ABSTRACT. We formulate a definition of symmetric derivatives of odd order for functions of two variables. Our definition is based on expanding in a Taylor's series a weighted average of the function about circles.

The definition is applied to derive results on Lebesgue summability for spherically convergent double trigonometric series.

1. Let \( f(t) \) be a function defined in a neighborhood of \( t_0 \in \mathbb{R} \). Let \( k \) be a natural number. We say that \( f \) has at \( t_0 \) a \( k \)th symmetric derivative with value \( a_k \) if the following holds:

If \( k = 2r \) is even,

\[
\frac{1}{2\pi} \left( f(t_0 + t) + f(t_0 - t) \right) = a_0 + \frac{a_2}{2!} t^2 + \cdots + \frac{a_{2r}}{(2r)!} t^{2r} + o(t^{2r})
\]
as \( t \to 0 \).

If \( k = 2r + 1 \) is odd,

\[
\frac{1}{2\pi} \left( f(t_0 + t) - f(t_0 - t) \right) = a_1 t + \frac{a_3}{3!} t^3 + \cdots + \frac{a_{2r+1}}{(2r+1)!} t^{2r+1} + o(t^{2r+1})
\]
as \( t \to 0 \).

If the limit in (1.1) or (1.2) exists only as \( t \to 0 \) through a set having 0 as a point of density, then we say \( f \) has a \( k \)th symmetric approximate derivative at \( t_0 \) equal to \( a_k \).

These definitions may be found in [7]. They have the following applications to termwise integrated trigonometric series. Let \( T: \sum_{n \in \mathbb{Z}} c_n e^{in\theta} \) be a trigonometric series in one variable.

**Theorem A.** If \( c_n \to 0 \) and \( T \) converges at \( \theta_0 \) to \( s \), then the function

\[
F(\theta) = \frac{c_0}{2} \theta^2 - \sum c_n \frac{e^{in\theta}}{n^2}
\]

has at \( \theta_0 \) a second symmetric derivative with value \( s \).
Theorem B. If \( c_n \to 0 \) and \( T \) converges at \( \theta_0 \) to a finite sum \( s \) then the function

\[
L(\theta) = c_0 \theta + \sum_{n} c_n e^{i n \theta}
\]

has at \( \theta_0 \) a first symmetric approximate derivative with value \( s \).

We are concerned in this paper with functions of two variables. We denote points of \( \mathbb{F}_2 \) by \( x = (x_1, x_2) = t e^{i \theta} \), and we write integral lattice points \( n = (n_1, n_2) \). We set \( n \cdot x = n_1 x_1 + n_2 x_2 \). We denote the Fourier series of a function \( F \) by \( S[F] \).

Suppose \( F(x) \) is defined in a neighborhood of \( x_0 \in \mathbb{F}_2 \). We say that \( F \) has at \( x_0 \) an \( r \)th generalized Laplacian equal to \( s \) if \( F \) is integrable over each circle \( |x-x_0| = t \), for \( t \) small, and if

\[
\frac{1}{2\pi} \int_{0}^{2\pi} F(x_0 + te^{i \theta}) d\theta = a_0 + \frac{a_2}{2!} t^2 + \cdots + \frac{s}{(2^r r!)^2} t^{2r} + o(t^{2r})
\]
as \( t \to 0 \). This definition is due to V. Shapiro [4] and forms a two dimensional analogue of (1.1) for symmetric derivatives of even order. In [3] and [4], it is used to establish two dimensional analogues of Theorem A.

The purpose of this paper is to give a two dimensional analogue of (1.2) for symmetric derivatives of odd order, and to apply it to Lebesgue summability for double trigonometric series.

2. We make the following definition. Let

\[
\Omega(\theta) = \cos \theta + \sin \theta.
\]

Let \( F(x) \) be defined in a neighborhood of \( x_0 \in \mathbb{F}_2 \), and suppose that \( F \) is integrable on each circle \( |x-x_0| = t \), for \( t \) small. Let \( k = 2r + 1 \) be an odd integer.

Definition. \( F \) has at \( x_0 \) a generalized symmetric derivative of order \( 2r + 1 \) with value \( s \) if

\[
\frac{1}{2\pi} \int_{0}^{2\pi} F(x_0 + te^{i \theta}) \Omega(\theta) d\theta = a_1 t + a_3 t^3 + \cdots + \frac{s}{2^{2r+1} r! (r+1)!} t^{2r+1} + o(t^{2r+1}),
\]
as \( t \to 0 \).

