SUMS OF SOLID $n$-SPHERES$^1$

BY

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ABSTRACT. We prove that the sum of two solid Antoine $n$-spheres ($n > 3$) by the identity on the boundary is homeomorphic to the $n$-sphere $S^n$.

1. Background.

1.1. Introduction. In 1952, R. H. Bing [3] showed that $S^3$ is obtained from sewing a solid Alexander horned sphere to itself with the identity homeomorphism on the horned sphere. He obtained this surprising result by showing that the decomposition space of an u.s.c. decomposition of $E^3$ into points and tame arcs associated with the sewing is (topologically) $E^3$.

In 1920–1921, M. L. Antoine [1], [2] described a wild Cantor set in $E^3$ that is now known as “Antoine’s necklace.” In 1951, W. A. Blankenship [6] constructed a wild Cantor set in $E^n$ ($n > 3$) which is a generalization of Antoine’s necklace. We call these sets “Antoine-Blankenship necklaces.” There are wild $n$-cells ($n > 3$) in $E^n$ containing Antoine (-Blankenship) Cantor sets [5], [6]. The technique for obtaining these $n$-cells is called tubing out and is illustrated in [5]. Hence there are wild $(n - 1)$-spheres, “Antoine (-Blankenship) spheres”, containing these same Cantor sets. Each of these wild $(n - 1)$-spheres has a nonsimply connected complementary domain in $S^n$. The union of such a wild $(n - 1)$-sphere with its bad complementary domain will be called a solid Antoine (-Blankenship) $n$-sphere.

In this paper we show that the identity sewing of a solid Antoine-Blankenship $n$-sphere with itself yields $S^n$. The author thanks the referee for comments and suggestions.

1.2. Definitions and notation. Euclidean $n$-space is denoted by $E^n$. We use $\rho$ for the Euclidean metric, Diam for diameter, Bd, Ext and Int for boundary, exterior and interior (point set or combinatorial), and $N(X, \varepsilon)$ for the (open)
e-neighborhood of X in \( E^n \), where \( \varepsilon > 0 \). An e-set is a set of diameter less than \( \varepsilon \) and an e-map or e-homeomorphism moves points less than \( \varepsilon \). The symbol \( \text{Cl} \) denotes topological closure.

We assume as familiar the notions of n-cells (\( B^0, B^1, B^2, \ldots \) and their homeomorphic images), n-spheres (\( S^0, S^1, S^2, \ldots \) and their homeomorphic images), complexes, manifolds, disks (2-cells), arcs (1-cells), simple closed curves (1-spheres), tamely (and flatly) embedded complexes, knotted arcs and simple closed curves, general position, and cut and paste techniques. We suggest [7] for a basic reference.

A torus is a 2-manifold homeomorphic with \( S^1 \times S^1 \); an n-torus is homeomorphic to \( S^1 \times S^1 \times \cdots \times S^1 \) where \( S^1 \) is a 1-sphere; a solid torus is homeomorphic to \( B^2 \times S^1 \); and a solid n-torus is homeomorphic to \( B^2 \times S^1 \times \cdots \times S^1 \).

A Cantor set is any set homeomorphic with the standard middle-third Cantor set. A Cantor set \( C \) is tame in a manifold \( M \) if \( C \) lies on a tame arc in \( M \). If \( C \) is not tame, then it is wild. A sequence of compact n-manifolds with boundary \( M_1 \supset \text{Int} M_1 \supset M_2 \supset \text{Int} M_2 \supset \ldots \) is called a defining sequence for a Cantor set \( C \) if \( \bigcap M_i = C \). If the components of \( M_i \) for each \( i \) are n-cells (solid n-tori, etc.), then we say that \( C \) is definable by n-cells (solid n-tori, etc.). We have the following useful characterization of tame Cantor sets [4], [11].

**Theorem 1.1.** A Cantor set in \( E^n \) is tame iff it is definable by n-cells.

We define a very simple kind of linking that will be sufficient for our purposes. We say that two unknotted simple closed curves \( J \) and \( K \) are linked in \( E^3 \) if there is a homeomorphism \( h \) on \( E^3 \) taking \( J \) and \( K \) onto the canonically linked pair \( h(J) = \{(x, y, z)|x^2 + y^2 = 1, z = 0\} \) and \( h(K) = \{(x, y, z)|(y - 1)^2 + z^2 = 1, x = 0\} \). We say that two disjoint compact sets \( X \) and \( Y \) link in \( E^3 \) if there exist unknotted simple closed curves \( J \subset X \) and \( K \subset Y \) such that \( J \) and \( K \) link.

A crumpled n-cell or solid n-sphere \( K \) is a space homeomorphic to the union of an \((n-1)\)-sphere (topologically embedded in \( S^n \)) and one of its complementary domains, \( U \). We call \( U \) the interior of \( K \) (Int \( K \)) and \( S \) the boundary of \( K \) (Bd \( K \)). If \( K_1 \) and \( K_2 \) are crumpled n-cells and \( h \): Bd \( K_1 \rightarrow \text{Bd} K_2 \) is a homeomorphism, then the space \( K_1 \cup_h K_2 \), called the sum of \( K_1 \) and \( K_2 \) by \( h \), is the space obtained from the disjoint union of \( K_1 \) and \( K_2 \) by identifying each point \( p \in \text{Bd} K_1 \) with \( h(p) \in \text{Bd} K_2 \). The homeomorphism \( h \) is called the sewing of \( K_1 \) and \( K_2 \). If each of \( K_1 \) and \( K_2 \) is an n-cell, then any sewing of \( K_1 \) to \( K_2 \) yields \( S^n \). This need not be the case if \( K_1 \) and \( K_2 \) are not n-cells.

