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A SINGULAR SEMILINEAR EQUATION IN $L^1(\mathbf{R})$

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ABSTRACT. Let β be a positive and nondecreasing function on $\mathbf{R}$. The boundary-value problem $\beta(u) - u'' = f$, $u'(\pm\infty) = 0$ is considered for $f \in L^1(\mathbf{R})$. It is shown that this problem can have a solution only if β is integrable near $-\infty$, and that if this is the case, then the problem has a solution exactly when $\int_{-\infty}^{\infty} f(x)\,dx > 0$.

In [5, Lemma 5.6] T. Kurtz proves that the problem $e^u - u'' = f$ has a solution $u \in C^2(\mathbf{R})$ satisfying $u'(\pm\infty) \equiv \lim_{x\to\pm\infty} u'(x) = 0$, whenever f is nonnegative, continuous, compactly supported, and not identically equal to zero. Herein we study more general problems of the form

$$(\mathrm{P}_f) \qquad \beta(u) - u'' \ni f, \qquad u'(\pm\infty) = 0,$$

where β is a maximal monotone graph in $\mathbf{R}$ (see, for example, Brezis [2, §I.8]). In particular, β can be any continuous, nondecreasing function on $\mathbf{R}$. If $0 \in \operatorname{int}\beta(\mathbf{R})$, this problem is well understood; see Benilan, Brezis and Crandall [1] and Proposition 1 below. When $\beta(\mathbf{R}) \subseteq (0,\infty)$, as for the case $\beta(u) = e^u$, Kurtz's result is the only one known to the authors; and his methods depend very strongly on the explicit form of $\beta(u) = e^u$. We characterize those maximal monotone graphs β with $\beta(\mathbf{R}) \subseteq (0,\infty)$ for which (P_f) has a solution for *some* $f \in L^1(\mathbf{R})$, and then show that for such β (P_f) has a solution if and only if $\int_{-\infty}^{\infty} f(x)\,dx > 0$. Thus our conclusions are sharp as regards possible β *and* f in (P_f).

Let us be more precise. If β is any maximal monotone graph and $f \in L^1_{\mathrm{loc}}(\mathbf{R})$, by a *solution* of (P_f) we understand a function u such that u and u' are locally absolutely continuous on $\mathbf{R}$, $f(x) + u''(x) \in \beta(u(x))$ a.e., and $u'(\pm\infty) = 0$. We denote by $D(\beta)$ the domain of β and by β^0 the minimal section of β; the function β^0 assigns to $r \in D(\beta)$ the element in $\beta(r)$ of least modulus (so $\beta = \beta^0$ if β is single-valued).

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The main result is

THEOREM 1. *Suppose β is a maximal monotone graph in* $\mathbf{R}$ *with* $\beta(\mathbf{R}) \subseteq (0, \infty)$. *Then the following are equivalent*:

(i) *If* $f \in L^1(\mathbf{R})$, (P_f) *has a solution exactly when* $\int_{-\infty}^{\infty} f(x)\,dx > 0$.

(ii) *There exists some* $f \in L^1(\mathbf{R})$ *for which* (P_f) *has a solution.*

(iii) *There is an* $a \in \mathbf{R}$ *for which* $(-\infty, a) \subseteq D(\beta)$ *and* $\int_{-\infty}^{a} \beta^0(x)\,dx < \infty$.

This result is of interest because if (i), (ii), or (iii) holds, then the (possibly multivalued) mapping $f + u'' \mapsto -u''$, u the solution to (P_f), defines an accretive operator in $L^1(\mathbf{R})$: see Lemma 4(c). This operator generates a semigroup of contractions on a subset of $L^1(\mathbf{R})$ associated with the nonlinear partial differential equation $u_t - (\phi(u))_{xx} = 0$, for $\phi = \beta^{-1}$. (See, for example, [4, §3].)

To obtain Kurtz's result from Theorem 1 we need only note that $\int_{-\infty}^{0} e^x\,dx < \infty$, and so (iii) is valid for $\beta(x) = e^x$. And conversely if, for example, $\beta(x) \geqslant -1/x$ for large negative x, the equivalence of (ii) and (iii) implies that (P_f) does not have a solution for any $f \in L^1(\mathbf{R})$.

PROOF OF THEOREM 1. We prove Theorem 1 by establishing (i) ⇒ (iii) ⇒ (ii) ⇒ (i), in that order; the implications are arranged in increasing levels of difficulty. We begin with some simple remarks.

