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HIGHER ALGEBRAIC K-THEORIES

BY

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ABSTRACT. A homotopy fibration is established relating the Volodin or BN-pair definition of algebraic K-theory to the theory defined by Quillen.

In [2] we outlined the construction of natural homomorphisms

$$K_*^Q \to K_*^{BN} \to K_*^V \to K_*^{KV}$$

between higher algebraic K-theories K_*^Q of [10] and [11], K_*^{BN} of [17], K_*^V of [16], and K_*^{KV} of [7] and [8]. This was one of the steps in proving the various definitions of higher K-theory are equivalent. It turns out they all agree—including the theory K_*^S of [14], [5], and [8]—provided one restricts to the category of regular rings when using K_*^{KV}. See [1], [2], [5], [8] and [18]. The purpose of this paper is to prove the following theorem, announced in [2], which yields the construction of $K_*^Q \to K_*^{BN}$.

THEOREM. *For any associative ring with identity A*

$$GL^{BN}(A) \to B\{U_F\}^+ \to BGL(A)^+$$

is a homotopy fibration.

For the reader's convenience and because the presentation of the BN-pair K-theory K_*^{BN} used here is slightly different from that of [17], we shall briefly recall the definition of GL^{BN} and $B\{U_F\}$ in the first section.

1. Preliminaries. Let $\{H_\alpha\}$ be the collection of hyperplanes in n-dimensional euclidean space R^n given by the condition $\alpha = 0$ where $\alpha = e_i - e_j$, $i \neq j$, is a linear root. Here e_i is the ith coordinate function. This determines a stratification of R^n whose strata F we call facettes as in [3]. By definition a facette of codimension k is a component of the complement in the union of the k-fold intersections of the H_α of the subset consisting of the union of the $(k+1)$-fold intersections. Let P^n be the set of facettes of R^n partially ordered by the condition that $F < G$ iff $F \subset \overline{G}$. We shall also let P^n denote the simplicial complex whose k-simplices are $(k+1)$-tuples $(F_0 < \cdots < F_k)$ where $F_i \in P^n$. P^n is a piecewise linear triangulation of the standard $(n-1)$-simplex. The stabilization map $R^n \to R^{n+1}$ defined by

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$$(*)\qquad (x_1,\ldots,x_n) \to (x_1,\ldots,x_n,x_n)$$

takes each facette F to a facette F' and preserves the relation "$<$". Thus we can consider P^n as a subset (or subcomplex) of P^{n+1} and we let $P^\infty = \bigcup_n P^n$. If $F \in P^n$, let $U_F \subset GL(n,A)$ be the subgroup generated by the elementary matrices $e_{ij}(\lambda)$ where $e_i - e_j > 0$ on F and $\lambda \in A$. Note that if $F \in P^\infty$ lies in P^n, then $U_F \subset GL(\infty,A)$ is the direct limit $U_F \to U_{F'} \to U_{F''} \to \cdots$.

Now for $1 \leqslant n \leqslant \infty$ let B_n be the realization of the simplicial space which in dimension $k \geqslant 0$ is the disjoint union of the spaces $(F_0 < \cdots < F_k) \times BU_{F_0}$ where $F_i \in P^n$. Then $B_\infty = \lim_{n\to\infty} B_n$ and by definition we let $B\{U_F\} = B_\infty$. The inclusions $BU_F \subset BGL(n,A)$ induce a map

$$B_n \to BGL(n,A)$$

for $1 \leqslant n \leqslant \infty$. Recall from [2] that $\pi_1 B\{U_F\} = St(A)$ and $\pi_1 B\{U_F\} \to \pi_1 BE(A)$ is just $St(A) \to E(A)$.

If $\alpha \cdot U_F$ and $\beta \cdot U_G$ are two left cosets in $GL(n,A)$ define

$$\alpha \cdot U_F < \beta \cdot U_G$$

to mean $F < G$ and $\alpha \cdot U_F \subset \beta \cdot U_G$. For $2 \leqslant n \leqslant \infty$ define G_n (resp. E_n) to be the simplicial complex where k-simplices are $(k+1)$-tuples

$$(\alpha_0 \cdot U_{F_0} < \cdots < \alpha_k \cdot U_{F_k})$$

where $F_i \in P^n$ and $\alpha_i \in GL(n,A)$, respectively $\alpha_i \in E(n,A) =$ the subgroup of elementary metrices. We have $G_\infty = \operatorname{ind\,lim}_n G_n$ and $E_\infty = \operatorname{ind\,lim}_n E_n$ and by definition we set

$$GL^{BN}(A) = G_\infty \text{ and } E^{BN}(A) = E_\infty.$$

The group $GL(n,A)$ acts on G_n by left multiplication: $\alpha \cdot (\alpha_0 \cdot U_{F_0} < \cdots < \alpha_k \cdot U_{F_k}) = (\alpha\alpha_0 \cdot U_{F_0} < \cdots < \alpha\alpha_k \cdot U_{F_k})$ and this restricts to an action of $E(n,A)$ on E_n. Moreover, $\pi_0 GL^{BN}(A) = K_1(A)$ and $GL^{BN}(A) = K_1(A) \times E^{BN}(A)$. See [17].