1.3. The Antoine (-Blankenship) sphere. The Antoine-Blankenship necklaces are definable by solid n-tori \((n \geq 3)\). The notation we use is that of [9]. First we define Antoine's necklace. Let \( M_1 \) be a solid 3-torus in \( E^3 \), represented by
the large solid 3-torus in Figure 1.1. Let \( A_1, \ldots, A_k \) be \( k \) solid 3-tori embedded in \( \text{Int } M_1 \), *linked around the factor* \( S^1 \) of \( M_1 \approx B^2 \times S^1 \) as indicated in Figure 1.1. The integer \( k \) may be arbitrarily large but at least 2. (For nice pictures, we use \( k = 3 \).) The embedding of the solid 3-tori \( \{ A_j \}_{j=1}^k \) in the solid 3-torus \( M_1 \) is called an *Antoine embedding*. Let \( M_2 = A_1 \cup \cdots \cup A_k \). We obtained \( M_2 \) by using an Antoine embedding of \( k \) solid 3-tori in \( M_1 \). This process iterates. If \( A \) is a component of \( M_i \), then the \( k \) components (\( k \) need not be the same at every stage) of \( M_{i+1} \) in \( A \) are Antoine embedded in \( A \), as indicated in Figure 1.1.

By choosing \( k \) large when constructing \( M_i \), we can make the components of \( M_i \) small in relation to the components of \( M_{i-1} \) (\( i > 2 \)). Hence the above process gives the defining sequence for a Cantor set \( C = \bigcap M_i \). The Cantor set \( C \) is called Antoine's necklace.

Now we describe the Antoine-Blankenship necklaces. Let \( P_i \) \((i = 1, \ldots, n - 2)\) be the natural projection of the solid \( n \)-torus \( A = B^2 \times S^1 \times \cdots \times S^1 \times \cdots \times S^1_{n-2} \) onto its factor space \( B^2 \times S^1 \). For \( n = 3 \), \( P_i = 1 \) and we get the usual definition of Antoine's necklace. For each integer \( i \in \{1, \ldots, n - 2 \} \) we will associate an embedding of \( k \) solid \( n \)-tori \( A_1, \ldots, A_k \) in \( A \). Since there exists a homeomorphism of \( A \) onto itself which interchanges the \( S^1 \) factors, the \( n - 2 \) embeddings are topologically equivalent. Let \( i \) be fixed. The factor space \( B^2 \times S^1_i \) is represented by the large solid 3-torus in Figure 1.2. The embeddings of the projections \( P_i(A_1), \ldots, P_i(A_k) \) of the solid \( n \)-tori \( A_1, \ldots, A_k \) are represented by the smaller solid 3-tori linked around the factor \( S^1_i \) in Figure 1.2.
The solid \( n \)-torus \( A_j \) is like \( A \) in \( n - 3 \) of its factors; in fact, if \( B^* \times S^* \) is the solid 3-torus \( P_k(A_j) \) in Figure 1.2, then \( A_j = B^* \times S_1^1 \times \cdots \times S_{i-1}^1 \times S^* \times S_{i+1}^1 \times \cdots \times S_{n-2}^1 \). The integer \( k \) of \( A_j \)'s may be arbitrarily large but always at least 2. (For nice pictures, we use \( k = 3 \).) The embedding of the solid \( n \)-tori \( \{A_j\}_{j=1}^k \) in the solid \( n \)-torus \( A \) is called an Antoine embedding. If \( A_1, \ldots, A_k \) are Antoine embedded in \( A \) using the \( i \)-th projection \( P_i \), then we say that \( A_1, \ldots, A_k \) are linked around the \( i \)-factor of \( A \).

In order that a sequence \( M_1 \supset \text{Int } M_1 \supset M_2 \supset \cdots \) be a defining sequence for a Cantor set, it is necessary that the diameters of the components of \( M_i \) get small as \( i \) gets large. For \( n = 3 \), by choosing \( k \) large when constructing \( M_i \), we can make the components of \( M_i \) small relative to the components of \( M_{i-1} \) (\( i \geq 2 \)). Hence for \( n = 3 \), we easily obtain a Cantor set, Antoine's necklace. More care is necessary for \( n > 3 \) to insure that the diameters of the components of \( M_i \) get small as \( i \) gets large. We take \( M_1 \) to be any solid \( n \)-torus in \( E^n \). We obtain \( M_2 \) by using an Antoine embedding of \( k \) solid \( n \)-tori in \( M_1 = A \). No matter how large the integer \( k \), a component \( A_j \) of \( M_2 \) can be made small in at most two of its factors. This becomes clear once we note that if \( A_1, \ldots, A_k \) are linked around the \( i \)-factor of \( A \), then

\[
A_j = B^* \times S_1^1 \times \cdots \times S_{i-1}^1 \times S^* \times S_{i+1}^1 \times \cdots \times S_{n-2}^1
\]

and

\[
A = B^2 \times S_1^1 \times \cdots \times S_{i-1}^1 \times S_i^1 \times S_{i+1}^1 \times \cdots \times S_{n-2}^1.
\]

That is, only the disk factor and the \( i \)-th circle factor may be made small which leaves \( A_j \) nearly as large as \( A \).
As we noted, if \( n = 3 \), then we can make all of the components of the second stage \( M_2 \) small. However, if \( n > 3 \), then we need \( n - 2 \) stages to make components small relative to the diameter of \( M_1 \). That is, we may link the solid \( n \)-tori of \( M_2 \) around the 1-factor of \( M_1 \), then in each component of \( A \) of \( M_2 \) we may link the solid \( n \)-tori of \( M_3 \) in \( A \) around the 2-factor of \( A \), etc. Thus the components of \( M_{n-1} \) may be made of arbitrarily small diameter. By repeating this we can insure that \( C = \bigcap M_i \) is a Cantor set.

The Cantor set \( C = C_n \) definable by \( n \)-manifolds in \( E^n \) is easily seen to be wild in \( E^n \). We obtain wild \( (n-1) \)-spheres \( \Sigma^{n-1} \) in \( E^n \) such that \( \Sigma^{n-1} \) is locally flat modulo \( C_n \) by a simple tubing out technique.

