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THE SPECTRAL GEOMETRY OF SYMMETRIC SPACES(1)

BY

PETER B. GILKEY

ABSTRACT. Let M be a compact Riemannian manifold without boundary. Let D be a differential operator on M. Let spec (D, M) denote the eigenvalues of D repeated according to multiplicity. Several authors have studied the extent to which the geometry of M is reflected by spec (D, M) for certain natural operators D. We consider operators D which are convex combinations of the ordinary Laplacian and the Bochner or reduced Laplacian acting on the space of smooth functions and the space of smooth one forms. We prove that is is possible to determine if M is a local symmetric space from its spectrum. If the Ricci tensor is parallel transported, the eigenvalues of the Ricci tensor are spectral invariants of M.

Introduction. Let M be a compact connected Riemannian manifold without boundary and let $D_0 = d^* d$ be the Laplacian acting on the space of smooth functions. Let spec (D_0, M) denote the set of eigenvalues $0 \leqslant \lambda_1 \leqslant \lambda_2 \cdots$; each eigenvalue is repeated according to the multiplicity. The basic question we will be considering is to what extent the geometry of the manifold M is reflected by spec (D_0, M) and by the spectra of certain other natural differential operators acting on M. Sakai [5] has proved

THEOREM (SAKAI). *Let M and M' be Einstein manifolds of dimension* 6. *Suppose that M and M' have the same Euler characteristic and that* spec (D_0, M) = spec (D_0, M'). *Then if M is a local symmetric space, so is M'.*

If we enlarge the class of differential operators which we are willing to consider, other geometrical properties of the manifold M are reflected by the spectrum. Let $D_p = d^* d + dd^*$ be the Laplacian acting on the space of smooth p-forms; let spec (D_p, M) denote the eigenvalues of D_p repeated according to multiplicity. Patodi [4] has proved

THEOREM (PATODI). *Suppose that* spec (D_p, M) = spec (D_p, M') *for* $p = 0$, 1, 2. *Then*:

(a) *if M has constant scalar curvature c, so does M'*;

(b) *if M is Einstein, so is M'*;

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(c) *if M has constant sectional curvature c, so does M'.*

Donnely [1] has applied this result of Patodi's to generalize Sakai's theorem as follows:

THEOREM (DONNELY). *Let* spec $(D_p, M) =$ spec (D_p, M') *for* $p = 0, 1, 2$. *If M is an Einstein local symmetric space, so is M'.*

In this paper we will remove the hypothesis that M is Einstein by considering a still more general class of natural differential operators on M. If M is a local symmetric space, the Ricci tensor is parallel transported. Consequently, the eigenvalues of the Ricci tensor do not depend upon the point of evaluation. We will show that if the Ricci tensor is parallel transported, then the eigenvalues of the Ricci tensor are also spectral invariants of the manifold.

1. The theorems discussed in the Introduction are all proved by computing certain asymptotic invariants of the spectrum. We will use the following notation: M is a compact connected Riemannian manifold of dimension m; let G denote the Riemannian metric on M and let $ds^2 = g_{ij}\,dx_i \circ dx_j$ in some system of coordinates. Indices $i, j, \ldots$ run from 1 through m and index a frame for the tangent bundle of M; we sum over repeated indices. Let g^{ij} denote the metric on T^*M and let $|d\,\mathrm{vol}|$ be the Riemannian measure.

Let V be a smooth vector bundle over M and let D: $C^\infty(V) \to C^\infty(V)$ be a second order differential operator with leading symbol given by the metric tensor. If we choose a system of local coordinates for M and a local frame for V, we can express D in the form

$$D = -(g^{ij}\partial^2/\partial x_i \partial x_j + M_k \partial/\partial x_k + N).$$

The M_k and N are square matrices which are not invariantly defined but depend upon the choice of frame and local coordinates.

Let V_x denote the fibre of V over x. For $t > 0$, $\exp(-tD)$ is a well-defined infinitely smoothing operator which is of trace class in $L^2(V)$. Let $K(t, D, x, y)$: $V_y \to V_x$ be the kernel function of $\exp(-tD)$. Then

$$\exp(-tD)u(x) = \int_M K(t, D, x, y)(y(y))\,|d\,\mathrm{vol}\,(y)|.$$

$K(t, D, x, y)$ is smooth in (t, x, y). Define

$$f(t, D, x) = \mathrm{Trace}_{V_x}(K(t, D, x, x)),$$

$$f(t, D) = \mathrm{Trace}_{L^2}(\exp(-tD)) = \int_M f(t, D, x)\,|d\,\mathrm{vol}\,(x)|.$$