If the limit in (2.1) exists only as \( t \) tends to 0 through a set \( E \) having 0 as a point of density, we will say \( F \) has at \( x_0 \) a generalized symmetric approximate derivative equal to \( s \).

3. The numerical value of the derivative is given by the following result.

Theorem 1. Suppose that \( F(x) \) and all partial derivatives of \( F \) of order
\[ s = \left( \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right) \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) F(x_0). \]

**Proof.** We may assume \( x_0 = 0 \). We apply Taylor's theorem. We write

\[
F(t e^{i\theta}) = \sum_{j=0}^{2r+1} \frac{1}{j!} \left( t \cos \theta \frac{\partial}{\partial x_1} + t \sin \theta \frac{\partial}{\partial x_2} \right)^j \frac{\partial^j}{\partial x_1^j} \left( \frac{\partial^j}{\partial x_2^j} \right) F(0)
\]

\[
+ \frac{1}{(2r+2)!} \left( t \cos \theta \frac{\partial}{\partial x_1} + t \sin \theta \frac{\partial}{\partial x_2} \right)^{2r+2} F(\mu e^{i\theta}),
\]

for some \( \mu \in (0, t) \).

\[
\frac{1}{2\pi} \int_0^{2\pi} F(t e^{i\theta}) \Omega(\theta) d\theta
\]

\[
= \sum_{j=0}^{2r+1} \frac{1}{j!} \int_0^{2\pi} \left( \cos \theta \frac{\partial}{\partial x_1} + \sin \theta \frac{\partial}{\partial x_2} \right)^j \frac{\partial^j}{\partial x_1^j} \left( \frac{\partial^j}{\partial x_2^j} \right) F(0) \Omega(\theta) d\theta
\]

\[
+ \frac{1}{(2r+2)!} \int_0^{2\pi} \left( \cos \theta \frac{\partial}{\partial x_1} + \sin \theta \frac{\partial}{\partial x_2} \right)^{2r+2} F(\mu e^{i\theta}) \Omega(\theta) d\theta
\]

\[
= \sum_{j=0}^{2r+1} a_j t^j + R_{2r+2}
\]

where

\[
a_j = \frac{1}{j!} \int_0^{2\pi} \sum_{m=0}^j \binom{j}{m} F(m, j - m) \cdot \cos^m \theta \sin^{j-m} \theta \Omega(\theta) d\theta
\]

Clearly \( a_j = 0 \) when \( j \) is even.

When \( j \) is odd,

\[
a_j = \frac{1}{j!} \sum_{m=0}^j \binom{j}{m} F(m, j - m) \cdot \frac{1}{2\pi} \int_0^{2\pi} \cos^m \theta \sin^{j-m} \theta \Omega(\theta) d\theta
\]

\[
= \frac{1}{j!} \sum_{m=0}^j \binom{j}{m} F(m, j - m) \left\{ \frac{1}{2\pi} \int_0^{2\pi} \cos^m \theta \sin^{j-m-1} \theta d\theta \right\}
\]

\[
+ \frac{1}{2\pi} \int_0^{2\pi} \cos^m \theta \sin^{j-m+1} \theta d\theta
\]

\[
= \frac{1}{j!} \sum_{m=0}^j \binom{j}{m} F(m, j - m) \{ c_{jm} + d_{jm} \}.
\]
Using reduction formulae we find,
\[
c_{jm} = \begin{cases} 
\frac{m!(j-m)!}{2^j((j+1)/2)!((m-1)/2)!(j-m/2)!} & \text{if } m \text{ is odd,} \\
0 & \text{if } m \text{ is even,}
\end{cases}
\]
and
\[
d_{jm} = \begin{cases} 
0 & \text{if } m \text{ is odd,} \\
\frac{m!(j-m+1)!}{2^{j+1}((j+1)/2)!((j-m+1)/2)!(m/2)!} & \text{if } m \text{ is even.}
\end{cases}
\]