Let \( C \) be a Cantor set in \( E^n \). Let \( M_1 \supset M_2 \supset \ldots \) be the defining sequence for \( C \). Let \( B_0 \) be an \( n \)-cell in the complement of \( M_1 \) (as usual, \( M_1 \) connected). Let \( \alpha \) be an arc from \( B_0 \) to \( M_1 \) such that \( \alpha \) intersects each in a single point. Thicken \( \alpha \) to an \( n \)-tube \( T_1 \) joining \( B_0 \) to \( M_1 \) and intersecting each in an \( (n-1) \)-cell in the boundary. Let \( \alpha_1, \ldots, \alpha_k \) be arcs in \( M_1 \) joining \( T_1 \cap M_1 \) to the \( k \) components \( A_1, \ldots, A_k \) of \( M_2 \), respectively. For each \( i, \alpha_i \) intersects \( \text{Bd} \ M_1 \) in a single point in \( T_1 \cap M_1 \). Thicken each of these arcs, obtaining a mutually exclusive collection of \( n \)-tubes \( T_{21}, \ldots, T_{2k_2} \) in \( \text{Int} M_1 \cup (T_1 \cap M_1) \)

\[ A_1 \]
\[ A_2 \]
\[ T_{21} \]
\[ T_{22} \]
\[ \text{Bd} \ B_0 \]

\[ \text{Figure 1.3} \]
such that, for each \( i \), \( T_{2i} \) joins \( T_i \) to \( A_i \) and intersects each in an \( (n-1) \)-cell in its boundary. Continuing in this fashion, we tube out from \( T_{21}, \ldots, T_{2k} \) to the components of \( M_3 \), and then to the components of \( M_4 \), etc. We let \( B_1 = B_0 \cup T_1 \), \( B_2 = B_1 \cup (\bigcup_{i=1}^{k_1} T_{2i}) \), \( B_3 = B_2 \cup (\bigcup_{i=1}^{k_2} T_{3i}) \), etc. For each \( m \), \( B_m \) is an \( n \)-cell and \( S_m = \text{Bd} B_m \) is an \( (n-1) \)-sphere. In the limit we have \( B_m \to B \), an \( n \)-cell containing \( C \), and \( S_m \to \Sigma^{n-1} = \text{Bd} B \), an \( (n-1) \)-sphere containing \( C \).

For \( n = 3 \), it is necessary to tube out carefully as indicated in Figure 1.3. The \( 3 \)-tubes \( T_{ij} \) are joined to the components of \( M_i \). A \( T_{ij} \) does not go through the hole of the component of \( M_{i-1} \) that contains it. [As each component \( A \) of \( M_{i-1} \) is a solid \( 3 \)-torus and \( \pi_1(A) \) is infinite cyclic, it is natural to consider the "hole" in \( A \).] No \( T_{ij} \) winds even halfway around the hole of \( M_{i-1} \). Hence there exists a nonempty family \( \mathcal{F} \) of countable dense subsets of the Cantor set \( C \) such that \( (E^3 - \Sigma^2) + F \) is 1-ULC if \( F \in \mathcal{F} \). (For \( n > 3 \), if \( F \) is any countable dense subset of the Antoine-Blankenship Cantor set \( C \), then \( (E^n - \Sigma^{n-1}) + F \) is 1-ULC.)

2. Identity sewings of Antoine solid \( n \)-spheres. We consider the Antoine or Antoine-Blankenship \( (n-1) \)-sphere \( S \) to be embedded in \( S^n \) and call \( K = S \cup U \) an Antoine crumpled \( n \)-cell or Antoine solid \( n \)-sphere, where \( U \) is the component of \( S^n - S \) that is not simply connected.

**Theorem 2.1.** If \( K \) is an Antoine solid \( n \)-sphere and \( h \) is the identity sewing on \( K \), then \( K \cup_h K \approx S^n \).

Suppose \( p \) is an ideal point at infinity such that \( E^n + p \) is homeomorphic to \( S^n \). When R. H. Bing sewed together two solid (Alexander) horned spheres [3], he described an upper semicontinuous decomposition of \( E^3 + p \) into tame arcs and points. He showed that the continuum obtained from sewing the solid horned spheres by the identity was homeomorphic to the decomposition space of \( E^3 + p \). Then he showed the decomposition was homeomorphic to \( S^3 \) by shrinking out the nondegenerate elements.

This is the plan of attack here. The bulk of the work is in showing that the decomposition space resulting from the identity sewing of Antoine solid \( n \)-spheres is homeomorphic to \( S^n \).

We prove the following version of Bing’s sewing theorem [3]:

**Theorem 2.2.** A continuum is homeomorphic with \( S^n \) if it is the sum of three mutually exclusive sets \( S, U^1, U^2 \) such that (1) there is a homeomorphism of \( S \cup U^1 \) onto an Antoine solid \( n \)-sphere that carries \( S \) onto the Antoine \( (n-1) \)-sphere and (2) there is a homeomorphism of \( S \cup U^1 \) onto \( S \cup U^2 \) that leaves each point of \( S \) fixed.
Proof. We first describe a decomposition of $E^n + p = E^n$ into tame arcs and points such that the resulting space $X = E^n / G$ is homeomorphic with $S \cup U^1 \cup U^2$. For the proof of $E^n / G \approx S^n$, which completes the proof of Theorem 2.2, see [3]. (Note that we will establish Bing’s shrinking criterion in §2.2.3.)

