ .

**THEOREM 1.** *Let  $\mathcal{F}$  be a family of measurable sets satisfying conditions (1), and suppose  $M_{\mathcal{F}}$  is of strong type  $(p, p)$ . That is,*

$$(4) \quad \|M_{\mathcal{F}}f\|_{L_p} \leq C\|f\|_{L_p}$$

for some constant  $C$  independent of  $f \in L_p(\mathbb{R}^n)$ ,  $1 \leq p < \infty$ . If  $f \in \mathcal{L}_p^\alpha$ ,  $\alpha \geq 0$ , then  $f \in \mathcal{I}_1^0(x)$  with respect to  $\mathcal{F}$  for  $B_{\alpha,p}$  almost every  $x \in \mathbb{R}^n$ .

**PROOF.** The result for  $\mathcal{L}_p^0 = L_p$  is precisely that detailed above, so let  $\alpha > 0$  and let  $f = G_\alpha * g \in \mathcal{L}_p^\alpha$ . We first make use of the strong type estimate (4) to prove the following capacity weak type estimate

$$(5) \quad B_{\alpha,p}\{M_{\mathcal{F}}f(x) > t\} \leq (C/t^p)\|f\|_{\mathcal{L}_p^\alpha}^p.$$

This follows by observing that  $M_{\mathcal{F}}f \leq G_\alpha * M_{\mathcal{F}}g$  and, hence,

$$\begin{aligned} B_{\alpha,p}\{M_{\mathcal{F}}f > t\} &\leq B_{\alpha,p}\{G_\alpha * M_{\mathcal{F}}g > t\} = B_{\alpha,p}\{G_\alpha * (M_{\mathcal{F}}g/t) > 1\} \\ &\leq t^{-p}\|M_{\mathcal{F}}g\|_{L_p}^p \leq Ct^{-p}\|g\|_{L_p}^p = Ct^{-p}\|f\|_{L_p^\alpha}^p \end{aligned}$$

using the definition of capacity and estimate (4). Secondly, we will show that any function  $f \in \mathcal{L}_p^\alpha$  can be approximated by continuous functions  $f_\varepsilon \in \mathcal{L}_p^\alpha$  such that

$$(6) \quad \|f_\varepsilon - f\|_{\mathcal{L}_p^\alpha} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0$$

and also that

$$(7) \quad B_{\alpha,p}\{x: |f_\varepsilon - f|(x) > \delta\} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0 \text{ for every } \delta > 0.$$

To this end we use a standard mollifier argument. Let  $\phi \in C_c^\infty(\mathbb{R}^n)$  be positive with  $\int_{\mathbb{R}^n} \phi = 1$ . Set  $\phi_\varepsilon(x) = \varepsilon^{-n}\phi(x/\varepsilon)$  and define

$$f_\varepsilon = \phi_\varepsilon * f = G_\alpha * (\phi_\varepsilon * g).$$

Then

$$\|f_\varepsilon - f\|_{\mathcal{L}_p^\alpha} = \|G_\alpha * (G_\varepsilon - G)\|_{\mathcal{L}_p^\alpha} = \|g_\varepsilon - g\|_{L_p},$$

and since  $g_\epsilon \rightarrow g$  in  $L_p$ , (6) is proved. Also notice that

$$B_{\alpha,p}\{x: |f_\epsilon(x) - f(x)| > \delta\} \leq B_{\alpha,p}\{x: G_\alpha * |g_\epsilon - g|(x) > \delta\} \\ \leq \delta^{-p} \|g_\epsilon - g\|_{L_g}^p \rightarrow 0 \text{ as } \epsilon \rightarrow 0$$

so that (7) holds as well. Now fix  $\eta > 0$  and choose  $\epsilon$  so small that  $\|f_\epsilon - f\|_{L_g}^p \leq \eta^{p+1}/C$ , where  $C$  is the constant of (4), such that

$$B_{\alpha,p}\{x: |f_\epsilon - f|(x) > \eta\} < \eta.$$

Then

$$\frac{1}{m(S)} \int_{S+x} |f(y) - f(x)| dy \leq \frac{1}{m(S)} \int_{S+x} |f(y) - f_\epsilon(y)| dy \\ + \frac{1}{m(S)} \int_{S+x} |f_\epsilon(y) - f_\epsilon(x)| dy + |f_\epsilon(x) - f(x)|.$$