Breaking the sum in (3.2) into two parts,

\[
(3.3)
\]

\[
a_j = \sum_{m=0; m \text{ odd}}^{j} \frac{1}{j!} \binom{j}{m} \frac{m!(j-m)!}{2^j((j+1)/2)!((m-1)/2)!(j-m/2)!} F(m,j-m)
\]

\[
+ \sum_{m=0; m \text{ even}}^{j} \frac{1}{j!} \binom{j}{m} \frac{m!(j-m+1)!}{2^{j+1}((j+1)/2)!((j-m+1)/2)!(m/2)!} F(m,j-m)
\]

\[= I + II.
\]

To simplify I, set \(s = (m-1)/2\).

\[
(3.4)
\]

\[
I = \frac{1}{2^j((j+1)/2)!} \sum_{s=0}^{(j-1)/2} \frac{1}{s!((j-1)/2-s)!} F(2s+1,j-2s-1)
\]

\[
= \frac{1}{2^j((j+1)/2)!((j-1)/2)!} \left( \sum_{s=0}^{(j-1)/2} \binom{j-1}{s} \right) F(2s+1,j-2s-1)
\]

\[
= \frac{1}{2^j((j+1)/2)!((j-1)/2)!} \frac{\partial}{\partial x_1} \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right)^{(j-1)/2} F(0).
\]

To simplify II, set \(s = m/2\).

\[
II = \sum_{m=0; m \text{ even}}^{j} \frac{j-m+1}{2 \cdot 2^j((j+1)/2)!((j-m+1)/2)!(m/2)!} F(m,j-m)
\]

\[
= \sum_{m=0; m \text{ even}}^{j} \frac{1}{2^j((j+1)/2)!((j-m-1)/2)!} F(m,j-m)
\]

\[
= \frac{1}{2^j((j+1)/2)!} \sum_{s=0}^{(j-1)/2} \frac{1}{((j-1)/2-s)!} F(2s,j-2s)
\]

\[
= \frac{1}{2^j((j+1)/2)!((j-1)/2)!} \frac{\partial}{\partial x_2} \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right)^{(j-1)/2} F(0).
\]

Combining (3.4) and (3.5), we get
For the remainder term,
\begin{equation}
R_{2r+2} = t^{2r+2} O(1) = o(t^{2r+1}).
\end{equation}
Substituting (3.6) and (3.7) into (3.1), the proof of Theorem 1 is complete.

4. We now apply the definition in (2.1) to deduce two dimensional versions of Lebesgue summability for spherically convergent double trigonometric series. The role of (2.1) in the extension of Theorem B to two dimensions is parallel to the role played by generalized Laplacians in the extension of Theorem A to two dimensions. Our proof is similar to the methods used in [5], where a different multi-dimensional analogue of Theorem B is given.

**Theorem 2.** Let \( T: \sum_{n \in \mathbb{Z}^2} c_n e^{i n \cdot x} \) be a double trigonometric series which converges spherically at \( x_0 \) to \( s \), \( s < \infty \). Suppose the coefficients of \( T \) satisfy
\begin{equation}
\sum_{n_1+n_2=0} |n|^\alpha c_n|^2 + \sum_{n_1+n_2 \neq 0} |n|^\alpha (n_1 + n_2)^{-2} |c_n|^2 < \infty,
\end{equation}
for some number \( \alpha > 1 \). Then the series
\begin{equation}
\sum_{n_1+n_2=0} \frac{1}{2}(x_1 + x_2) c_n e^{i n \cdot x} + \sum_{n_1+n_2 \neq 0} \frac{-i c_n}{n_1 + n_2} e^{i n \cdot x}
\end{equation}
converges spherically a.e. on \( T^2 \) to a function \( L(x) \) which has at \( x_0 \) a first generalized symmetric approximate derivative equal to \( s \).

**Theorem 3.** Suppose \( \sum_{n \in \mathbb{Z}^2} c_n e^{i n \cdot x} \) converges spherically at \( x_0 \) to \( s \), \( s < \infty \). Suppose there are functions \( L_1(x) \) and \( L_2(x) \) such that
\begin{equation}
\sum_{n_1+n_2=0} c_n e^{i n \cdot x} = S[L_1]
\end{equation}
and
\begin{equation}
\sum_{n_1+n_2 \neq 0} \frac{-i c_n}{n_1 + n_2} e^{i n \cdot x} = S[L_2].
\end{equation}
Let \( L(x) = \frac{1}{2}(x_1 + x_2) L_1(x) + L_2(x) \). Then \( L(x) \) has at \( x_0 \) a first generalized symmetric approximate derivative with value \( s \).