2.1. Decomposition of $E^n + p$. We describe an upper semicontinuous decomposition $G'$ of $E^n$. With the addition of the ideal point at infinity, $p$, $G'$ extends to a decomposition $G$ of $S^n$. Suppose $\theta$ is an $(n - 1)$-hyperplane in $E^n$. The plane $\theta$ separates $E^n$ into two components $H'$ and $H''$ which we think of as the “upper half” and the “lower half.” Roughly, the arcs are those obtained by locating two Cantor sets on opposite sides of $\theta$ and joining the corresponding points with arcs locally PL modulo the end points. The decomposition is definable by $n$-dimensional dogbones or $n$-dimensional solid double tori as discussed in [9]. We denote the manifolds used to describe the Cantor set above $\theta$ (in $H'$) with primed letters and the manifolds used to describe the Cantor set below $\theta$ (in $H''$) by double primed letters. These Cantor sets are distinct copies of the Antoine Cantor set for $n = 3$ or the Antoine-Blankenship Cantor set for $n > 4$, the Cantor sets that are used to build the Antoine solid $n$-spheres.

The $n$-manifolds $M'_i$, $M''_i$ are solid $n$-tori, i.e., connected manifolds homeomorphic to $B^2 \times S^1 \times \cdots \times S^1_{n-2}$. Join $M'_i$ and $M''_i$ with an $n$-tube $T_i$ so that $T_i$ intersects each of $M'_i$ and $M''_i$ in an $(n - 1)$-cell in the boundary of each. $T_i$ is constructed as a thin regular neighborhood of a PL arc joining $M'_i$ to $M''_i$. Let $M_i$ denote the union $M'_i \cup T_i \cup M''_i$. Each component $A'_i$ of $M'_i$ is connected to a unique component $A''_i$ of $M''_i$ by an $n$-tube $T_i$ in Int $M_i$ so that $T_i$ runs straight through the tube $T_i$ of $M_i$ and $T_i$ intersects each of $M'_i$ and $M''_i$ in an $(n - 1)$-cell which lies in $A'_i$ and $A''_i$, respectively. We denote the union $A'_i \cup T_i \cup A''_i$ by $A_i$. The $T_i$'s are mutually exclusive and the union of the $r$ $n$-manifolds $A_1, \ldots, A_r$ is $M_2$. Inductively, if $A = A' \cup T \cup A''$ is a component of $M_k$ where $A'$ and $A''$ are solid $n$-tori and $T$ is an $n$-tube joining $A'$ and $A''$, then the $r$ components $(A_i)_i=1$ of $M_{k+1}$ in $A$ are obtained by connecting each of the $r$ solid $n$-tori $A'_i$ in $A'$ to a unique solid $n$-torus $A''_i$ in $A''$ by an $n$-tube $T_i$ in Int $A$ that runs straight through $T_i$ intersects each of $A'_i$ and $A''_i$ in an $(n - 1)$-cell and misses all other components of $A'$ and $A''$. The $T_i$ are mutually exclusive, and constructed as thin regular neighborhoods of PL arcs.

The manifolds $M_i$ are located so that they are symmetric with respect to $\theta$. Moreover, each component of $M_i$ intersects $\theta$ in an $(n - 1)$-cell. It is natural to think of each component of $M_i$ as being made up of an upper handle $A'$, a lower handle $A''$, and a stem $T$. We shall consider the image of an upper handle (lower handle, stem) under a homeomorphism on $E^n$ to be the upper handle (lower handle, stem) of the image.
A component of $M_i$ in $S^3$ is a solid double torus or a cube with two handles. Although it is not easy to “see” a component of $M_i$ in $S^4$, it is quite easy to visualize the spine of any component of $M_i$. It is the union of two PL toroidal surfaces and a PL arc joining them. This follows from the fact that a handle is of the form $B^2 \times S^1 \times S^1$ with spine homeomorphic to $S^1 \times S^1$ and the stem is a 4-tube with spine an arc. In general, for $n > 4$ the spine of a component of $M_i$ is the union $W' \cup Y \cup W''$ where $W'$, $W''$ are handle spines homeomorphic to $S^1 \times \cdots \times S^1$ and $Y$ is the stem spine, an arc. $Y$ intersects each of $W'$ and $W''$ in a point, which we call a node of the spine. [For $n > 3$, each component of $M_i$ is called an $n$-dogbone.]

The nondegenerate elements of our upper semicontinuous decomposition $G'$ of $E^n$ are the components of $\cap M_i$. We let $G$ be the extension of $G'$ to $S^n$. Let $H$ be the set of nondegenerate elements in $G$, $H^* = \cup H$. As we described the manifolds $M_i$, we could easily have described homeomorphisms $\gamma \gamma \gamma$ between $\gamma \gamma \gamma \gamma \gamma G$ and each of the two parts of $\gamma \gamma \gamma \gamma \gamma / G$, i.e., the part that intersects $\gamma \gamma \gamma\gamma\gamma\gamma$ and the part that intersects $\gamma \gamma \gamma\gamma\gamma$. The technique resembles that used in describing an Antoine $(n - 1)$-sphere or solid $n$-sphere. Hence $\gamma \gamma \gamma \gamma \gamma / G$ is homeomorphic with $S \cup U^1 \cup U^2$ where $G$ consists of points of $E^n + p - \gamma \gamma \gamma \gamma \gamma \gamma M_i$ and components of $\gamma \gamma \gamma M_i$.


To introduce the reader to the shrinking technique for $n > 4$, we first obtain a basic lemma for the case $n = 3$. Note that for $n = 3$, Theorem 2.2 also follows from [10].

In the description of the Antoine-Blankenship Cantor sets, the number $k$ of components $\{A_j\}_{j=1}^k$ of $M_{i+1}$ in a component $A$ of $M_i$ may be arbitrarily large and must be at least two. For simplicity, we always take $k = 3$ in the figures.