Note that the last term is  $< \eta$  except on a set of  $B_{\alpha,p}$  capacity  $< \eta$ . The first term is bounded by  $M_{\mathfrak{F}}(|f - f_\epsilon|)$ , which, according to (5) is less than  $\eta$  except on a set of  $B_{\alpha,p}$  capacity less than  $(C/\eta^p) \|f - f_\epsilon\|_{L_g}^p < \eta$ . The middle term becomes negligible as  $\delta(S) \rightarrow 0$  since  $f_\epsilon$  is continuous. Altogether, we conclude

$$\overline{\text{Lim}}_{\delta(S) \rightarrow 0} \frac{1}{m(S)} \int_{S+x} |f(y) - f(x)| dy < 2\eta$$

except on a set of  $B_{\alpha,p}$  capacity less than  $2\eta$ . Since  $\eta$  is arbitrary, this proves the theorem. Notice that we have specified the constant polynomial  $P_x$  to be  $f(x)$ , which is defined, of course,  $B_{\alpha,p}$  almost everywhere. Q.E.D.

**COROLLARY 1.** *If  $f \in \mathcal{L}_p^\alpha$  with  $\alpha > 0$  and  $1 < p < \infty$ , then  $f \in i_1^0(x)$  with respect to  $\mathfrak{F}$  for  $B_{\alpha,p}$  almost every  $x$  if*

1.  $\mathfrak{F} = \mathfrak{B}$ , the family of all balls centered at the origin,
2.  $\mathfrak{F} = \mathfrak{R}$ , the family of all rectangles containing the origin,
3.  $\mathfrak{F} = \mathfrak{V}$ , the family defined in §1.

**PROOF.** In view of Theorem 1, we need only verify hypothesis (4) for each family. For  $\mathfrak{B}$ , this result is well known and appears, for example, in Stein [ST]. For  $\mathfrak{R}$ , estimate (4) is proved in [JMZ].

The proof of (4) for the family  $\mathfrak{V}$  follows from the weak type (1, 1) estimate

$$(8) \quad m\{x: M_{\mathfrak{V}}f(x) > t\} \leq (C/t) \|f\|_{L_1}$$

for  $f \in L_1(\mathbf{R}^n)$ . This estimate is proved in [R]. To prove (4) from this, we use the familiar argument which appears, for example, on p. 7 of [ST]. Set

$$f_s(x) = \begin{cases} f(x) & \text{if } |f(x)| \geq s/2, \\ 0 & \text{otherwise.} \end{cases}$$

Then  $M_{\mathfrak{V}}f(x) \leq M_{\mathfrak{V}}f_s(x) + s/2$  and so

$$m\{x: M_{\mathfrak{V}}f(x) > s\} \leq m\{x: M_{\mathfrak{V}}f_s(x) > s/2\} \leq (2C/s)\|f_s\|_{L_1}$$

by (8) since  $f_s \in L_1(\mathbb{R}^n)$ . If we set  $\lambda(s) = m\{x: M_{\mathfrak{V}}f(x) > s\}$ , then

$$\begin{aligned} \int_{\mathbb{R}^n} |M_{\mathfrak{V}}f|^p dx &= p \int_0^\infty s^{p-1} \lambda(s) ds \leq p \int_0^\infty 2Cs^{p-2} \|f_s\|_{L_1} ds \\ &= 2Cp \int_0^\infty s^{p-2} \int_{|f|>s/2} |f(x)| dx ds = 2Cp \int_{\mathbb{R}^n} |f(x)| \int_0^{2|f(x)|} s^{p-2} ds dx \\ &= \frac{2Cp}{p-1} \int_{\mathbb{R}^n} |f(x)| 2|f(x)|^{p-1} dx = \frac{4Cp}{p-1} \|f\|_{L_p}^p \end{aligned}$$

which verifies (4) and completes the proof of the corollary.