Let β satisfy the assumption of Theorem 1. Define

$$L^1(\mathbf{R})_+ = \left\{ f \in L^1(\mathbf{R}) \,\middle|\, \int_{-\infty}^{\infty} f(x)\,dx > 0 \right\}.$$

We note first of all that $f \in L^1(\mathbf{R})_+$ is a necessary condition for the solvability of (P_f). If u solves (P_f), then $f + u'' \in \beta(u)$ implies $f + u'' > 0$ a.e. and so

$$0 < \int_{-\infty}^{\infty} f(x) + u''(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x) + u''(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx,$$

since $\lim_{R\to\infty} u'(\pm R) = 0$. Moreover this same calculation and Fatou's Lemma imply $f + u'' \in L^1(\mathbf{R})$ and $\|f + u''\|_1 \leqslant \|f\|_1$ ($\|\ \|_p$ denoting the norm in $L^p(\mathbf{R})$, $1 \leqslant p \leqslant \infty$). Thus if u is a solution of (P_f), $u'' \in L^1(\mathbf{R})$ and $\|u''\|_1 \leqslant 2\|f\|_1$.

DEFINITION. $\mathfrak{L}$ is the linear subspace of functions u defined on $\mathbf{R}$ such that u and u' are locally absolutely continuous, $u'' \in L^1(\mathbf{R})$, and $u'(\pm\infty) = 0$.

We have proved that if u solves (P_f) for some $f \in L^1(\mathbf{R})$, then $f \in L^1(\mathbf{R})_+$, $u \in \mathfrak{L}$, and $\|u''\|_1 \leqslant 2\|f\|_1$, $\|f + u''\|_1 \leqslant \|f\|_1$. Also $u \in \mathfrak{L}$ clearly implies $\|u'\|_\infty \leqslant \|u''\|_1$.

PROOF OF (i) ⇒ (iii). If $f \in L^1(\mathbf{R})_+$ and u solves (P_f), then $u' \in L^\infty(\mathbf{R})$ by the preceding; and so there is a positive constant c such that $u(x) \geqslant cx$ for $x \leqslant -1$. Furthermore,

$$f(x) + u''(x) \geqslant \beta^0(u(x)) \geqslant \beta^0(cx) > 0$$

a.e. for $x < -1$, since β^0 is positive and nondecreasing. Therefore

$$\|f\|_1 \geqslant \|f + u''\|_1 \geqslant \int_{-\infty}^{-1} \beta^0(cx)\,dx = \frac{1}{c}\int_{-\infty}^{-c} \beta^0(y)\,dy;$$

and (iii) follows.

PROOF OF (iii) ⇒ (ii). This is a bit more subtle. Suppose $\int_{-\infty}^{a} \beta^0(x)\,dx < \infty$. We claim there is a continuously differentiable function $g\colon (-\infty, a] \to \mathbf{R}$ satisfying

(1) $g \geqslant 1$, g nonincreasing, $\lim_{x\to-\infty} g(x) = \infty$, $\int_{-\infty}^{a} \beta^0(x)g(x)\,dx < \infty$.

Let us for the moment assume that such a g exists. Define $v\colon (-\infty,-1] \to \mathbf{R}$ by

$$\begin{cases} v'(x) = 1/g(v(x)), & x < -1, \\ v(-1) = a - 1. \end{cases}$$

Since g is positive, nonincreasing, and continuously differentiable, v is increasing, convex, and twice continuously differentiable. In addition, it is clear that $v(x) \to -\infty$ as $x \to -\infty$, because g is bounded above on compact sets. Since $g(x) \to \infty$ when $x \to -\infty$, $v'(-\infty) = 0$. Moreover

$$v'' \in L^1(-\infty,-1)$$

and

$$\int_{-\infty}^{-1} \beta^0(v(x))\,dx = \int_{-\infty}^{a-1} \beta^0(y)\frac{1}{v'(v^{-1}(y))}\,dy = \int_{-\infty}^{a-1} \beta^0(y)g(y)\,dy < \infty.$$

Let u be any even, twice continuously differentiable function on $\mathbf{R}$ which satisfies $u(x) = v(x)$ for $x < -1$ and $u < a$ everywhere. Then, by the construction, $f(x) \equiv u''(x) + \beta^0(u(x)) \in L^1(\mathbf{R})$ and $u'(\pm\infty) = 0$, u is a solution of (P_f).