5. Before starting the proofs of Theorems 2 and 3 we establish the following result. Here \( J_\nu(z) \) represents the Bessel function of the first kind of order \( \nu \).
\begin{equation}
J_\nu(z) = \frac{1}{\pi i} \int_0^{\pi} e^{iz \cos \varphi} \cos (\nu \varphi) d\varphi.
\end{equation}

**Lemma.** Let \( x = te^{i \theta} \in E_2 \) and let \( n = (n_1, n_2) \in \mathbb{Z}^2 \), with \( |n| \neq 0 \). Define
\[ g_n(x) = \begin{cases} \frac{-ie^{jn_1x}}{n_1 + n_2} & \text{if } n_1 + n_2 \neq 0, \\ \frac{1}{2}(x_1 + x_2)e^{jn_1x} & \text{if } n_1 + n_2 = 0. \end{cases} \]

Then
\[
\frac{1}{2\pi} \int_0^{2\pi} g_n(e^{i\theta})\Omega(\theta) \, d\theta = J_1(|n|t) \frac{|n|}{|n|}. \]

**Proof.** Let \( n_1/|n| = \cos \varphi, \ n_2/|n| = \sin \varphi. \)

We first consider \( g_n(x) \) for \( n_1 + n_2 \neq 0. \)

\[
\frac{1}{2\pi} \int_0^{2\pi} g_n(e^{i\theta})\Omega(\theta) \, d\theta
= \frac{|n|}{(n_1 + n_2)^2} \cdot \frac{1}{2\pi i} \int_0^{2\pi} \exp \left\{ i|n| \left( \frac{n_1}{|n|} \cos \theta + \frac{n_2}{|n|} \sin \theta \right) \right\} 
\cdot (\cos \theta + \sin \theta) \left( \frac{n_1}{|n|} + \frac{n_2}{|n|} \right) \, d\theta
\]

\[
= \frac{|n|}{(n_1 + n_2)^2} \cdot \frac{1}{2\pi} \int_0^{2\pi} e^{i|n|\cos(\theta - \varphi)} (\cos (\theta - \varphi) + \sin (\theta + \varphi)) \, d\theta
\]

\[
= \frac{|n|}{(n_1 + n_2)^2} \cdot \frac{1}{2\pi} \int_0^{2\pi} e^{i|n|\cos(\theta - \varphi)} \cos (\theta - \varphi) \, d\theta
\]

\[
+ \frac{|n|}{(n_1 + n_2)^2} \cdot \frac{1}{2\pi} \int_0^{2\pi} e^{i|n|\cos(\theta - \varphi)} \sin (\theta + \varphi) \, d\theta
\]

\[
= A_1 + B_1.
\]

\[
A_1 = \frac{|n|}{(n_1 + n_2)^2} J_1(|n|t).
\]

Let \( \mu = \theta - \varphi. \)

\[
B_1 = \frac{|n|}{(n_1 + n_2)^2} \frac{1}{2\pi i} \int_0^{2\pi} e^{i|n|\cos \mu} \sin (\mu + 2\varphi) \, d\mu
\]

\[
= \frac{|n|}{(n_1 + n_2)^2} \cos 2\varphi \frac{1}{2\pi i} \int_0^{2\pi} e^{i|n|\cos \mu} \sin \mu \, d\mu
\]

\[
+ \frac{|n|}{(n_1 + n_2)^2} \sin 2\varphi \frac{1}{2\pi i} \int_0^{2\pi} e^{i|n|\cos \mu} \cos \mu \, d\mu
\]

\[
= 0 + \frac{|n|}{(n_1 + n_2)^2} \sin (2\varphi) J_1(|n|t) = \frac{|n|}{(n_1 + n_2)^2} \frac{2n_1n_2}{|n|^2} J_1(|n|t). \]
Combining,

\[ \frac{1}{2\pi} \int_0^{2\pi} g_n(te^{i\theta})\Omega(\theta) \, d\theta \]

\[ = A_1 + B_1 = \left(1 + \frac{2n_1 n_2}{|n|^2}\right) \frac{|n|}{(n_1 + n_2)^2} J_1(|n|) = \frac{J_1(|n|)}{|n|}. \]