Now let $G = E(A)$ and define $E\{\alpha \cdot U_F\}$ to be the pullback of the diagram

$$\begin{tikzcd}
E\{\alpha \cdot U_F\} \arrow[r, dashed, "i"'] \arrow[d, dashed] & EG \arrow[d, "\pi"] \\
B\{U_F\} \arrow[r, "j"'] & BG
\end{tikzcd}$$

Here we let EG be the realization of the simplicial set whose k-simplices are $(k+1)$-tuples $(g_0, \ldots, g_k)$. The universal principal G-bundle $\pi: EG \to BG$ is defined by

$$\pi(g_0, \ldots, g_k) = (g_0^{-1} g_1, \ldots, g_{k-1}^{-1} g_k).$$

See [2]. Let $E(\alpha \cdot U_F) \subset EG$ denote the contractible subcomplex whose k-simplices are those $(k+1)$-tuples for which $g_i \in \alpha \cdot U_F$ for $0 \leqslant i \leqslant k$. Then $E\{\alpha \cdot U_F\}$ is the realization of the simplicial space which in dimension $k \geqslant 0$ is the disjoint union of the spaces

$$(\alpha_0 \cdot U_{F_0} < \cdots < \alpha_k \cdot U_{F_k}) \times E(\alpha_0 \cdot U_{F_0})$$

where $F_i \in P^\infty$. Hence by [10, Lemma for Theorem B] the natural map $E\{\alpha \cdot U_F\} \to E^{BN}(A)$ is a homotopy equivalence. Since EG is contractible the "nine-lemma" [15] implies

$$E\{\alpha \cdot U_F\} \to B\{U_F\} \to BG$$

is a homotopy fibration and so

$$E^{BN}(A) \to B\{U_F\} \to BG$$

is a homotopy fibration. Similarly letting $G = GL(A)$ there is a homotopy fibration

$$GL^{BN}(A) \to B\{U_F\} \to BGL(A).$$

We must show this remains a fibration when the "plus-construction" is performed on the second and third spaces. Since the universal cover of $BGL(A)^+$ is $BE(A)^+$ and since $\pi_1 BGL(A)^+ = K_1(A)$, to prove the main theorem it suffices to prove

THEOREM 1. *For any associative ring with identity A*

$$E^{BN}(A) \to B\{U_F\}^+ \to BE(A)^+$$

is a homotopy fibration.

The idea of the proof is to consider the diagram

$$\begin{tikzcd} E^{BN}(A) \arrow[r] \arrow[d] & B\{U_F\} \arrow[r] \arrow[d] & BE(A) \arrow[d] \\ X \arrow[r] & B\{U_F\}^+ \arrow[r, "j"'] & BE(A)^+ \end{tikzcd}$$

where X is the homotopy theoretic fiber of the map j. Suppose we can verify that:

(I) $E^{BN}(A)$ is a connected H-space such that $E(A) = \pi_1 BE(A)$ acts trivially on $H_*(E^{BN}(A))$, and

(II) X is a connected H-space.

Note that $BE(A)^+$ is simply connected and so its fundamental group acts trivially on $H_*(X)$. Then since the "plus-construction" preserves homology, the Comparison Theorem for the spectral sequence of a fibration [9] implies $E^{BN}(A) \to X$ is a homology equivalence. Hence it is a homotopy equivalence by [4, Lemma 6.2]. Condition (I) will be established in §2 and §3; (II) will be shown in §4 by seeing that $B\{U_F\}^+ \to BE(A)^+$ is an H-map and so by the proof of Theorem 2 of [13] its homotopy fiber is an H-space.

For convenience we state the following lemma of [18, Lemma 3.3]. Let K denote a partially ordered set and also the corresponding simplicial complex. Let f: $K \to P^n$ be a map of partially ordered sets. Thus for each vertex v of K, $f(v)$ is a facette of R^n. Now let g: $K \to P^n$ be any map of sets (not necessarily order preserving). Give $K \times I$ the standard triangulation as a partially ordered set where $I = \{0, 1\}$ with $0 < 1$. Together f and g define a map

$$\{\text{vertices of } K \times I\} \to P^n$$

which does not necessarily preserve order except on $K \times 0$.