It is well known [6] that as $t \to 0+$ that $f(t, D, x)$ has an asymptotic expansion of the form

$$f(t, D, x) \sim (4\pi t)^{-m/2} \sum_{n=0}^{\infty} A_n(D, x) t^n.$$

The coefficients $A_n(x, D)$ are smooth functions of x which can be computed functorially in terms of the derivatives of the total symbol of the differential operator D; $A_n(D, x)$ is a local invariant of D. Let

$$A_n(D) = \int_M A_n(D, x)|d\,\mathrm{vol}\,(x)|;$$

then

$$f(t, D) \sim (4\pi t)^{-m/2} \sum_{n=0}^{\infty} A_n(D)t^n.$$

If V has a smooth inner product $(\ ,\ )$ on each fibre and if D is selfadjoint with respect to the fibre metric, let $\{\lambda_\nu, \theta_\nu\}_{\nu=1,\infty}$ be a complete spectral decomposition of D into an orthonormal basis of eigensections θ_ν and corresponding eigenvalues λ_ν. For such a D, we can express

$$f(t, D, x) = \sum_\nu \exp(-t\lambda_\nu)(\theta_\nu, \theta_\nu)(x) \sim (4\pi t)^{-m/2} \sum_n A_n(D, x)t^n,$$

$$f(t, D) = \sum_\nu \exp(-t\lambda_\nu) \sim (4\pi t)^{-m/2} \sum_n A_n(D)t^n.$$

The integrated invariants $A_n(D)$ depend only on the asymptotic behavior of the series $\sum_\nu \exp(-t\lambda_\nu)$ and are therefore spectral invariants. Although we will only be computing the invariants $A_n(D, x)$ for selfadjoint operators, it is technically convenient to have them defined for operators which are not necessarily selfadjoint.

In [3] we defined the invariants $B_n(D, x)$ and derived explicit formulas for B_0, B_2, B_4, B_6. In the notation of this paper;

$$B_{2n}(D, x) = (4\pi)^{-m/2} A_n(D, x) \quad \text{and} \quad B_{2n+1}(D, x) = 0.$$

We use the notation A_n in this paper rather than B_{2n} since it is more classical. We describe these formulas using the following notation: let ∇_g be the Levi-Civita connection on TM; extend ∇_g to tensors of all types. Let $e = \{e_1, \dots, e_m\}$ be an orthonormal frame for TM defined in a neighborhood of some $x_0 \in M$. We identify TM with the dual T^*M using the metric G. Let

$$R_{ijkl} = G(\{\nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i} - \nabla_{[e_i, e_j]}\}e_k, e_l)$$

be the components of the curvature tensor of the Levi-Civita connection. Let ∇ be any connection on the vector bundle V. Define the reduced or Bochner Laplacian D^∇ by the diagram:

$$D^\nabla \colon C^\infty(V) \xrightarrow{\nabla} C^\infty(T^*M \otimes V) \xrightarrow{\nabla_g \otimes 1 + 1 \otimes \nabla} C^\infty(T^*M \otimes T^*M \otimes V)$$
$$\xrightarrow{-G \otimes 1} C^\infty(V).$$

In normal coordinates, $D^{\nabla} = -g^{ij}\nabla_i\nabla_j$. We proved earlier [2]

THEOREM 1.1. *Given D: $C^{\infty}(V) \to C^{\infty}(V)$ which is a second order differential operator with leading symbol given by the metric tensor, there is a unique connection ∇ on V such that $E = D^{\nabla} - D$ is a 0th order operator.*

Let $D = D^{\nabla} - E$ where E is an invariantly defined endomorphism of V. If $D_0 = d^*d$ is the Laplacian acting on the space of smooth functions, the connection ∇ induced by D_0 is the flat connection and the endomorphism E is zero. Let

$$W_{ij} = \nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i} - \nabla_{[e_i,e_j]}\colon V \to V$$

be the curvature of the connection on V. For each (i,j), W_{ij} is a square matrix. If θ is a tensor field, let $\theta_{;i}$ denote covariant differentiation in the direction e_i. We denote multiple covariant differentiation by $\theta_{;i_1\ldots i_s}$.

Let $\rho_{ij} = R_{ikjk}$ and $\tau = R_{ijij} = \rho_{ii}$. We are using a different sign convention from that used by Sakai in the definition of the curvature tensor; this changes some of the signs in our formulas. Let

$$\rho^n = \rho_{i_1i_2}\rho_{i_2i_3}\cdots\rho_{i_{n-1}i_n}\rho_{i_ni_1}$$

be the trace of the nth power of the Ricci tensor. Let

$$(\nabla\rho)^2 = \rho_{ij;k}\rho_{ij;k}, \qquad (\nabla R)^2 = R_{ijkl;n}R_{ijkl;n},$$
$$(\nabla\tau)^2 = \tau_{;i}\tau_{;i}, \qquad R^2 = R_{ijkl}R_{ijkl}.$$

We sum over repeated indices. In [3] we derived formulas for $A_n(D,x)$ in terms of the $R_{ijkl;\ldots}$, $W_{ij;\ldots}$, $E_{;\ldots}$ tensors for $n = 0, 1, 2, 3$. Since only the integral of these local formulas is a spectral invariant, we simplify the formulas of [3] by integrating by parts.