In the case \( n_1 + n_2 = 0 \),

\[ \frac{1}{2\pi} \int_0^{2\pi} g_n(te^{i\theta})\Omega(\theta) \, d\theta \]

\[ = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2}(t \cos \theta + t \sin \theta)e^{i|n|\cos(\theta - \varphi)} \, d\theta \]

\[ = \frac{t}{4\pi} \int_0^{2\pi} \cos \theta \sin \theta e^{i|n|\cos(\theta - \varphi)} \, d\theta \]

\[ = \frac{t}{4\pi} \int_0^{2\pi} e^{i|n|\cos(\theta - \varphi)} \, d\theta + \frac{t}{4\pi} \int_0^{2\pi} 2 \cos \theta \sin \theta e^{i|n|\cos(\theta - \varphi)} \, d\theta \]

\[ = A_2 + B_2. \]

\[ A_2 = \frac{1}{2} \tau_0(|n|). \]

\[ B_2 = \frac{t}{4\pi} \int_0^{2\pi} \sin 2(\mu + \varphi)e^{i|n|\cos \mu} \, d\mu \]

\[ = \cos (2\varphi) \frac{t}{4\pi} \int_0^{2\pi} \sin (2\mu)e^{i|n|\cos \mu} \, d\mu \]

\[ + \sin (2\varphi) \frac{t}{4\pi} \int_0^{2\pi} \cos (2\mu)e^{i|n|\cos \mu} \, d\mu \]

\[ = 0 - \sin (-\pi/2) \frac{1}{2} \tau_2(|n|) = \frac{1}{2} \tau_2(|n|). \]

Combining \( A_2 \) and \( B_2 \),

\[ \frac{1}{2\pi} \int_0^{2\pi} g_n(te^{i\theta})\Omega(\theta) \, d\theta = \frac{1}{2} \left( \tau_0(|n|) + \tau_2(|n|) \right) = \frac{J_1(|n|)}{|n|} \]

by a formula from [1, p. 12]. Thus the proof of the Lemma is complete.

6. Proof of Theorem 3. We will assume, as we may, that \( x_0 = 0 \) and \( s = 0 \). We must show

\[ \lim_{t \to 0} \frac{1}{2\pi t} \int_0^{2\pi} L(te^{i\theta})\Omega(\theta) \, d\theta = 0. \]

Set

\[ \text{...} \]
\[ L_1(x, r) = \sum_{n_1 + n_2 = 0} c_n e^{in \cdot x} e^{-|n|r}, \]
\[ L_2(x, r) = \sum_{n_1 + n_2 \neq 0} \frac{-ic_n}{n_1 + n_2} e^{in \cdot x} e^{-|n|r} \]

and let \( L(x, r) = \frac{1}{2} (x_1 + x_2) L_1(x, r) + L_2(x, r) \). Using results found in [6], for example, we obtain

\[
\lim_{r \to 0} \int_{T_2} |L(x) - L(x, r)| \, dx \\
\leq \lim_{r \to 0} \int_{T_2} |L_1(x) - L_1(x, r)| \, dx + \lim_{r \to 0} \int_{T_2} |L_2(x) - L_2(x, r)| \, dx \\
= 0.
\]

Choose a sequence \( \mu_k \) decreasing to 0 such that

\[
\int_{T_2} |L(x) - L(x, \mu_k)| \, dx \leq 2^{-3k-1}.
\]

Let

\[
C_k = \left\{ t \in (0, 1) \mid \int_0^{2\pi} |L(te^{it\theta}) - L(te^{it\theta}, \mu_k)| \, d\theta > 2^{-k} \right\}.
\]

Then

\[
2^{-3k-1} \geq \int_0^1 t \, dt \int_0^{2\pi} |L(te^{it\theta}) - L(te^{it\theta}, \mu_k)| \, d\theta \\
\geq \int_{C_k} t2^{-k} \, dt > \int_0^{C_k} t2^{-k} \, dt \\
= 2^{-k-1} |C_k|^2.
\]