It remains to prove the existence of g with the properties (1). Select a sequence $\{a_n\}_{n=1}^{\infty}$ which satisfies $a_n < a_{n-1} < a$ for $n = 1, 2, \ldots$ and $\int_{-\infty}^{a_n} \beta^0(x)\,dx < 1/n^2$. Now take g to be any nonincreasing continuously differentiable function so that $g(a_n) = \sqrt{n}$, $n = 1, 2, \ldots$, and $g = 1$ on $[a_1, a]$. Then

$$\int_{-\infty}^{a} \beta^0(x)g(x)\,dx = \int_{a_1}^{a} \beta^0(x)\,dx + \sum_{n=1}^{\infty} \int_{a_{n+1}}^{a_n} \beta^0(x)g(x)\,dx$$
$$\leqslant \int_{a_1}^{a} \beta^0(x)\,dx + \sum_{n=1}^{\infty} \sqrt{n+1} \int_{a_{n+1}}^{a_n} \beta^0(x)\,dx$$
$$\leqslant \int_{a_1}^{a} \beta^0(x)\,dx + \sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n^2} < \infty;$$

g has the desired properties.

PROOF OF (ii) $\Rightarrow$ (i). This implication is the most difficult and its proof requires several steps. The lemmas following outline the program.

LEMMA 1. *Let* $f, g \in L^1(\mathbf{R})_+$ *and* $\int_{-\infty}^{\infty} f(x)\,dx > \int_{-\infty}^{\infty} g(x)\,dx$. *If* (P_g) *has a solution, then so does* (P_f).

LEMMA 2. *If* (ii) *holds, then*

$$\left\{ f \in L^1(\mathbf{R})_+ \,|\, \exists g \in L^1(\mathbf{R})_+, \int_{-\infty}^{\infty} f(x)\,dx > \int_{-\infty}^{\infty} g(x)\,dx, \text{ and } (\mathrm{P}_g) \text{ has a solution} \right\} = L^1(\mathbf{R})_+.$$

The combined implications of Lemmas 1–2 prove that (ii) $\Rightarrow$ (i). If (ii) is valid, Lemmas 1 and 2 demonstate that (P_f) has a solution for all $f \in L^1(\mathbf{R})_+$. Again we prove these results in order of ascending difficulty.

PROOF OF LEMMA 2. Choose $f \in L^1(\mathbf{R})$ so that (P_f) has a solution u; by (ii) there is at least one such f (and in fact $f \in L^1(\mathbf{R})_+$). Now for fixed $\varepsilon > 0$ we prove that there is some $g \in L^1(\mathbf{R})_+$, $\|g\|_1 \leqslant \varepsilon$, for which (P_g) also has a solution. If δ, $M > 0$, define $u_{\delta,M}(x) \equiv u(\delta x) - M$. Then $u_{\delta,M}$ solves $(\mathrm{P}_{f_{\delta,M}})$, where

$$f_{\delta,M}(x) \equiv \beta^0(u_{\delta,M}(x)) - (u_{\delta,M})''(x) = \beta^0(u(\delta x) - M) - \delta^2 u''(x).$$

We have $\|u''_{\delta,M}\|_1 = \delta\|u''\|_1 \leqslant \varepsilon/2$ for a fixed δ small enough. Moreover $\lim_{M\to\infty} \beta^0(u(\delta x) - M) = 0$ since $\beta^0(x) \to 0$ as $x \to -\infty$ (otherwise (ii) could not hold). By the Dominated Convergence Theorem we can choose M so large that $\|\beta^0(u(\delta x) - M)\|_1 \leqslant \varepsilon/2$. Then $g \equiv f_{\delta,M}$ satisfies $\|g\|_1 \leqslant \varepsilon$.

Therefore (P_g) has a solution for g's with arbitrarily small L^1-norm. Now take any $f \in L^1(\mathbf{R})_+$ and let g be as above and satisfy $\int_{-\infty}^{\infty} f(x)\,dx > \|g\|_1$. Then $\int_{-\infty}^{\infty} f(x)\,dx > \int_{-\infty}^{\infty} g(x)\,dx$. The proof is complete.

For the proof of Lemma 1 we require another Lemma 3(a) below. (Parts (b) and (c) are included for interest's sake.)