THEOREM 1.2. *Let $D = D^{\nabla} - E$; then*:

(a) $A_0(D) = \dim(V)\cdot\mathrm{vol}(M)$;

(b) $A_1(D_0) = \int_M -\tau/6\,|d\,\mathrm{vol}|$,

$A_1(D) = \dim(V)\cdot A_1(D_0) + \int_M \mathrm{Tr}(E)\,|d\,\mathrm{vol}|$;

$$A_2(D_0) = \frac{1}{360}\int_M (5\tau^2 - 2\rho^2 + 2R^2)\,|d\,\mathrm{vol}|,$$

(c) $A_2(D) = \dim(V)\cdot A_2(D_0)$

$$+\frac{1}{360}\int_M \mathrm{Tr}(30W_{ij}W_{ij} - 60\tau E + 180E^2)\,|d\,\mathrm{vol}|;$$

$$A_3 D_0) = \frac{1}{9\cdot 7!}\int_M \{-142(\nabla\tau)^2 - 26(\nabla\rho)^2 - 7(\nabla R)^2 - 35\tau^3 + 42\tau\rho^2 - 42\tau R^2 + 36\rho^3 - 20\rho_{ij}\rho_{kl}R_{ikjl} + 8\rho_{ij}R_{ikln}R_{jkln} - 24R_{ijkn}R_{ijuv}R_{knuv}\}\,|d\,\mathrm{vol}|.$$

(d) $$A_3(D) = \dim(V)\cdot A_3(D_0) + \frac{1}{360}\int_M \operatorname{Tr}\{-4W_{ij;k}W_{ij;k} + 2W_{ij;j}W_{ik;k} - 12W_{ij}W_{jk}W_{ki} - 6R_{ijkn}W_{ij}W_{kn} + 4\rho_{jk}W_{jn}W_{kn} - 5\tau W_{kn}W_{kn} + (5\tau^2 - 2\rho^2 + 2R^2 + 30W_{ij}W_{ij})E - 4\rho_{jk}E_{;jk} + 10\tau_{;k}E_{;k} - 30\tau E^2 - 30E_{;k}E_{;k} + 60E^3\}\,|d\,\mathrm{vol}|.$$

2. Let $D_p = (d^*d + dd^*)$ be the Laplacian acting on the space of smooth p-forms. We must determine the connection ∇^p and the endomorphism E^p induced by D_p to apply the formulas of Theorem 1.2. Let $D_p^\nabla = -g^{ij}\nabla_i\nabla_j$ be the Bochner Laplacian defined by the Levi-Civita connection. The operators D_p and D_p^∇ both have the same second order symbol; $\bar{E}^p = D_p^\nabla - D_p$ is at most a first order operator. The first order symbol of $\bar{E}^p$ is invariantly defined; it can be computed functorially as a linear combination of the first derivatives of the metric. In normal coordinates, the first derivatives of the metric vanish at the centre. Therefore $\bar{E}^p = D_p^\nabla - D_p$ is a 0th order operator. By Theorem 1.1, $\nabla = \nabla^p$ and $\bar{E}^p = E^p$.

We compute E^p as follows: let

$$D = \bigoplus_p D_p, \qquad D^\nabla = \bigoplus_p D_p^\nabla, \qquad E = \bigoplus_p E^p.$$

Let $\{e_1,\ldots,e_m\}$ be an orthonormal frame for TM defined near $x_0 \in M$. We have identified TM with T^*M using the metric. If $I = (i_1,\ldots,i_p)$ for $1 \leqslant i_1 < \cdots < i_p \leqslant m$, let

$$e_I = e_{i_1}\wedge\cdots\wedge e_{i_p}, \qquad |I| = p,$$

$$\nabla_{e_i}(e_I) = \Gamma_{iIJ}e_J \qquad (\Gamma_{iIJ} = 0 \text{ if } |I| \neq |J|),$$

$$G(\{\nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i} - \nabla_{[e_i,e_j]}\}e_I, e_J) = G(W_{ij}e_I, e_J) = R_{ijIJ} \qquad (R_{ijIJ} = 0 \text{ if } |I| \neq |J|).$$