LEMMA. *There is a triangulation $(K \times I)'$ of the simplicial complex $K \times I$ as a partially ordered set which refines the standard triangulation of $K \times I$ leaving $K \times 0$ unchanged, and there is an order preserving map w: $(K \times I)' \to P^n$ of the vertices of this new triangulation such that*

(a) $w|K \times 0 = f$;

(b) *if v is a vertex of $K \times 1$ in the new and also in the old triangulation, then $w(v) = g(v)$;*

(c) *if g: $K \to P^n$ is order preserving, then $K \times 1$ with the new triangulation is just a copy of K;*

(d) *if $\sigma = (v_0 < \cdots < v_k)$ is a simplex of the standard triangulation of $K \times I$, v is a vertex in σ of the new triangulation, and $e_{ij}(\lambda)$ lies in $U_{w(v_s)}$ for $0 \leqslant s \leqslant k$, then*

$$e_{ij}(\lambda) \in U_{w(v)}.$$

2. H-**space structure on** $E^{BN}(A)$. In this section we show the direct sum homomorphism

$$\alpha \oplus \beta = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

from $E(m,A) \times E(n,A)$ to $E(m+n,A)$ induces an H-space structure on $E^{BN}(A)$. Compare [16].

For this it will be convenient to describe facettes $F \in P^n$ in terms of partitions of the set $\{e_1, \ldots, e_n\}$ of standard dual basis vectors for R^n. We write

$$F = X_1 | X_2 | \cdots | X_r$$

to mean that F is determined by the conditions

$$\begin{aligned} e_i - e_j &= 0 \qquad \text{if } e_i, e_j \in X_\alpha, \\ e_i - e_j &> 0 \qquad \text{if } e_i \in X_\alpha, e_j \in X_\beta, \text{ and } \alpha < \beta. \end{aligned}$$

If $n = \infty$ we require that each X_α is finite for $1 \leqslant \alpha < r$. Let $m, n < \infty$ and let $F = X_1 | \cdots | X_r$ and $G = Y_1 | \cdots | Y_s$ lie in P^m and P^n respectively. Define

$$F \oplus G = X_1 | \cdots | X_r | Y_1' | \cdots | Y_s'$$

where Y_j' is obtained from Y_j by adding n to the indices of the e_i to get a subset of $\{e_{m+1}, \ldots, e_{m+n}\}$. If $F_1 < F_2$ and $G_1 < G_2$, then $F_1 \oplus G_1 < F_2 \oplus G_2$.

We shall let $\Delta \in P^n$ be the diagonal facette defined by setting all $e_i - e_j = 0$. There is another stabilization map $F \to F \oplus \Delta$ from P^m to P^{m+n} which is not quite the same as n repetitions $F \to F^{(n)}$ of $(*)$ of §1. However, $F^{(n)} < F \oplus \Delta$ for all $F \in P^m$ and if $\alpha \cdot U_F < \beta \cdot U_G$, then there is a commutative square

$$\begin{array}{ccc} (\alpha \oplus 1) \cdot U_{F(n)} & < & (\alpha \oplus 1) \cdot U_{G(n)} \\ \wedge & & \wedge \\ (\alpha \oplus 1) \cdot U_{F \oplus \Delta} & < & (\alpha \oplus 1) \cdot U_{G \oplus \Delta} \end{array}$$

which shows [12] that the two stabilization maps $E_m \to E_{m+n}$ defined respectively by

$$\alpha \cdot U_F \to (\alpha \oplus 1) \cdot U_{F(n)} \quad \text{and} \quad \alpha \cdot U_F \to (\alpha \oplus 1) \cdot U_{F \oplus \Delta}$$

are homotopic. Note that the second stabilization does not take the base point U_Δ of E_m to the base point U_Δ of E_{m+n}. However, consider the contractible complex P^m as embedded in E_m by the correspondence $F \to U_F$. Then stabilization $E_m \to E_{m+n}$ via $F \to F \oplus \Delta$ takes P^m to P^{m+n} and hence determines a base point preserving map well defined up to base point preserving homotopy. From now on in this section we use the second stabilization.