A set of $(n - 1)$-hyperplanes $P_1, \ldots, P_s$ parallel to $\theta$ with $P_i$ between $P_{i-1}$ and $P_{i+1}$ ($i = 2, 3, \ldots, s - 1$) is said to be in good position relative to $h(M_k)$ for a homeomorphism $h$ on $E^n$ if for each component $A$ of $h(M_k)$,

1. each handle of $A$ may intersect at most one hyperplane,
2. if $P_i$ intersects $A$ in the stem of $A$, then $P_i \cap A$ is an $(n - 1)$-cell, and
3. if $A_\mu$ is a component of $A \cap h(M_{k+m})$, $m = 1, 2, \ldots$, then if $P_i$ intersects $A$ in the stem for $i \in \{2, 3, \ldots, s - 1\}$, then $P_i \cap A_\mu$ is an $(n - 1)$-cell in the stem of $A_\mu$.

We shall consider the images of the planes $P_1, \ldots, P_s$ under a homeomorphism $f$ on $E^n$ to be in good position relative to $fh(M_k)$ if $P_1, \ldots, P_s$ are in good position relative to $h(M_k)$.

In Lemmas 2.3 and 2.4, we use four planes. For notational simplicity, we use $P_a, P_b, P_c,$ and $P_d$ for $P_1, P_2, P_3,$ and $P_4,$ respectively.

2.2.1. Antoine solid 3-spheres: A basic lemma. We do not shrink the arcs of $H$ in one stage of the defining sequence, but we follow a procedure that will
eventually shrink the arcs. The shrinking procedure is outlined in the proof of the following lemma.

**Lemma 2.3.** If \( h' \) is a homeomorphism of \( E^3 \) and \( P_a, P_b, P_c, \) and \( P_d \) are 2-hyperplanes in good position relative to \( h'(M_i) \), then there exists a homeomorphism \( h \) of \( E^3 \) such that (1) \( h = 1 \) on \( E^3 - h'(M_i) \), (2) \( h = 1 \) between \( P_b \) and \( P_c \), (3) \( h \) takes each component of \( h'(M_{i+1}) \) onto a set which intersects at most one of \( P_a \) and \( P_d \), and (4) \( P_a, P_b, P_c, \) and \( P_d \) are in good position relative to \( hh'(M_{i+1}) \).

**Proof.** It suffices to consider one component \( A \) of \( h'(M_i) \), i.e., if \( A_1, A_2, \ldots, A_k \) are components of \( h'(M_{i+1}) \) in \( A \), it suffices to obtain a homeomorphism \( g \) of \( E^3 \) such that (1) \( g = 1 \) on \( E^3 - A \), (2) \( g = 1 \) between \( P_b \) and \( P_c \), (3) \( g \) takes each \( A_j \) onto a set which intersects at most one of \( P_a \) and \( P_d \), and (4) \( P_a, P_b, P_c, \) and \( P_d \) are in good position relative to

\[
g(A_1 \cup A_2 \cup \cdots \cup A_k).
\]

There are two cases to consider; either no plane intersects a handle of \( A \) or

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**Figure 2.1**
The idea of the proof is contained in the following sequence of pictures in which the $A_j$'s are represented by their 1-spines. The effect of $g$ is that $g(A_2), g(A_3), \ldots, g(A_k)$ do not intersect $P_a$ and $g(A_1), g(A_2), \ldots, g(A_{k-1})$ do not intersect $P_d$.

We need to consider two sets of handle pictures: (a) A plane intersects a handle. (b) A plane does not intersect a handle. The planes are illustrated by the shaded disks.

Although the figures are sufficient to describe the action of $g$, we give another description of $g$ which is analogous to the $g$ used later in sewing an Antoine solid $n$-sphere to itself by the identity. The homeomorphism $g$ can be defined as the composition of two homeomorphisms: $g_1$, which rearranges the handles of $\{A_j\}$ in the handles of $A$, and $g_2$, which pulls the handles of the $A_j$ toward the stem of $A$.

We may assume that the spines of the $A_j$'s are very close to the spine of $A$. It will be convenient later to consider $g_1$ and $g_2$ as moving spines or neighborhoods of spines. The map $g_1$ acts on the handles of $A$ moving the spines of the upper (lower) handles of $A_2, \ldots, A_k$ ($A_1, \ldots, A_{k-1}$) into a close neighborhood of the upper (lower) node of the spine of $A$. The map $g_2$ shortens the spine of the stem of $A$, pulling the neighborhoods of the nodes of the spine of $A$ beyond the outermost planes intersecting $A$. Alternatively, one might consider the stem of $A$ as reaching out and engulfing small ball neighborhoods of the nodes of the spine of $A$, and then pulling them back into the stem.

We now have that (1) $g = 1$ on $E^3 - A$, (2) $g = 1$ between $P_b$ and $P_c$, and (3) $g$ takes each $A_j$ onto a set which intersects at most one of $P_a$ and $P_d$.

Consider a component $A_j$ of $h'(M_{i+1}) \cap A$. Since $g(A_j) \cap (P_b \cup P_c) = A_j \cap (P_b \cup P_c)$ and the nodes of the spine of $A_j$ are close to the nodes of the spine of $A$, it follows that $P_a$ and $P_d$ do not intersect the stem of $g(A_j)$. The map $g$ pulls the nodes of the spine of $A$ beyond $P_a$ and $P_d$. We easily obtain $g$ so that conditions (2) and (3) of good position are met. Hence, the planes $P_a, P_b, P_c, P_d$ are in good position relative to $g(A_j \cap h'(M_{i+1}))$.

2.2.2. Antoine solid $n$-spheres: A basic lemma. In this section we prove a lemma that is the higher dimensional analogue of Lemma 2.3.

**Lemma 2.4.** If $h'$ is a homeomorphism of $E^n$ and $P_a, P_b, P_c,$ and $P_d$ are $(n - 1)$-hyperplanes in good position relative to $h'(M_i)$, then there exists a homeomorphism $h$ of $E^n$ such that (1) $h = 1$ on $E^n - h'(M_i)$, (2) $h = 1$ between $P_b$ and $P_c$, (3) $h$ takes each component of $h'(M_{i+n-2})$ onto a set which intersects at most one of $P_a$ and $P_d$, and (4) $P_a, P_b, P_c, P_d$ are in good position relative to $hh'(M_{i+n-2})$.