Let $\mathrm{Clif}(T^*M)$ denote the Clifford bundle over M. It is defined from the tensor algebra on T^*M by the relations:

$$e_i * e_i = -1, \qquad e_i * e_j = -e_j * e_i \quad \text{for } i \neq j.$$

There is a functorial vector bundle isomorphism between $\mathrm{Clif}(T^*M)$ and the exterior algebra $\Lambda(T^*M)$; this map is not an algebra morphism. This identification defines $*$ multiplication on $\Lambda(T^*M)$. Let $\alpha \in \Lambda(T^*M)$ and $\beta \in T^*M$. Let $e(\beta)$ denote exterior and $i(\beta)$ denote interior multiplication. Then $\beta^*\alpha = (e(\beta) - i(\beta))(\alpha)$. We use Clifford multiplication to construct a first order operator

$$d_0\colon C^\infty(\Lambda(T^*M)) \xrightarrow{\nabla} C^\infty(T^*M \otimes \Lambda(T^*M)) \xrightarrow{*} C^\infty(\Lambda(T^*M)).$$

The leading symbols of d_0 and $(d + d^*)$ agree; the difference is a functorially defined operator of order 0 which can be expressed functorially as a linear combination of the first derivatives of the metric. This difference must vanish so $d_0 = d + d^*$.

We will denote differentiation in the direction e_i by "$/i$".

$$\begin{aligned}
(d + d^*)(f_I e_I) &= f_{I/i} e_i * e_I + f_I \Gamma_{iIJ} e_i * e_J,\\
(d + d^*)^2(f_I e_I) &= f_I \Gamma_{iIJ/j} e_j * e_i * e_J + \cdots,\\
D^\nabla(f_I e_I) &= f_I \Gamma_{iIJ/i} e_i * e_i * e_J + \cdots.
\end{aligned}$$

We have omitted terms in the first derivatives of the metric and terms in the derivatives of f_I. Therefore,

$$\begin{aligned}
E(f_I e_I) &= (D^\nabla - D)(f_I e_I)\\
&= -\sum_{i<j} f_I(\Gamma_{iIJ/j} - \Gamma_{jIJ/i}) e_j * e_i * e_J\\
&\qquad + \text{terms in the first derivatives of the metric.}
\end{aligned}$$

In normal coordinates, the first derivatives of the metric vanish and $\Gamma_{iIJ/j} - \Gamma_{jIJ/i} = -R_{ijIJ}$. Since $R_{iiIJ} = 0$ we can let the sum range over all possible indices.

THEOREM 2.1. (a) *The connection defined by D is the Levi-Civita connection.* (b) *The endomorphism E defined by D is*

$$E(e_I) = \tfrac{1}{2} R_{ijIJ} e_j * e_i * e_J = \tfrac{1}{2} e_j * e_i * W_{ij}.$$

D restricts to an operator on $C^\infty(\Lambda^p(T^*M))$; E must restrict to an endomorphism of $\Lambda^p(T^*M)$. This implies many of the terms in $e_j * e_i * W_{ij}$ must cancel; this cancellation is due to the Bianchi identities.

We apply Theorem 2.1 to compute E^1:

$$E^1(e_k) = \tfrac{1}{2} e_j * e_i * R_{ijkl} e_l.$$

Since $E^1\colon \Lambda^1(T^*M) \to \Lambda^1(T^*M)$, one of the indices in the pair (i,j) must be l. This expression is symmetric in (i,j) so we may assume $i = l$.

$$E^1(e_k) = -R_{ljki}e_j = \rho_{jk}e_j.$$

Similarly,

$$\begin{aligned} E^2(e_k \wedge e_l) &= \tfrac{1}{2}e_j * e_i * W_{ij}(e_k \wedge e_l) \\ &= \tfrac{1}{2}e_j * e_i * (R_{ijkn}e_n \wedge e_l + R_{ijln}e_k \wedge e_n). \end{aligned}$$

Since E^2 preserves $\Lambda^2(T^*M)$, one of the indices in the pair (i,j) must be either n or l in the first expression, and one of the indices in the pair (i,j) must be either k or n in the second expression.

$$\begin{aligned} E^2(e_k \wedge e_l) &= -e_j * (R_{njkn}e_l - R_{ljkn}e_n + R_{kjln}e_n - R_{njln}e_k) \\ &= \rho_{kn}e_n \wedge e_l + \rho_{ln}e_k \wedge e_n - 2R_{knlp}e_n \wedge e_p \\ &= \rho_{kn}e_n \wedge e_l + \rho_{ln}e_k \wedge e_n - R_{klnp}e_n \wedge e_p. \end{aligned}$$

If $P(G)$ is an invariant in the derivatives of the metric, let

$$P(M) = \int_M P(G)\,|d\text{ vol}|.$$

For example, $l(M) = \text{vol}(M)$. If ε is real, let

$$D_p^\varepsilon = \varepsilon D_p + (1-\varepsilon)D_p^\nabla.$$

The connection induced by D_p^ε is the Levi-Civita connection and the corresponding endomorphism is εE^p. We combine the expressions for E^1 and E^2 with the formulas of Theorem 1.2 to compute