**3. Higher order differentiability.** In this section we shall discuss the concept of higher order differentiability with respect to irregular families. Theorem 2 will imply, for example, that  $\mathcal{L}_p^\alpha$  functions are in  $t_1^k(x)$  with respect to  $\mathfrak{R}$  for  $B_{\alpha-k,p}$  almost every  $x \in \mathbb{R}^n$ . Unfortunately, the additional assumption we shall make in Theorem 2 does not apply to arbitrary families  $\mathfrak{V}$ , and the question of higher order differentiability with respect to such a family remains open.

**THEOREM 2.** *Let  $\mathfrak{F}$  be a family of measurable sets satisfying conditions (1) and suppose in addition that  $\mathfrak{F}$  is closed under dilations. That is, if  $0 < t < 1$ ,  $t \cdot S = \{ty: y \in S\}$  is in  $\mathfrak{F}$  whenever  $S$  is. Suppose  $M_{\mathfrak{F}}$  satisfies condition (4).*

*If  $f \in \mathcal{L}_p^\alpha$ ,  $\alpha \geq 1$ , and if  $k \leq \alpha$  is an integer, then  $f \in t_1^k(x)$  with respect to  $\mathfrak{F}$  for  $B_{\alpha-k,p}$  almost every  $x \in \mathbb{R}^n$ .*

In order to prove the theorem, we shall make use of the following result, which is an improvement of a similar result proved in [GZ].

**LEMMA 1.** *Let  $f \in \mathcal{L}_p^\alpha$  with  $1 \leq \alpha < \infty$ . Then, for  $B_{\alpha,p}$  almost every  $x \in \mathbb{R}^n$ ,  $f$  is absolutely continuous on  $H^{n-1}$  almost every ray emanating from  $x$ . Moreover, on such a ray,*

$$f(x+z) - f(x) = \int_0^1 \nabla f(x+tz) \cdot z dt.$$

**PROOF OF LEMMA 1.** Let  $f = g_\alpha * g \in \mathcal{L}_p^\alpha$  and let  $f_i = G_\alpha * g_i$  be  $C^\infty$  mollifiers of  $f$ . Then  $\|f_i - f\|_{\mathcal{L}_p^\alpha} \rightarrow 0$  as  $i \rightarrow \infty$ , and there is a subsequence  $\{f_j\}$  that converges for  $B_{\alpha,p}$  almost every  $x$ . That is, for some set  $E$  with  $B_{\alpha,p}(E) = 0$ ,

$$(9) \quad f_j(x) \rightarrow f(x) \quad \text{for every } x \in \mathbb{R}^n \setminus E.$$

For this subsequence,  $\nabla f_j \rightarrow \nabla f$  in  $\mathcal{L}_p^{\alpha-1}$  (see [ST, Lemma 3, p. 136]), and therefore

$$\nabla f_j - \nabla f = G_{\alpha-1} * h_j$$

where  $h_j \rightarrow 0$  in  $L_p(\mathbb{R}^n)$ . But then

$$G_1 * |\nabla f_j - \nabla f|(x) = G_1 * |G_{\alpha-1} * h_j|(x) \leq G_\alpha * |h_j|(x).$$

Since  $h_j \rightarrow 0$  in  $L^p(\mathbb{R}^n)$ ,  $G_\alpha * |h_j|(x) \rightarrow 0$  for  $B_{\alpha,p}$  almost every  $x$ , hence so does  $G_1 * |\nabla f_j - \nabla f|(x)$  by the above inequality. For any such point  $x$  we have

$$\begin{aligned} G_1 * |\nabla f_j - \nabla f|(x) &= \int_{\mathbb{R}^n} G_1(z) |\nabla f_j(x-z) - \nabla f(x-z)| dz \\ &\geq \int_{B(0,R)} G_1(z) |\nabla f_j(x-z) - \nabla f(x-z)| dz \\ &\quad \cdot \int_{\partial B(0,1)} \int_0^R \frac{G_1(ry)}{|ry|^{n-1}} |\nabla f_j(x-ry) - \nabla f(x-ry)| dr dH^{n-1}(y) \end{aligned}$$