A generalized $n$-annulus is an $n$-manifold homeomorphic with $S^1_1 \times \cdots \times$
A generalized \((n, k)\)-annulus is an \(n\)-manifold homeomorphic with \(S^n_1 \times \cdots \times S^n_{n-k} \times I_1 \times \cdots \times I_k\).

**Proof (Lemma 2.4).** It suffices to consider one component \(A\) of \(h'(M_i)\). The components of \(h'(M_i+1)\) in \(A\) are denoted by \(A_1, \ldots, A_u\), the components of \(h'(M_i+2)\) in \(A\) by \(A_{11}, \ldots, A_{u_1}\) where \(A_{j_0} \subset A_{j_1}\), and the components of \(h'(M_i+n-2)\) in \(A\) by \(A_{j_1} \cap \cdots \cap A_{j_n}\). If \(A_a\) represents a component of \(h'(M_i+u)\), then \(a\) is a finite sequence of integers of length \(u\). The components of \(h'(M_{i+u+1})\) in \(A\) are then denoted by \(A_{a_1}, \ldots, A_{a_{vo}}\). We assume that \(A\) intersects all four planes.

As in the proof of Lemma 2.3, it suffices to find a homeomorphism \(g\) of \(E^n\) such that (1) \(g = 1\) on \(E^n - A\), (2) \(g = 1\) between \(P_b\) and \(P_c\), (3) \(g\) takes each component of \(h'(M_{i+n-2})\) in \(A\) onto a set which intersects at most one of the two hyperplanes \(P_a\) and \(P_d\), and (4) \(P_a, P_b, P_c,\) and \(P_d\) are in good position relative to \(g(A \cap h'(M_{i+n-2}))\).

The idea of the proof is contained in a sequence of pictures. In the figures, \(n = 4\) and the manifolds \(A_a\) are represented by their spines. In general, a partition of the \((n-2)\)-dimensional portion of the spine of \(A_a\), a component of \(h'(M_{i+n-3})\), is used to indicate the approximate location of the spines of \(A_{a_1}, A_{a_2}, \ldots, A_{a_{vo}}\).

In the figure, a handle spine of \(A\) represented by \(S^n_1 \times S^n_2\) (we drop the superscripts for the remainder of this proof) and the handles of \(A_{11}, A_{22}, A_{33}\) in that handle of \(A\) are shown linked around the \(S^n_1\)-factor. In general if the handles of \(h'(M_{i+1})\) link around the \(S^n_{\beta_1}\)-factor of \(h'(M_i)\) for some \(\beta_1\), then the handles of \(h'(M_{i+2})\) link around the \(S^n_{\beta_2}\)-factor of \(h'(M_{i+1})\), where \(\beta_2 = \beta_1 + 1\). To save notation, we shall write \(i\) for \(\beta_i\). With this in mind, we have that the handles of \(h'(M_{i+j})\) link around the \(S^n_{\beta_j}\)-factor of the handles of \(h'(M_{i+j-1})\) for \(j = 1, 2, \ldots\).

The map \(g\) is obtained as the composition of two homeomorphisms, \(g_1\) and \(g_2\). For each \(A_a\) in \(h'(M_{i+n-3}) \cap A\), \(g_1\) rearranges the handles of \(A_{a_1}, \ldots, A_{a_{vo}}\) in the handles of \(A_{a_1}\). For each component \(A_a\) of \(h'(M_{i+n-3}) \cap A\), \(g_2\) pulls the handles of \(A_{a_1}, \ldots, A_{a_{vo}}\) into or toward the stem of \(A_{a_1}\), achieving the desired relationship with the \((n-1)\)-hyperplanes \(P_a, P_b, P_c,\) and \(P_d\). Moreover, \(g_1\) is fixed outside \(h'(M_{i+n-3}) \cap A\) and \(g_2\) is fixed outside \(A\).

We now set up the machinery necessary to describe \(g_1\) and \(g_2\). Consider a component \(A_a\) of \(h'(M_{i+n-3})\) in \(A\) containing \(A_{a_1}, \ldots, A_{a_{vo}}\) that are in turn components of \(h'(M_{i+n-2})\). Let the spine of \(A_a\) be the union of \(Z_a, Z'_a\), and \(X_a\) where \(Z_a, Z'_a\) are the spines of the upper and lower handles, respectively. Each of \(Z_a\) and \(Z'_a\) is homeomorphic to \(S^n_1 \times \cdots \times S^n_{n-2}\). The set \(X_a\) is an arc meeting \(Z_a\) and \(Z'_a\) in points \(p_a\) and \(p'_a\), respectively. Similarly, let the spine of \(A\) be \(Z \cup Z' \cup X\) with nodes \(\{p, p'\}\).

Let \(R_a, R'_a\) be PL generalized \((n-2)\)-annular neighborhoods of \(p_a, p'_a\) in \(Z_a, Z'_a\), respectively. If \(Z_a\) is represented by \(S^n_1 \times \cdots \times S^n_{n-3} \times S^n_{n-2}\) [the
Figure 2.2

The asterisk notation is used to indicate that the handles of the components of $h'(M_{j+n-3})$ are small in the first $(n - 3)$ factors and still large in the last factor. Also, if $Z$ is represented by $S_1 \times \cdots \times S_{n-2}$, then each $S_i^*$ is represented by an interval $I_i$ in $S_i$, $i = 1, \ldots, n - 3$; see Figure 2.2], then $R_a$ is represented by $S_1^* \times \cdots \times S_{n-3}^* \times I$ where $I$ is an appropriate interval in $S_{n-2}$. Since we may assume that the spine of $h'(M_{j+1})$ is in a very small neighborhood of the spine of $h'(M_j)$ for all $j$, we may assume that $R_a$ is close to a set $D_a \subset Z$ represented by $I_1 \times \cdots \times I_{n-3} \times I$, where $I_k$ is the interval in $S_k$ corresponding to $S_k^*$. The neighborhood $R_a'$ has a similar representation; see Figure 2.3. Clearly the set $D_a \subset Z$ $(D_a' \subset Z')$ is an $(n - 2)$-cell.