THEOREM 2.2. *Let* $\bar{A}_n(D) = A_n(D) - \dim(V)A_n(D_0)$; *then*

(a)
$$\begin{aligned} \bar{A}_1(D_1^\varepsilon) &= \varepsilon\tau(M), \\ \bar{A}_1(D_2^\varepsilon) &= \varepsilon(m-2)\tau(M); \end{aligned}$$

$$\bar{A}_2(D_1^\varepsilon) = \frac{1}{360}\{-30R^2 - 60\varepsilon\tau^2 + 180\varepsilon^2\rho^2\}(M),$$

(b)
$$\begin{aligned} \bar{A}_2(D_2^\varepsilon) = \frac{1}{360}\{&-30(m-2)R^2 - 60(m-2)\varepsilon\tau^2 \\ &+ 180\varepsilon^2(\tau^2 + (m-6)\rho^2 + R^2)\}(M); \end{aligned}$$

$$\bar{A}_3(D_1^\varepsilon) = \frac{1}{360}\{-2(\nabla\tau)^2 + 8(\nabla\rho)^2 + (\nabla R)^2 + 5\tau R^2 - 8\rho^3$$
$$+8\rho_{ij}\rho_{kl}R_{ikjl} + 2\rho_{ij}R_{ikln}R_{jkln} + 3R_{ijkn}R_{ijuv}R_{knuv}$$
$$+\varepsilon(12(\nabla\tau)^2 + 5\tau^3 - 2\tau\rho^2 + 2\tau R^2 - 30\rho_{ij}R_{ikln}R_{jkln})$$
$$+\varepsilon^2(-30(\nabla\rho)^2 - 30\tau\rho^2) + 60\varepsilon^3\rho^3\}(M);$$

$$\bar{A}_3(D_2^\varepsilon) = \frac{1}{360}\{(m-2)(-2(\nabla\tau)^2 + 8(\nabla\rho)^2 + (\nabla R)^2 + 5\tau R^2 - 8\rho^3)$$
$$\text{(c)} \qquad +(m-2)(8\rho_{ij}\rho_{kl}R_{ikjl} + 2\rho_{ij}R_{ikln}R_{jkln} + 3R_{ijkn}R_{ijuv}R_{knuv})$$
$$+\varepsilon((m-2)(12(\nabla\tau)^2 + 5\tau^3 - 2\tau\rho^2) + (2m-34)\tau R^2)$$
$$+\varepsilon(30(6-m)\rho_{ij}R_{ikln}R_{jkln} - 30R_{ijkn}R_{ijuv}R_{knuv})$$
$$+\varepsilon^2(-30(\nabla\tau)^2 - 30(m-6)(\nabla\rho)^2 - 30(\nabla R)^2)$$
$$+\varepsilon^2(-30\tau^3 - 30(m-6)\tau\rho^2 - 30\tau R^2)$$
$$+\varepsilon^3(180\tau\rho^2 + 60(m-10)\rho^3 - 360\rho_{ij}\rho_{kn}R_{ikjn})$$
$$+\varepsilon^3(360\rho_{ij}R_{ikln}R_{jkln} - 60R_{ijkn}R_{ijuv}R_{knuv})\}(M).$$

We will need one more result concerning the invariants A_n for general n. Let $D\colon C^\infty(V) \to C^\infty(V)$ be a second order differential operator with leading symbol given by the metric tensor. Let $D^\varepsilon = \varepsilon D + (1-\varepsilon)D^\nabla$.

THEOREM 2.3. *We may decompose $A_n(D^\varepsilon, x) = \sum_{k=0}^n \varepsilon^k A_{n,k}(D,x)$. Furthermore, $A_{n,n}(D,x) = (1/n!)\operatorname{Tr}(E^n)$.*

The proof is by inspection of the formulas of Theorem 1.2 for $n = 0, 1, 2, 3$. For the general case, we use some results of [2]. Consider the collection of endomorphism valued tensors:

$$R_{i_1i_2i_3i_4;i_5\ldots}I, \qquad W_{i_1i_2;i_3\ldots}, \qquad E_{;i_1\ldots}.$$

Let $K(t,D,x,y)$ be the kernel function for $\exp(-tD)$; there is an asymptotic series for $K(t,D,x,x)$ as $t \to 0+$ of the form

$$K(t,D,x,x) \sim (4\pi t)^{-m/2}\sum_{n=0}^\infty E_n(D,x)t^n.$$

The endomorphisms $E_n(D,x)$ are local invariants of the operator D; $\operatorname{Tr}(E_n(D,x)) = A_n(D,x)$.