LEMMA 3. (a) *Let* $v \in \mathfrak{L}$, *and* $p \in L^\infty(\mathbf{R})$ *be locally Lipschitz continuous and nondecreasing. Then* $p'(v)v'^2 \in L^1(\mathbf{R})$ *and*

$$\int_{-\infty}^{\infty} p(v(x))v''(x) + p'(v(x))v'(x)^2\,dx = 0.$$

(b) *Let*

$$\text{Sign}\ r = \begin{cases} \{1\}, & r > 0, \\ [-1, 1]; & r = 0, \\ \{-1\}, & r < 0. \end{cases}$$

If $a \in L^\infty(\mathbf{R})$, $v \in \mathfrak{L}$, *and* $a(x) \in \text{Sign}\ v(x)$ *a.e., then* $\int_{-\infty}^{\infty} v'')a(x)\,dx \leqslant 0$.

(c) *If* $f, \hat{f} \in L^1(\mathbf{R})_+)u, \hat{u}$ *are solutions of* (P_f) *and* $(\mathrm{P}_{\hat{f}})$, *respectively, then* $\|(f + u'') - (\hat{f} + \hat{u}'')\|_1 \leqslant \|f - \hat{f}\|_1$.

Proof of Lemma 3. We adapt arguments used in [1] and [3] to this simple case. If $R > 0$, then

$$\int_{-R}^{R} p(v(x))v''(x) + p'(v(x))v'(x)^2\,dx = p(v(R))v'(R) - p(v(-R))v'(-R).$$

Since $p \in L^\infty(\mathbf{R})$ and $v'(\pm\infty) = 0$, (a) follows from Fatou's Lemma by letting $R \to \infty$ above.

To obtain (b), apply (a) with $p(s) = p_n(s) = p_0(ns)$, where $p_0(s) = s$ for $|s| \leqslant 1$ and $p_0(s) = \text{sign}\,s$ for $|s| \geqslant 1$. Then by (a) $\int_{-\infty}^{\infty} p_n(v)v''\,dx \leqslant 0$. But $p_n(v) \to \text{sign}_0(v)$, where $\text{sign}_0\, s = \text{sign}\,s$ for $s \neq 0$, $\text{sign}_0\, 0 = 0$. Therefore we can send $n \to \infty$ to conclude

$$\int_{[v>0]} v''(x)\,dx - \int_{[v<0]} v''(x)\,dx \leqslant 0$$

($[v > 0] \equiv \{x | v(x) > 0\}$, etc.). Finally $v'(x) = 0$ a.e. on $[v = 0]$ and so $v''(x) = 0$ a.e. on this set (the derivative of any absolutely continuous function v vanishes a.e. on $[v = c]$ for any $c \in \mathbf{R}$). If $a(x) \in \text{Sign}\,v(x)$ a.e., we therefore have

$$\begin{aligned}\int_{-\infty}^{\infty} a(x)v''(x)\,dx &= \int_{[v>0]} v''(x)\,dx - \int_{[v<0]} v''(x)\,dx + \int_{[v=0]} a(x)v''(x)\,dx \\ &= \int_{[v>0]} v''(x)\,dx - \int_{[v<0]} v''(x)\,dx \leqslant 0.\end{aligned}$$

(It is not hard to prove that equality actually holds.) To prove (c) let

$$a(x) = \begin{cases} 1 & \text{on } [f + u'' > \hat{f} + \hat{u}''] \cup [u > \hat{u}], \\ 0 & \text{on } [f + u'' = \hat{f} + \hat{u}''] \cap [u = \hat{u}], \\ -1 & \text{on } [f + u'' < \hat{f} + \hat{u}''] \cup [u < \hat{u}]. \end{cases}$$

Then a is well defined since β is monotone, $a(x) \in \text{Sign}(u - \hat{u})(x)$ a.e., and $a(f + u'' - (\hat{f} + \hat{u}'')) = |f + u'' - (\hat{f} + \hat{u}'')|$ a.e. By (b)

$$\|f + u'' - (\hat{f} + \hat{u}'')\|_1 = \int_{-\infty}^{\infty} a(f - \hat{f})\,dx + \int_{-\infty}^{\infty} a(u - \hat{u})''\,dx$$
$$\leqslant \int_{-\infty}^{\infty} a(f - \hat{f})\,dx \leqslant \|f - \hat{f}\|_1,$$

and (c) is proved.