where the last step is integration in polar coordinates. Using the estimate  $G_1(z) \geq C|z|^{n-1}$  for  $|z| < R$  (see [ST, p. 132]) we conclude from the above that

$$\int_{\partial B(0,1)} \int_0^R |\nabla f_j(x-ry) - \nabla f(x-ry)| dr dH^{n-1}(y) \rightarrow 0$$

for  $B_{\alpha,p}$  almost every  $x$ . For such a point  $x$ , we may choose a further subsequence  $f_k$  such that

$$(10) \quad \int_0^R |\nabla f_k(x-ry) - \nabla f(x-ry)| dr \rightarrow 0$$

for  $H^{n-1}$  almost every  $y \ni \partial B(0,1)$ . Moreover we may assume that  $x-ry \notin E$  for any such point  $y$ ,  $0 < r < R$  (see [GZ]). If  $z = x-ry$  is any such point, since  $f_k \in C^\infty(\mathbb{R}^n)$  we have

$$f_k(x+z) - f_k(x) = \int_0^1 \nabla f_k(x+tz) \cdot z dt.$$

Letting  $k \rightarrow \infty$ , in view of (9) and (10) we may replace  $f_k$  by  $f$  in the above. Finally, choose a sequence  $R_j \rightarrow \infty$  to prove the lemma. Q.E.D.

**PROOF OF THEOREM 2.** Let  $f \in \mathcal{L}_p^\alpha$ ,  $\alpha \geq 1$ , and let  $k \leq \alpha$  be an integer. Let  $P_x(y)$  be the Taylor polynomial of degree  $k$ , i.e.

$$P_x(y) = \sum_{0 \leq |\beta| \leq k} \frac{D^\beta f(x)}{\beta!} (y - x)^\beta.$$

Here  $\beta = (\beta_1, \dots, \beta_n)$  is a multi-index,

$$|\beta| = \sum_{j=1}^n \beta_j, \quad D^\beta = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \dots \partial x_n^{\beta_n}}, \quad \beta! = \beta_1! \beta_2! \dots \beta_n!,$$

and  $z^\beta = z_1^{\beta_1} z_2^{\beta_2} \dots z_n^{\beta_n}$ . Note that if  $|\beta| \leq k$ , then  $D^\beta f \in \mathcal{L}_p^{\alpha-k}$ , and since such functions are defined up to sets of  $B_{\alpha-k,p}$  capacity zero,  $P_x(y)$  is defined for  $B_{\alpha-k,p}$  almost all  $x \in \mathbb{R}^n$ .

Let  $R_x(y) = f(y) - P_x(y)$  and notice that if  $|\gamma| = k$  then

$$D^\gamma R_x(y) = D^\gamma f(y) - D^\gamma f(x).$$

Since  $D^\gamma f \in \mathcal{L}_p^{\alpha-k}$ , we know from Theorem 1 that

$$(11) \quad \overline{\text{Lim}}_{\delta(S) \rightarrow 0; S \in \mathfrak{F}} \frac{1}{m(S)} \int_{S+x} |D^\gamma R_x(y)| dy = 0$$

at  $B_{\alpha-k,p}$  almost every  $x$ , whenever  $|\gamma| = k$ . Now let  $\nu$  be a multi-index such that  $|\nu| = k - 1$ . Since  $D^\nu f \in \mathcal{L}_p^{\alpha-k}$ , we know from Lemma 1 that for  $B_{\alpha-k,p}$  almost every point  $x$ ,  $D^\nu f$  is absolutely continuous on  $H^{n-1}$  almost every ray emanating from  $x$ . Hence on such a ray,  $D^\nu R_x(y)$  is also absolutely continuous and

$$|D^\nu R_x(x + z) - D^\nu R_x(x)| \leq \int_0^1 |\nabla(D^\nu R_x)(x + tz) \cdot z| dt.$$