Hence, \( |C_k| \leq 2^{-k} \). Thus if we let

\[
T = (0, 1) - \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} C_k,
\]

then \( |T| = 0 \) and, outside of \( T \),

\[
\lim_{k \to \infty} \int_0^{2\pi} |L(te^{it\theta}) - L(te^{it\theta}, \mu_k)| \, d\theta = 0,
\]

so

\[
\lim_{k \to \infty} \int_0^{2\pi} |L(te^{it\theta})\Omega(\theta) - L(te^{it\theta}, \mu_k)\Omega(\theta)| \, d\theta = 0.
\]

Thus, for almost all \( t \in (0, 1) \),

\[
(6.1) \quad \lim_{k \to \infty} \frac{1}{2\pi} \int_0^{2\pi} L(te^{it\theta}, \mu_k)\Omega(\theta) \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} L(te^{it\theta})\Omega(\theta) \, d\theta.
\]

For \( t \in (0, 1) \), define
(6.2) \[ \varphi(t) = \lim_{k \to \infty} \frac{1}{2\pi} \int_0^{2\pi} L(te^{i\theta}, \mu_k) \Omega(\theta) \, d\theta. \]

Then, applying the Lemma,

\[ \varphi(t) = \lim_{k \to \infty} \frac{1}{2\pi} \int_0^{2\pi} \lim_{R \to \infty} \left( \sum_{|n| < R} c_n s_n(te^{i\theta}) e^{-|n|\mu_k} \right) \cdot \Omega(\theta) \, d\theta \]

(6.3) \[ = \lim_{k \to \infty} \lim_{R \to \infty} \sum_{|n| < R} t^{-1} c_n \cdot \frac{1}{2\pi} \int_0^{2\pi} s_n(te^{i\theta}) \Omega(\theta) \, d\theta e^{-|n|\mu_k} \]

\[ = \lim_{k \to \infty} \lim_{R \to \infty} \sum_{|n| < R} c_n \frac{J_1(|n|t)}{|n|t} e^{-|n|\mu_k}. \]

Let \( S_u = \sum_{|n| < u} c_n \). Then, summing by parts,

\[ \sum_{|n| < R} c_n \frac{J_1(|n|t)}{|n|t} e^{-|n|\mu_k} \]

(6.4) \[ = - \int_0^R S_u \frac{d}{du} \left( \frac{J_1(ut)}{ut} e^{-up_k} \right) \, du + S_R \frac{J_1(Rt)}{Rt} e^{-R\mu_k}. \]

Since \( S_R = o(1) \) as \( R \to \infty \), and using the identity \( d(t^{-\nu}J_\nu(t))/dt = -t^{-\nu}J_{\nu+1}(t) \), we get

\[ S_R \frac{J_1(Rt)}{Rt} e^{-R\mu_k} \to 0 \]

as \( R \to \infty \). Hence the last term on the right side of (6.4) drops out, and

\[ \lim_{R \to \infty} \sum_{|n| < R} c_n \frac{J_1(|n|t)}{|n|t} e^{-|n|\mu_k} \]

\[ = - \int_0^\infty S_u \frac{d}{du} \left( \frac{J_1(ut)}{ut} e^{-up_k} \right) \, du \]

\[ = - \int_0^\infty S_u \left\{ \frac{J_2(ut)}{u} e^{-up_k} - \mu_k \frac{J_1(ut)}{ut} e^{-up_k} \right\} \, du. \]

Returning to (6.3),

\[ \varphi(t) = - \lim_{k \to \infty} \int_0^\infty S_u \frac{J_2(ut)}{u} e^{-up_k} \, du + \lim_{k \to \infty} \mu_k \int_0^\infty S_u \frac{J_1(ut)}{ut} e^{-up_k} \, du \]

\[ = - \lim_{k \to \infty} \int_0^\infty S_u \frac{J_2(ut)}{u} e^{-up_k} \, du. \]

We claim

(6.5) \[ \int_{\rho}^{2\rho} |\varphi(t)| \, dt = o(\rho) \quad \text{as} \quad \rho \to 0. \]
For,
\[
\int_{2}^{2^p} |\varphi(t)| \, dt = \int_{2}^{2^p} \left| \lim_{k \to \infty} \int_{0}^{\infty} S_u \frac{J_2(ut)}{u} e^{-u^p} \, du \right| \, dt
\]
\[
\leq \int_{2}^{2^p} \int_{0}^{\infty} \left| S_u \frac{J_2(ut)}{u} \right| \, du \, dt = \int_{2}^{2^p} \int_{0}^{\infty} \left| S_u \frac{J_2(ut)}{u} \right| \, dt \, du
\]
\[
= \int_{0}^{2^p} \int_{2}^{2^p} \left| S_u \frac{J_2(ut)}{u} \right| \, dt \, du + \int_{2^p}^{\infty} \int_{2}^{2^p} \left| S_u \frac{J_2(ut)}{u} \right| \, dt \, du
\]
\[
= P + Q.
\]