Since the spine of $h'(M_{j+1})$ is in a very small neighborhood of the spine of $h'(M_j)$, $j = 1, 2, \ldots$, we can assume that each generalized annulus $R_a$ ($R_a'$) is in a PL $n$-cell neighborhood, $B_a$ ($B_a'$), of the $(n - 2)$-cell, $D_a$ ($D_a'$), in $Z$ ($Z'$). (Consider a regular neighborhood of $D_a$ ($D_a'$).) Moreover, $B_a$ and $B_a'$ are in Int $A$. Since the handles of $A$ do not intersect $P_b$ and $P_c$ and $p_a$ ($p_a'$) is in a handle of $A_a$, $B_a$ ($B_a'$) does not intersect $P_b$ ($P_c$). With a minor adjustment (a slight rotation of the $R_a$ ($R_a'$) or a shortening of the interval factor of $R_a$ ($R_a'$),
we can assume that the $D_a$ ($D'_a$) and hence the $B_a$ ($B'_a$), for all $A_a$ components of $h'(M_{i+n-3}) \cap A$, form a pairwise disjoint collection. See Figure 2.3.

We can adjust $B_a$ and $X_a$ so that $B_a$ does not intersect the stem of $A_a$ for $\beta \neq \alpha$ and so that $X_a$ intersects $\text{Bd } B_a$ in a single point. We make the same adjustments for $X_a$ and $B'_a$. Then there exists a collection of pairwise disjoint open $n$-cells $(U_a)$ such that $(\text{stem } A_a) \cup B_a \cup B'_a \subset U_a \subset \text{Int } A$.

Now we define $g_1$. Let $g_1$ be a homeomorphism on $E^n$, that is the identity outside $h'(M_{i+n-3}) \cap A$ and between $P_b$ and $P_c$, that moves neighborhoods of the spines of the upper (lower) handles of $A_{a_1}, \ldots, A_{a_k}$ ($A'_{a_1}, \ldots, A'_{a_k-1}$) into a small neighborhood of $R_a$ ($R'_a$). This is possible since the handles of $A_{a_1}, \ldots, A_{a_k}$ are linked around the $S_{n-2}$-factor of $A_a$. We can assume that the parts of the stems of $g_1(A_{a_2}), \ldots, g_1(A_{a_k})$ above $P_b$ ($g_1(A'_{a_1}), \ldots, g_1(A'_{a_k-1})$ below $P_c$) are very close to $X_a$. (The small neighborhoods of $R_a$ ($R'_a$) are in $B_a$ ($B'_a$).) Note here that although

$$g_1(A_{a_2} \cup \cdots \cup A_{a_k-1}) \subset U_a,$$

$A_{a_1}$ or $A_{a_k}$ may have a handle that intersects $E^n - U_a$. 

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We define $g_2$ in pieces. If $P_a$ intersects $A_a$, then we obtain an $(n-1)$-cell $P^a$ as a slice of the stem of $A_a$ between $P_a$ and $P_b$. We choose $P^a$ sufficiently close to $P_b$ so that $P^a$ separates $P_a \cap A$ from $P_b$. Certainly the cell $P^a$ separates the upper handle of $A_a$ from the lower handle. Let $\hat{A}_a$ be the component of $A_a - P^a$ containing the upper handle of $A_a$ and let $\hat{A}_b$ be the component of $A_a - P^b$ containing the lower handle of $A_a$. We define a homeomorphism $f_a$ on $E^n$ such that $f_a = 1$ on $\hat{A}_a$, $f_a = 1$ outside $U_a$, and $f_a(A_a - \hat{A}_a) \supset N(X_a) \cup B_a$ for $N(X_a)$ a neighborhood of $X_a$ in $A_a$, by letting the stem of $A_a$ below $P^a$ "engulf" $B_a$. Then $f_a^{-1}$ pulls $B_a$ into the stem of $A_a$, below $P^a$, and hence below $P_a$. If $P_d$ intersects $A_a$, we define $f_d$ to have a similar effect on the lower handle of $A_a$. If $P_a$ ($P_d$) does not intersect $A_a$, then we let $f_a = 1$ ($f_d = 1$).

By combining $f_a^{-1}$ and $f_d^{-1}$, we obtain $g_a$ for each $A_a$ in $A$. We obtain $g_2$ by piecing together the $g_a$.

Now we consider $g = g_2g_1$. It is clear that $g = 1$ outside $A$, and between $P_b$ and $P_c$. The map $g_1$ acts on each component $A_a$ of $h'(M_{i+n-3}) \cap A$, moving the spines of the upper (lower) handles of $A_{a_1}, \ldots, A_{a_k}$ into a neighborhood of the upper (lower) node of the spine of $A_a$. The map $g_2$ shortens the spine of the stem of each $A_a$, if necessary, pulling the neighborhoods of the nodes of the spine of each $A_a$ beyond the outermost planes intersecting each $A_a$. Hence $g = g_2g_1$ carries each component of $h'(M_{i+n-2})$ in $A$ off one of the planes $P_a$ or $P_d$.