Since this estimate holds for almost every  $z \in \mathbb{R}^n$ , we may integrate both sides with respect to  $z$  over a set  $S \in \mathfrak{F}$ , yielding

$$\begin{aligned} \int_S |D^\nu R_x(x + z) - D^\nu R_x(x)| dz &\leq \int_S \int_0^1 |\nabla(D^\nu R_x)(x + tz) \cdot z| dt dz \\ &\leq \int_0^1 \int_S \delta(S) |\nabla(D^\nu R_x)(x + tz)| dz dt \\ &= \int_0^1 \frac{\delta(S)}{t^n} \int_{t \cdot S} |\nabla(D^\nu R_x)(x + z)| dz dt. \end{aligned}$$

The second inequality above is obtained by interchanging the order of integration and using the Schwartz inequality, and the final step is just the change of variables  $z \mapsto tz$ . We now divide both sides by  $m(S)\delta(S)$  and use the transformation  $x + z \mapsto z$ , noting that  $t^n m(S) = m(t \cdot S)$ .



$$\begin{aligned} & \frac{1}{\delta(S)} \frac{1}{m(S)} \int_{S+x} |D^p R_x(z) - D^p R_x(0)| dz \\ & \leq \int_0^1 \frac{1}{m(t \cdot S)} \int_{t \cdot S+x} |\nabla(D^p R_x)(z)| dz dt. \end{aligned}$$

Since all derivatives of  $R_x$  of order  $k$  satisfy (11) and since for  $0 < t < 1$ ,  $t \cdot S \in \mathcal{F}$  with  $\delta(t \cdot S) \leq \delta(S)$ , we conclude the existence of a number  $\delta$  such that the right side of the above is less than  $\varepsilon$  whenever  $\delta(S) < \delta$ . Since  $D^p R_x(0) = 0$ , this implies

$$(12) \quad \overline{\text{Lim}}_{\delta(S) \rightarrow 0; S \in \mathcal{F}} \frac{1}{\delta(S)} \frac{1}{m(S)} \int_{S+x} |D^p R_x(y)| dy = 0$$

for  $B_{\alpha-k,p}$  almost every  $x$ , whenever  $|\nu| = k - 1$ . This estimate replaces (11) for derivatives of order  $k$ . Now let  $\eta$  be a multi-index with  $|\eta| = k - 2$ . Arguing just as above we conclude that

$$(13) \quad \overline{\text{Lim}}_{\delta(S) \rightarrow 0; S \in \mathcal{F}} \frac{1}{\delta(S)^2} \frac{1}{m(S)} \int_{S+x} |D^\eta R_x(y)| dy = 0$$

and inductively that

$$\overline{\text{Lim}}_{\delta(S) \rightarrow 0; S \in \mathcal{F}} \frac{1}{\delta(S)^k} \frac{1}{m(S)} \int_{S+x} |D^0 R_x(y)| dy = 0$$

which proves the theorem. Q.E.D.

**COROLLARY 2.** *If  $f \in \mathcal{L}_p^\alpha$  with  $\alpha \geq 1$ ,  $1 < p < \infty$ , and if  $k \leq \alpha$  is an integer, then  $f \in \mathcal{I}_p^k(x)$  with respect to  $\mathcal{R}$  for  $B_{\alpha-k,p}$  almost every  $x \in \mathbb{R}^n$ .*

The exponent 1 in Theorems 1 and 2 can be improved if one is willing to accept a larger exceptional set. For example, if  $1 < q < p$  and  $k \leq \alpha - 1$ , then one can show that  $f \in \mathcal{I}_q^k(x)$  for  $B_{1,s}$  almost every  $x$ , where  $s < p$  satisfies the equation  $sp/(p - s) = np/(n - p)(q - 1)$ . This follows from the observation that if  $|\gamma| = k$ ,  $|D^\gamma f|^q \in W^{1,s}(\mathbb{R}^n) = \mathcal{L}_s^1$ . Applying Theorem 1 to  $|D^\gamma f|^q$  we obtain estimate (11) with exponent  $q$  for  $B_{1,s}$  almost every  $x$ . Making the appropriate changes in the rest of the argument now proves the claim.

The question as to whether the exponent 1 can be improved in the general case without altering the capacity seems to be open. Saks' example [S], seems to indicate that one cannot expect  $f$  to lie in  $\mathcal{I}_p^k(x)$  even almost everywhere.