By applying H. Weyl's theorem [7] we showed [2] that E_n could be expressed in terms of contractions of various noncommutative polynomial expressions in the $R_{ijkl;\ldots}$, $W_{ij;\ldots}$, and $E_{;\ldots}$ tensors. For example, E_2 has the form

$$E_2(D,x) = A_2(D_0,x)I + A_1(D_0,x)E + E^2/2 + (W_{ij}W_{ij} + 2E_{;kk})/12.$$

Let

$$\operatorname{ord}(R_{i_1 i_2 i_3 i_4; j_1 \cdots j_s}) = \operatorname{ord}(W_{i_1 i_2; j_1 \cdots j_s})$$
$$= \operatorname{ord}(E_{;j_1 \cdots j_s}) = 2 + s.$$

E_n is homogeneous of order $2n$ in these tensors. By replacing E by εE, we can express

$$E_n(D^\varepsilon, x) = \sum_{k=0}^{n} \varepsilon^k E_{n,k}(D, x).$$

The coefficient of ε^n is a multiple of $(E)^n$ by the homogeneity property. Therefore,

$$A_n(D^\varepsilon, x) = \sum_{k=0}^{n} \varepsilon^k A_{n,k}(D, x) \quad \text{for } A_{n,k} = \operatorname{Tr}(E_{n,k}).$$
$$A_{n,n} = c_n \operatorname{Tr}((E)^n).$$

We complete the proof of Theorem 2.3 by evaluating the constant c_n.

Consider the operator $D - \varepsilon I$. The corresponding endomorphism is $E + \varepsilon I$. By the functional calculus,

$$\exp(-t(D - \varepsilon I)) = \exp(\varepsilon t)\exp(-tD),$$
$$K(t, D - \varepsilon I, x, y) = \exp(t\varepsilon) K(t, D, x, y).$$

Therefore,

$$\sum_{n=0}^{\infty} E_n(D - \varepsilon I, x) t^n \sim \exp(t\varepsilon) \sum_{n=0}^{\infty} E_n(D, x) t^n$$
$$\sim \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} \varepsilon^k E_k(D, x)/k! \right\} t^n.$$

By comparing powers of t in the two asymptotic expansions we compute

$$E_n(D - \varepsilon I, x) = \sum_{k=0}^{n} \varepsilon^k E_k(D, x)/k!.$$

We compute $E_n(D - \varepsilon I)$ by replacing E by $E + \varepsilon I$. Therefore, the coefficient of $(E)^n$ in E_n is $E_0(D, x)/n! = 1/n!$ by the normalization which we have chosen. This completes the proof of Theorem 2.3.

We apply Theorem 2.3 to the operator D_1^ε: the endomorphism E^1 is the Ricci tensor up to a possible sign; $\operatorname{Tr}((E^1)^n) = \rho^n$.

THEOREM 2.4.

$$A_n(D_1^\varepsilon) = \sum_{k=0}^{n} \varepsilon^k A_{n,k}(D_1),$$
$$A_{n,n}(D_1) = (\rho^n/n!)(M).$$

3. In this section, we will apply the combinatorial theorems which were proved in the first two sections. We will show that certain geometrical properties are reflected by the spectrum. Let M and M' be two Riemannian manifolds.

LEMMA 3.1. *Let* $p > 0$ *be given and suppose* $\operatorname{spec}(D_p^\varepsilon, M) = \operatorname{spec}(D_p^\varepsilon, M')$ *for* $n+1$ *distinct values of* ε. *Then* $A_{n,k}(D_p)(M) = A_{n,k}(D_p)(M')$ *for* $k = 0, \ldots, n$.

PROOF. Since the two spectra are the same, $A_n(D_p^\varepsilon)(M) = A_n(D_p^\varepsilon)(M')$ for $n+1$ distinct values of ε. Therefore,

$$\sum_{k=0}^{n} \varepsilon^k A_{n,k}(D_p)(M) = \sum_{k=0}^{n} \varepsilon^k A_{n,k}(D_p)(M'),$$

$$\sum_{k=0}^{n} \varepsilon^k (A_{n,k}(D_p)(M) - A_{n,k}(D_p)(M)) = 0.$$

Since this polynomial vanishes for $n+1$ distinct values of ε, the coefficients vanish identically.