PROOF OF LEMMA 1. Suppose $f, g \in L^1(\mathbf{R})_+$ and $\int_{-\infty}^{\infty} f(x)\,dx > \int_{-\infty}^{\infty} g(x)\,dx$. Assume (P_g) has a solution. To prove that then (P_f) has a solution we employ the following result of Benilan, Brezis and Crandall [1, §4]:

PROPOSITION 1. *Suppose* γ *is a maximal monotone graph in* $\mathbf{R}$ *with* $0 \in \gamma(0)$ *and* $0 \in \operatorname{int}\gamma(\mathbf{R})$. *Then for every* $f \in L^1(\mathbf{R})$ *there is a function* v *such that*

(a) $v, v' \in L^\infty(\mathbf{R})$ *and* $v'' \in L^1(\mathbf{R})$,
(b) $f(x) + v''(x) \in \gamma(v(x))$ *a.e.*,
(c) $v'(\pm\infty) = 0$, $\|v'\|_\infty \leqslant \|v''\|_1 \leqslant 2\|f\|_1$.

REMARK 1. At this point there is a discontinuity in our presentation: except for Proposition 1 the discussion does not assume the reader to be familiar with [1] or [3]. The interested reader should attempt to prove Proposition 1 for himself, at least for the special case when γ is continuous. (This one-dimensional proposition does not require the machinery of [1].)

Proposition 1 allows us to solve as follows certain problems approximating (P_f).

For $0 < \lambda < \sup \beta(\mathbf{R})$ there is a number $r_\lambda \in D(\beta)$ with $\lambda \in \beta(r_\lambda)$. Set $\beta^\lambda(x) \equiv \beta(x + r_\lambda) - \lambda$; then β^λ satisfies the assumptions on γ in Proposition 1. And so there exists a w_λ satisfying (a), (b), (c), with β^λ in place of γ. Define $u_\lambda \equiv w_\lambda + r_\lambda$. Then we have

(2)
$$\text{(a) } u_\lambda, u_\lambda' \in L^\infty(\mathbf{R}), \quad u_\lambda'' \in L^1(\mathbf{R}),$$
$$\text{(b) } f(x) + u_\lambda''(x) \in \beta^\lambda(w_\lambda(x)) = \beta(u_\lambda(x)) - \lambda \quad \text{a.e.},$$
$$\text{(c) } u_\lambda'(\pm\infty) = 0, \quad \|u_\lambda'\|_\infty \leqslant \|u_\lambda''\|_1 \leqslant 2\|f\|_1.$$

The solution u of (P_f) will be constructed as the limit of the u_λ as $\lambda \searrow 0$. First we show the u_λ decreases as λ decreases. Let p be a smooth, nondecreasing function defined on $\mathbf{R}$ such that $p(x) = 0$ for $x \geqslant 0$, $p(x) < 0$ for $-1 < x < 0$, $p(x) = -1$ for $x \leqslant -1$. Now $(u_\lambda - u_\eta)'' \in (\beta(u_\lambda) - \beta(u_\eta)) + \eta - \lambda$; and so, by the monotonicity of β,

$$p(u_\lambda - u_\eta)(u_\lambda - u_\eta)'' \geqslant (\eta - \lambda)p(u_\lambda - u_\eta).$$

Lemma 3(a) implies $\int_{-\infty}^{\infty} p(u_\lambda - u_\eta)(u_\lambda - u_\eta)''\,dx \leqslant 0$. Letting $\lambda > \eta$ we conclude that $u_\lambda \geqslant u_\eta$ a.e.

To discover a (pointwise) lower bound for the u_λ we recall that the problem (P_g) has a solution v:

$$(P_g) \qquad g(x) + v(x)'' \in \beta(v(x)) \quad \text{a.e.}, \qquad v'(\pm\infty) = 0.$$

As in the preceding we construct approximate functions v_λ which satisfy conditions like (2), with g replacing f. The v_λ, like the u_λ, decrease as $\lambda \searrow 0$. In addition, the v_λ are bounded from below by v; this is proved by the same method as above.