We use the relations \(|J_\rho(t)| \leq c t^\rho\) for \(0 < t < 2\), and \(|J_\rho(t)| \leq c t^{-1/2}\) for \(t > 1\).

In the interval of integration involving \(P\), \(|ut| \leq 2\), so \(|u^{-1} J_2(ut)| \leq c u^2\).

\[
P = \int_{0}^{2^p} \int_{2}^{2^p} o(1) O(u^2) \, dt \, du = o(p).
\]

In the interval of integration for \(Q\), \(ut > 1\), so \(|J_2(ut)| \leq c(ut)^{-1/2}\).

\[
Q = \int_{2^p}^{\infty} \int_{2}^{2^p} o(1) u^{-1} O(u t)^{-1/2} \, dt \, du = o(p).
\]

Thus the claim is established.

We complete the proof of Theorem 2 as follows. Let

\[
(6.6) \quad \int_{2-\epsilon_n}^{2} |\varphi(t)| \, dt = 2^{-n} \epsilon_n,
\]

where \(\epsilon_n \to 0\) as \(n \to \infty\). Let \(E_n = \{t \in [2^{-n-1}, 2^{-n}]: |\varphi(t)| > \sqrt{\epsilon_n}\}\). Then

\[
\int_{2-\epsilon_n}^{2} |\varphi(t)| \, dt \geq |E_n| \sqrt{\epsilon_n},
\]

so using (6.6), \(2^{-n} \epsilon_n \geq \sqrt{\epsilon_n} |E_n|\), and \(|E_n| \leq 2^{-n} \sqrt{\epsilon_n}\). Now let \(E = T - \bigcup_{n=1}^{\infty} E_n\). Then \(E\) has 0 as a point of density. In \(E\), \(\varphi(t) \to 0\), and \(\varphi(t) = \frac{1}{(2\pi t)} \int_{0}^{2\pi} L(e^{i\theta}) \Omega(\theta) \, d\theta\). Thus, the theorem is established.

7. Proof of Theorem 2. Let

\[
T_R(x) = \sum_{|n| < R; n_1 + n_2 = 0} \frac{1}{2}(x_1 + x_2)c_n e^{i n x} + \sum_{|n| < R; n_1 + n_2 \neq 0} \frac{-i c_n}{n_1 + n_2} e^{i n x}.
\]

The condition (4.1) insures that \(L(x) = \lim_{R \to \infty} T_R(x)\) exists a.e. on each circle \(|x| = t\). This is a consequence of Theorem 1 of [2]. Moreover, by Theorem 2 of [2], \(\int_{0}^{2\pi} \sup_{R} |T_R(e^{i\theta})| \, d\theta < \infty\), so
\[
\frac{1}{2\pi t} \int_0^{2\pi} L(te^{i\theta}) \Omega(\theta) \, d\theta = \lim_{R \to \infty} \frac{1}{2\pi t} \int_0^{2\pi} T_R(te^{i\theta}) \Omega(\theta) \, d\theta
\]
\[
= \lim_{R \to \infty} \sum_{|n| < R} c_n \cdot \frac{1}{2\pi t} \int_0^{2\pi} g_n(te^{i\theta}) \Omega(\theta) \, d\theta
\]
\[
= \lim_{R \to \infty} \sum_{|n| < R} c_n \frac{J_1([n]|t)}{|n|t}.
\]

We now let
\[
\varphi(t) = \frac{1}{2\pi t} \int_0^{2\pi} L(te^{i\theta}) \Omega(\theta) \, d\theta.
\]

Summing by parts,
\[
\varphi(t) = \int_0^u S_u \frac{J_2(u)}{u} \, du.
\]

The verification of the claim (6.5) and the completion of the proof follow exactly the lines of the completion of the proof of Theorem 3.

REFERENCES


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