We apply the formulas we have derived previously to show

THEOREM 3.2. *Let*

$$\operatorname{spec}(D_0, M) = \operatorname{spec}(D_0, M') \quad \textit{and} \quad \operatorname{spec}(D_1^\varepsilon, M) = \operatorname{spec}(D_1^\varepsilon, M')$$

for 3 *distinct values of* ε. *Then*:

(a) $1(M) = 1(M')$, $\tau(M) = \tau(M')$, $\rho^2(M) = \rho^2(M')$, $\tau^2(M) = \tau^2(M')$, $R^2(M) = R^2(M')$;

(b) *if M has constant scalar curvature c, so does M'*;

(c) *if M is Einstein, so is M'*;

(d) *if M has constant sectional curvature c, so does M'.*

PROOF. We use Theorem 2.2 to prove (a):

$$1(M) = A_0(D_0)(M) = A_0(D_0)(M') = 1(M'),$$
$$\tau(M) = -6A_1(D_0)(M) = -6A_1(D_0)(M') = \tau(M').$$

By Lemma 3.1, $A_{2,k}(D_1)(M) = A_{2,k}(D_1)(M')$ for $k = 0, 1, 2$. We apply Theorem 2.2 to compute $A_{2,1}(D_1)$ and $A_{2,2}(D_1)$:

$$\rho^2(M) = 2A_{2,2}(D_1)(M) = 2A_{2,2}(D_1)(M') = \rho^2(M'),$$
$$\tau^2(M) = -6A_{2,1}(D_1)(M) = -6A_{2,1}(D_1)(M') = \tau^2(M').$$

Finally, since $A_2(D_0)(M) = A_2(D_0)(M')$,

$$(5\tau^2 - 2\rho^2 + 2R^2)(M) = (5\tau^2 - 2\rho^2 + 2R^2)(M').$$

This implies $R^2(M) = R^2(M')$ and completes the proof of (a).

We complete the proof of Theorem 3.2 by expressing the geometrical assumptions about M as integral conditions. M has constant scalar curvature c iff $\tau = -2c$. This is equivalent to assuming that $(\tau + 2c)^2(M) = 0$. Since $(\tau + 2c)^2 = \tau^2 + 4c\tau + 4c^2$, we apply (a) to show that $(\tau + 2c)^2(M') = 0$ which implies M' has constant scalar curvature c. If $m = 2$, M and M' are automatically Einstein. We suppose $m > 2$; M is Einstein iff $\rho_{jk} = c\delta_{jk}$ or equivalently $(\rho_{jk} - c\delta_{jk})^2(M) = 0$. Since $(\rho_{jk} - c\delta_{jk})^2 = \rho^2 - 2c\tau + mc^2$,

$$(\rho_{jk} - c\delta_{jk})^2(M') = 0.$$

This implies M' is Einstein. Finally, M has constant sectional curvature iff $R_{ijkl} = c\delta_{ik}\delta_{jl} - c\delta_{il}\delta_{jk}$, or equivalently,

$$0 = (R_{ijkl} - c\delta_{ik}\delta_{jl} + c\delta_{il}\delta_{jk})^2(M) = (R^2 - 4\tau c + 2c^2(m^2 - m))(M) = 0.$$

By (a) this invariant also vanishes for M' so M' has constant sectional curvature c.

The Ricci tensor ρ_{ij} will be invariant under parallel translation iff $(\nabla\rho)^2(M) = 0$; for such a Riemannian manifold, the eigenvalues of ρ_{ij} are independent of the point of the manifold.

THEOREM 3.3. *Let* $\operatorname{spec}(D_0, M) = \operatorname{spec}(D_0, M')$ *and* $\operatorname{spec}(D_1^\varepsilon, M) = \operatorname{spec}(D_1^\varepsilon, M')$ *for* $m + 1$ *distinct values of* ε. *If* $(\nabla\rho)^2(M) = 0$, *then* $(\nabla\rho)^2(M') = 0$ *and the eigenvalues of the Ricci tensors on M and on M′ are the same.*

PROOF. By Lemma 3.1, $A_{n,k}(D_1)(M) = A_{n,k}(D_1)(M')$ for $0 \leqslant k \leqslant n \leqslant m$. By Theorem 2.2,

$$((\nabla\rho)^2 + \tau\rho^2)(M) = -12A_{3,2}(D_1)(M) = -12A_{3,2}(D_1)(M') = ((\nabla\rho)^2 + \tau\rho^2)(M').$$

Since $\rho = 0$ on M, M has constant scalar curvature c. M' has constant scalar curvature c by Theorem 3.2. Therefore,

$$\tau\rho^2(M) = -2c\rho^2(M) = -2c\rho^2(M') = \tau\rho^2(M).$$

This implies $0 = (\nabla\rho)^2(M) = (\nabla\rho)^2(M')$.