We claim that there is some $x_0 \in \mathbf{R}$ such that $\{u_\lambda(x_0)\}$ is bounded. If not, then $u_\lambda(x) \to -\infty$ as $\lambda \searrow 0$ for every $x \in \mathbf{R}$. Subtract the equation satisfied by v_λ from that satisfied by u_λ:

$$(3) \qquad f(x) - g(x) + (u_\lambda(x) - v_\lambda(x))'' \in \beta(u_\lambda(x)) - \beta(v_\lambda(x)).$$

Multiply this by $p(u_\lambda(x) - v_\lambda(x))$ (p as defined above), recall the monotonicity of β, and integrate:

$$\int_{-\infty}^{\infty} (f(x) - g(x))p(u_\lambda(x) - v_\lambda(x)) + (u_\lambda(x) - v_\lambda(x))''p(u_\lambda(x) - v_\lambda(x))\,dx \geqslant 0.$$

By Lemma 3(a), we have

$$(4) \qquad \int_{-\infty}^{\infty} (f(x) - g(x))p(u_\lambda(x) - v_\lambda(x))\,dx \geqslant 0.$$

For fixed x, $u_\lambda(x) \to -\infty$ and $v_\lambda(x)$ is bounded; therefore $p(u_\lambda(x) - v_\lambda(x)) \to -1$. So the Dominated Convergence Theorem applied to (4) leads to $\int_{-\infty}^{\infty} (g(x) - f(x))\,dx \geqslant 0$. However this contradicts the assumption on f and g. Hence there is some x_0 for which $\{u_\lambda(x_0)\}$ is bounded; and this implies, since $\|u'_\lambda\|_\infty \leqslant 2\|f\|_1$, that the u_λ are bounded uniformly on compact sets. They thus converge monotonically and uniformly on compact sets to a limit $u \equiv \lim_{\lambda\searrow 0} u_\lambda$.

Furthermore $u_\lambda(x)'' + \lambda + f(x) \in \beta(u_\lambda(x))$ and $u_\eta(x)'' + \eta + f(x) \in \beta(u_\eta(x))$ a.e. implies $u''_\lambda + \lambda \leqslant u''_\eta + \eta$ if $u_\lambda < u_\eta$. Since $u''_\lambda = u''_\eta$ a.e. on $[u_\lambda = u_\eta]$, $u''_\lambda + \lambda \leqslant u''_\eta + \eta$ a.e. Also $u''_\lambda(x) + \lambda > -f(x)$ a.e. because $0 < \beta^0(u_\lambda(x)) \leqslant u_\lambda(x)'' + \lambda + f(x)$. It follows that the u''_λ converge in $L^1_{\text{loc}}(\mathbf{R})$ to u'' as $\lambda \searrow 0$, and therefore that $f + u'' \in \beta(u)$ a.e.

We must show that $u'(\pm\infty) = 0$. Since $\|u''_\lambda\|_1 \leqslant 2\|f\|_1$ by (2), Fatou's Lemma implies $u'' \in L^1(\mathbf{R})$, and therefore $u'(+\infty)$ and $u'(-\infty)$ exist. It suffices to prove that $u'(-\infty) = 0$, the same equality for $u'(+\infty)$ following by similar arguments. Since $u \leqslant u_\lambda$ and $u'_\lambda(-\infty) = 0$, $u'(-\infty) \geqslant 0$. We multiply both sides of (3) by $p(u_\lambda - v_\lambda)$ as before and integrate:

$$\int_{-\infty}^{y} (v_\lambda(x) - u_\lambda(x))'' p(u_\lambda(x) - v_\lambda(x))\,dx$$
$$\leqslant \int_{-\infty}^{y} (f(x) - g(x)) p(u_\lambda(x) - v_\lambda(x))\,dx$$
$$\leqslant \int_{-\infty}^{y} |f(x) - g(x)|\,dx.$$

Integrate by parts on the left and recall that $u'_\lambda(-\infty) = v'_\lambda(-\infty) = 0$:

$$[v'_\lambda(y) - u'_\lambda(y)]p(u_\lambda(y) - v_\lambda(y)) \leqslant \int_{-\infty}^{y} |f(x) - g(x)|\,dx. \tag{5}$$

Since $u''_\lambda \to u''$ in $L^1_{\text{loc}}(\mathbf{R})$, $u'_\lambda \to u'$ in $C(\mathbf{R})$; and similarly for the v_λ. So for every y we can pass to the limit as $\lambda \searrow 0$ in (5) to deduce

$$[v'(y) - u'(y)]p(u(y) - v(y)) \leqslant \int_{-\infty}^{y} |f(x) - g(x)|\,dx. \tag{6}$$

Suppose that $u'(-\infty) > 0$. Then for all y less than some number, $u(y) < v(y) - 1$ and so $p(u(y) - v(y)) = -1$. Thus sending $y \to -\infty$ in (6) implies $u'(-\infty) \leqslant v'(-\infty) = 0$, a contradiction. Therefore $u'(-\infty) = 0$, and the proof is complete.