Finally, in the definition of strong  $L_p$  differentiability (see (2)), one may be tempted to replace  $\delta(S)^k$  by  $m(S)^{k/n}$ . We conclude by exhibiting an example which shows that  $\delta(S)^k$  cannot be replaced by  $m(S)$  raised to any power.

Let  $f(x, y) = x^2$  and consider the family  $\mathcal{F} = \{R_\varepsilon\}_{\varepsilon > 0}$  of rectangles of the form  $R_\varepsilon = \{(x, y) : -\varepsilon < x < \varepsilon, -\varepsilon^3 < y < \varepsilon^3\}$ . After mollification outside a

larger set, we may assume that  $f \in \mathcal{L}_p^\alpha(\mathbf{R}^2)$  for all  $\alpha \geq 0$ ,  $p \geq 1$ . The Taylor polynomial of degree one at the origin is simply  $P_0(y) \equiv 0$ , and it is easily verified that

$$\frac{1}{m(R_\varepsilon)^{1/2}} \frac{1}{m(R_\varepsilon)} \int_{R_\varepsilon} |f(x, y)| dx dy = \frac{1}{(4\varepsilon^4)^{1/2}} \frac{1}{4\varepsilon^4} \frac{4\varepsilon^6}{3} = \frac{1}{6} \rightarrow 0.$$

Similar behavior is exhibited at points other than the origin. Moreover, by modifying the rectangles  $R_\varepsilon$ , one can show it is impossible to replace  $\delta(S)^k$  by  $m(S)$  to any power.

## REFERENCES

- [A] David R. Adams, *Maximal operators and capacity*, Proc. Amer. Math. Soc. **34** (1972), 152–156. MR **50** #2807.
- [BZ] T. Bagby and W. P. Ziemer, *Pointwise differentiability and absolute continuity*, Trans. Amer. Math. Soc. **191** (1974), 129–148. MR **49** #9129.
- [BF] H. Busemann and W. Feller, *Zur Differentiation der Lebesgueschen Integrale*, Fund. Math. **22** (1934), 226–256.
- [CZ] A. P. Calderón and A. Zygmund, *Local properties of solutions of elliptic partial differential equations*, Studia Math. **20** (1961), 171–225. MR **25** #310.
- [D] Daniel J. Deignan, *Boundary regularity of weak solutions to a quasilinear parabolic equation*, Doctoral Dissertation, Indiana University, 1974.
- [FZ] H. Federer and W. P. Ziemer, *The Lebesgue set of a function whose distribution derivatives are  $p$ -th power summable*, Indiana Univ. Math. J. **22** (1972), 139–158.
- [GZ] Ronald Gariepy and W. P. Ziemer, *Behavior at the boundary of solutions of quasilinear elliptic equations*, Arch. Rational Mech. Anal. **56** (1974/75), 372–384. MR **50** #7807.
- [JMZ] B. Jessen, J. Marcinkiewicz and A. Zygmund, *Note on the differentiability of multiple integrals*, Fund. Math. **25** (1935), 217–234.
- [M1] N. G. Meyers, *A theory of capacities for potentials of functions in Lebesgue classes*, Math. Scand. **26** (1970), 255–292 (1971). MR **43** #3474.
- [M2] ———, *Taylor expansion of Bessel potentials*, Indiana Univ. Math. J. **23** (1973/74), 1043–1049. MR **50** #980.
- [M] A. P. Morse, *Perfect blankets*, Trans. Amer. Math. Soc. **61** (1947), 418–442. MR **8**, 571.
- [R] N. Riviere, *Singular integrals and multiplier operators*, Ark. Mat. **9** (1971), 243–278.
- [S] S. Saks, *Remark on the differentiability of the Lebesgue indefinite integral*, Fund. Math. **22** (1934), 257–261.
- [ST] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, N. J., 1970. MR **44** #7280.
- [Z1] A. Zygmund, *On the differentiability of multiple integrals*, Fund. Math. **23** (1934), 143–149.
- [Z2] ———, *Trigonometric series*, 2nd ed., Cambridge Univ. Press, New York, 1959. MR **21** #6498.

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