Since $1(M) = 1(M')$, we may make a change of scale to assume without loss of generality that $1(M) = 1(M') = 1$. Let

$$P(M)(\lambda) = \det(\rho_{jk} - \lambda_{jk}) = \sum_{i=0}^{m} \lambda^i c_i(\rho_{jk})(M);$$

$c_i(\rho_{jk})$ are the characteristic classes of the matrix ρ. Let $\rho^n = \operatorname{Tr}((\rho)^n)$; by Theorem 2.4,

$$\rho^n(M) = n!\,A_{n,n}(D_1)(M) = n!\,A_{n,n}(D_1)(M') = \rho^n(M')$$

for $0 \leqslant n \leqslant m$. It is well known that we can express the invariants $c_i(\rho)$ in terms of the invariants ρ^n for $0 \leqslant n \leqslant m$. Since $1(M) = 1(M') = 1$,

$$c_i(1,\rho,\ldots,\rho^m)(M) = c_i(1(M),\rho(M),\rho^2(M),\ldots,\rho^m(M))$$
$$= c_i(1(M'),\rho(M'),\rho^2(M'),\ldots,\rho^m(M')) = c_i(1,\ldots,\rho^m)(M').$$

This implies that $P(M)(\lambda) = P(M')(\lambda)$. λ is an eigenvalue of the Ricci tensor iff $P(M)(\lambda) = 0$ so the two sets of eigenvalues are the same.

We generalize Donnely's theorem [1] as follows:

THEOREM 3.4. *Let*

$$\text{spec}\,(D_0,M) = \text{spec}\,(D_0,M'), \quad \text{spec}\,(D_2,M) = \text{spec}\,(D_2,M'),$$

and

$$\text{spec}\,(D_1^\varepsilon,M) = \text{spec}\,(D_1^\varepsilon,M')$$

for 4 distinct values of ε. If M is a local symmetric space, so is M'.

We use Patodi's theorem [4] and Theorem 3.4 to derive Donnely's result: let M be Einstein symmetric and let spec $(D_p,M) =$ spec (D_p,M') for $p = 0, 1, 2$. If $m = 2$, M is symmetric implies τ is constant on M. This shows τ is constant on M' so M' is symmetric. We may therefore suppose that $m > 2$. M is Einstein implies M' is Einstein. $E^1(M) = E^1(M') = cI$, $D_1^\varepsilon(M) = D_1(M) = (1-\varepsilon)cI$, $D_1^\varepsilon(M') = D_1(M') + (1-\varepsilon)cI$. Since spec $(D_1,M) =$ spec (D_1,M'), spec $(D_1^\varepsilon,M) =$ spec (D_1^ε,M') for all ε. We apply Theorem 3.4 to show M' must be symmetric as well.

We prove Theorem 3.4 as follows: M is symmetric implies τ is constant on M. By Theorem 3.2, τ is constant on M'. Therefore by Theorem 3.2,

$$(\nabla\tau)^2(M) = (\nabla\tau)^2(M') = 0, \qquad \tau^3(M) = \tau^3(M'),$$
$$\tau\rho^2(M) = c\rho^2(M) = c\rho^2(M') = \tau\rho^2(M'),$$
$$\tau R^2(M) = cR^2(M) = cR^2(M') = \tau R^2(M').$$

By Lemma 3.1, $A_{3,k}(D_1,M) = A_{3,k}(D_1,M')$ for $k = 0, 1, 2, 3$. This shows $A_3(D_1,M) = A_3(D_1,M')$. We use the identities derived above together with the identities $A_{3,k}(D_1,M) = A_{3,k}(D_1,M')$ for $k = 1, 2, 3$ to show

$$\rho_{jk}R_{jnmp}R_{knmp}(M) = \cdots(M'), \quad (\nabla\rho)^2(M) = (\nabla\rho)^2(M') = 0,$$
$$\rho^3(M) = \rho^3(M').$$

We use the notation "$\cdots$" to indicate that the same invariant is to be applied to both M and M' in the equation.

We use the 7 identities derived above together with the identities $A_3(D_p,M) = A_3(D_p,M')$ for $p = 0, 1, 2$ to show

$$-7(\nabla R)^2 - 20\rho_{jk}\rho_{kl}R_{ijkl} - 24R_{ijkn}R_{ijuv}R_{knuv}(M) = \cdots (M'),$$

$$(\nabla R)^2 + 8\rho_{jk}\rho_{kl}R_{ijkl} + 3R_{ijkn}R_{ijuv}R_{knuv}(M) = \cdots (M'),$$

$$(m-32)(\nabla R)^2 + (8m-376)\rho_{jk}\rho_{kl}R_{ijkl} + (3m-96)R_{ijkn}R_{ijuv}R_{knuv}(M) = \cdots (M').$$

The matrix of coefficients in these 3 equations is nonsingular. We can solve this system of equations to show

$$(\nabla R)^2(M) = (\nabla R)^2(M').$$

Since M is symmetric, $(\nabla R)^2(M) = 0$. This shows $(\nabla R)^2(M') = 0$ so M' is symmetric.

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Department of Mathematics, Princeton University, Princeton, New Jersey 08540

Current address: Mathematics Institute, University of Warwick, Coventry, Warwickshire, England CV4-7AL