REMARK 2. We record some additional facts about solutions u of (P_f) and the map $f \in L^1(\mathbf{R})_+ \mapsto Tf = f + u''$. First, T is a contraction by Lemma 3(c). Next, if u is a solution of (P_f), then $u(\pm\infty) = -\infty$. Indeed, if there is a sequence x_n, $|x_n| \to \infty$ and $u(x_n) \geqslant -A$ for some A, then $u(x) \geqslant -A - \|u'\|_\infty$ on $|x - x_n| \leqslant 1$ and measure$([u(x) \geqslant -A - \|u'\|_\infty]) = \infty$. But $\beta^0(u(x)) \geqslant \beta^0(-A - \|u'\|_\infty) > 0$ on this set, contradicting $\beta^0(u(x)) \in L^1(\mathbf{R})$. Second, if u and $\hat{u}$ are solutions of (P_f), then $Tf = f + u'' = f + \hat{u}''$ implies $u' - \hat{u}'$ is a constant. Since $u'(\pm\infty) = \hat{u}'(\pm\infty)$, $u' = \hat{u}'$. Thus $u = \hat{u} + c$ for some $c \in \mathbf{R}$, $c \geqslant 0$ without loss of generality. Now $Tf(x) \in \beta(\hat{u}(x)) \cap \beta(\hat{u}(x) + c)$ a.e. Since $\hat{u}(x) \to -\infty$ as $|x| \to \infty$, we can choose x so that $u(x)$ is a point of strict increase of β^0, $\beta^0(\hat{u}(x)) < \beta^0(\hat{u}(x) + r)$ for $r > 0$. For this x we conclude that $c = 0$. Finally, if $f, \hat{f} \in L^1(\mathbf{R})_+$, then

$$\int_{-\infty}^{\infty} (Tf - T\hat{f})^+\,dx \leqslant \int_{-\infty}^{\infty} (f - \hat{f})^+\,dx, \tag{7}$$

$$m \leqslant f \leqslant M \text{ a.e. implies } m \leqslant Tf \leqslant M \text{ a.e.,} \tag{8}$$

and

$$f \in L^1(\mathbf{R})_+ \text{ implies } \int_{-\infty}^{\infty} j(Tf)\,dx \leqslant \int_{-\infty}^{\infty} j(f)\,dx \tag{9}$$

for every convex lower-semicontinuous function $j\colon \mathbf{R} \to [0, \infty]$ satisfying $j(0) = 0$. The estimates (7) (which imply that T is order preserving) and (8) may be proved directly in a fashion similar to Lemma 3. Alternatively, according to [1], (7), (8) and (9) hold for the mappings $T_\lambda: f \to f + u''_\lambda$, where u_λ is as in (2), and one just lets λ tend to zero. Also, (7) and (8) imply (9) by results of [3].

Added in proof. In a paper to appear in the Israel Journal of Mathematics, S. Fisher shows (among other things) that Theorem 1 remains correct if $\beta \in C(\mathbf{R})$; $\beta(-\infty) = 0$, $\beta > 0$ and $\beta \notin L^1(\mathbf{R})$. We also thank Professor Fisher for a useful remark.

References

1. Ph. Benilan, H. Brezis and M. G. Crandall, *A semilinear elliptic equation in* $L^1(\mathbf{R}^N)$, MRC Technical Summary Report #1526; Ann. Scuola Norm. Sup. Pisa (4) **2** (1975), 521–555.

2. H. Brezis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espace de Hilbert*, North-Holland Studies, no. 5, North-Holland, Amsterdam, 1973. MR **50** #1060.

3. H. Brezis and W. Strauss, *Semi-linear second-order elliptic equations in* L^1, J. Math. Soc. Japan **25** (1973), 565–590. MR **49** #826.

4. M. G. Crandall, *An introduction to evolution governed by accretive operators*, Dynamical Systems: An International Symposium, Vol. 1, L. Cesari, J. K. Hale, J. P. LaSalle, Editors, Academic Press, New York, 1976, 131–165.

5. T. Kurtz, *Convergence of sequences of semigroups of nonlinear operators with an application to gas kinetics*, Trans. Amer. Math. Soc. **186** (1973), 259–272. MR **49** #1256.

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