We now obtain that $P_a$, $P_b$, $P_c$, and $P_d$ be in good position relative to $g(A \cap h'(M_{i+n-2}))$. Let $A_{a_1}$ be a component of $A \cap h'(M_{i+n-2})$. Since $g_2g_1 = 1$ between $P_b$ and $P_c$, we have that each of $P_b \cap g_2g_1(A_{a_1})$ and $P_c \cap g_2g_1(A_{a_1})$ is an $(n-1)$-cell in the stem of $g_2g_1(A_{a_1})$. If $P_a$ ($P_d$) does not intersect $A$, then $P_a$ ($P_d$) does not intersect $g_2g_1(A_{a_1})$. If $P_a$ ($P_d$) does intersect $A$, then since the nodes of the spine of $g_1(A_{a_1})$ are close to the nodes of the spine of $A_a$, it follows that $P_a$ ($P_d$) does not intersect the stem of $g(A_{a_1} = g_2g_1(A_{a_1})$. (The map $g$ moves the upper node of the spine of $A_{a_1}$ "below" $P_a$.)

Then, since $P_b$ and $P_c$ do not intersect the handles of $g(A_{a_1})$, and the handles of $g(A_{a_1}) \cap g(M_{i+n-2+m})$, $m = 1, 2, \ldots$, are in the handles of $g(A_{a_1})$, the set of planes $P_a$, $P_b$, $P_c$, and $P_d$ is in good position relative to $g(A_{a_1})$.

2.2.3. Shrinking criterion. We now show that we have Bing's shrinking criterion [3].

**Lemma 2.5.** If $U$ is an open set in $E^n$ containing $H^* = \bigcup H$ and $\varepsilon$ is a positive number, then there is a homeomorphism $h$ of $E^n$ such that (1) $h = 1$ on $E^n - U$ and (2) for any $g \in H$, $\text{Diam}(h(g)) < \varepsilon$.

**Proof.** Since $H^* \subset U$, by a compactness argument, there exists an integer $j$ such that $M_j \subset U$. As we observed in §1, the handles must eventually get small, hence there is an integer $k > j$ such that the diameter of each handle in
$M_k$ is less than $\epsilon/4$. We will show that there is a finite set of $(n - 1)$-hyperplanes $P_1, \ldots, P_s$ that are in good position relative to $M_m$ for $m > k$ and such that each component of $M_k - \bigcup_{i=1}^s P_i$ has a diameter less than $\epsilon/4$.

We obtain the planes $P_1, \ldots, P_s$. The stem of each component $A$ of $M_k$ is of the form $\Phi_A(B^{n-1} \times I)$ where $B^{n-1}$ is the standard $(n - 1)$-cell, $I$ is the interval $[0, 1]$, and $\Phi_A$ is a homeomorphism. Moreover, there exists an integer $t_A > 0$ such that each component of $A - \bigcup_{i=0}^{t_A} \Phi(B^{n-1} \times i/t_A)$ has diameter less than $\epsilon/4$. As $M_k$ has only finitely many components, there exists an integer $s = \max\{t_A \mid A \text{ is a component of } M_k\}$. Let $P_1', \ldots, P_s'$ be $(n - 1)$-hyperplanes parallel to $\theta$ and intersecting the stem of $M_1$. We can assume that each plane $P_i'$ intersects the stem of each component of $M_k$ in exactly one $(n - 1)$-cell. (Consider $P_1', \ldots, P_s'$ to be located close to $\theta$. Let $f$ be a homeomorphism on $E^n$ so that for each component $A$ of $M_k$, $f(A)$ is in a close regular neighborhood $N(A)$ of the spine of $A$. We can obtain $f$ so that $f$ pulls $A$ into $N(A)$ radially. Since the stem spine of each $A$ intersects $\theta$ in a single point and the $P_i'$ are near $\theta$, we may assume that $N(A) \cap P_i' (i = 1, \ldots, s)$ is an $(n - 1)$-cell. We can replace $M_k$ by $(f^{-1}(A) \setminus A)$ is a component of $M_k$.) We index the planes $P_1', \ldots, P_s'$ so that they are in good position relative to $M_1$, and hence relative to $M_k$, $k > 1$.

There exists a homeomorphism $\Psi$ on $E^n$ such that $\Psi = 1$ outside $M_{k-1}$ and $\Psi(P_i') = P_i$ $(i = 1, \ldots, s)$ is the desired collection of hyperplanes. (For each component $A$ of $M_k$, $\Psi$ takes a subcollection of the $(n - 1)$-cells $P_i' \cap A$ onto the collection $\{\Phi_A(B^{n-1} \times i/t_A) \mid i = 0, \ldots, t_A\}$.)

From Lemmas 2.3, 2.4 there is a homeomorphism $h_1$ on $E^n$ such that $h_1 = 1$ outside $M_2$ and between $P_2$ and $P_{t_1}$ and $h_1$ takes each component of $M_{k+n-2}$ onto a set that intersects at most one of $P_1$ and $P_s$. Since $h_1$ preserves good position, we can apply Lemmas 2.3, 2.4 again to each component of $h_1(M_{k+n-2})$ using $P_1, P_2, P_{t_1}$, or $P_2, P_3, P_{t_2}$, or $P_3, P_{t_2}, P_s$, or $P_s$, $P_{t_3}$ to obtain a homeomorphism $h_2$ on $E^n$ such that $h_2 = 1$ outside $h_1(M_{k+n-2})$ and $h_2 h_1$ takes each component of $M_{k+2n-4}$ onto a set which intersects at most $s - 2$ of the planes $P_1, \ldots, P_s$. In $s - 3$ steps we observe that each component of $h_2 \cdot \cdots \cdot h_2 h_1(M_{k+(s-3)(n-2)})$ hits at most three of the hyperplanes $P_1, \ldots, P_s$ and hence has diameter less than $\epsilon$. Since every nondegenerate element of $G$ is a subset of $M_{k+(s-3)(n-2)}$, (2) is satisfied. Property (1) follows since $h$ is fixed outside $M_j$. The proof of Lemma 2.5 is completed.

**Bibliography**


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