If $\alpha_0 \cdot U_{F_0} < \alpha_1 \cdot U_{F_1}$ in E_m and $\beta_0 \cdot U_{G_0} < \beta_1 \cdot U_{G_1}$ and E_n, then $(\alpha_0 \oplus \beta_0) \cdot U_{F_0 \oplus G_0} < (\alpha_1 \oplus \beta_1) \cdot U_{F_1 \oplus G_1}$ in E_{m+n}. This gives a map

$$\gamma_{m,n}: E_m \times E_n \to E_{m+n}$$

which does not preserve base point; but since $\gamma_{m,n}(P^n \times P^n) \subset P^{m+n}$, it does determine a base point preserving map well defined up to base point preserving homotopy.

PROPOSITION 2. *The diagrams* $(n \geqslant 2)$

$$(**) \qquad \begin{array}{ccc} E_n \times E_n & \longrightarrow & E_{2n} \times E_{2n} \\ \Big\downarrow \gamma_{n,n} & & \Big\downarrow \gamma_{2n,2n} \\ E_{2n} & \longrightarrow & E_{4n} \end{array}$$

are commutative up to base point preserving homotopy and give rise to an H-space structure on $E^{BN}(A)$.

PROOF OF PROPOSITION 2. The left-hand restriction map $\Phi = \gamma_{n,n}: E_n \times U_\Delta \to E_{2n}$ is defined by $\alpha \cdot U_F \to (\alpha \oplus 1) \cdot U_{F\oplus\Delta}$. The main step will be to show that Φ is homotopic by a base point preserving homotopy to the right-hand restriction $\Psi = \gamma_{n,n}: U_\Delta \times E_n \to E_{2n}$, which is given by the correspondence $\alpha \cdot U_F \to (1 \oplus \alpha) \cdot U_{\Delta\oplus F}$. The two maps $E_n \times E_n \to E_{4n}$ of $(**)$ are induced by the homomorphisms

$$\text{(a)} \qquad (\alpha,\beta) \to \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\text{(b)} \qquad (\alpha,\beta) \to \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The homotopy commutativity of $(**)$ is obtained by applying essentially the same argument for $\Phi \sim \Psi$ to the "second and third rows and columns." Finally, the $\gamma_{n,n}$ are telescoped together to give the H-space structure on $E^{HN}(A)$.

The proof that $\Phi \sim \Psi$ will be based on the matrix identities

$$\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix},$$

$$\begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix} = \begin{pmatrix} 1 & \alpha^{-1} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -\alpha & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \alpha^{-1} \\ 0 & 1 \end{pmatrix}$$
$$\cdot \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

together with the following six commutative squares wherein if x, y, and z are matrices, then $x \to^{z} y$ means $y = x \cdot z$:

$$\begin{tikzcd}[column sep=huge]
\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \arrow[r, "\begin{pmatrix} \delta & 0 \\ 0 & 1 \end{pmatrix}"] \arrow[d, swap, "\text{(i)}\ \begin{pmatrix} 1 & \alpha^{-1} \\ 0 & 1 \end{pmatrix}"] & \begin{pmatrix} \alpha\delta & 0 \\ 0 & 1 \end{pmatrix} \arrow[d, "\begin{pmatrix} 1 & \delta^{-1}\alpha^{-1} \\ 0 & 1 \end{pmatrix}"] \\
\begin{pmatrix} \alpha & 1 \\ 0 & 1 \end{pmatrix} \arrow[r, "\begin{pmatrix} \delta & 0 \\ 0 & 1 \end{pmatrix}"] \arrow[d, swap, "\text{(ii)}\ \begin{pmatrix} 1 & 0 \\ -\alpha & 1 \end{pmatrix}"] & \begin{pmatrix} \alpha\delta & 1 \\ 0 & 1 \end{pmatrix} \arrow[d, "\begin{pmatrix} 1 & 0 \\ -\alpha\delta & 1 \end{pmatrix}"] \\
\begin{pmatrix} 0 & 1 \\ -\alpha & 1 \end{pmatrix} \arrow[r, "\begin{pmatrix} \delta & 0 \\ 0 & 1 \end{pmatrix}"] \arrow[d, swap, "\text{(iii)}\ \begin{pmatrix} 1 & \alpha^{-1} \\ 0 & 1 \end{pmatrix}"] & \begin{pmatrix} 0 & 1 \\ -\alpha\delta & 1 \end{pmatrix} \arrow[d, "\begin{pmatrix} 1 & \delta^{-1}\alpha^{-1} \\ 0 & 1 \end{pmatrix}"] \\
\begin{pmatrix} 0 & 1 \\ -\alpha & 0 \end{pmatrix} \arrow[r, "\begin{pmatrix} \delta & 0 \\ 0 & 1 \end{pmatrix}"] \arrow[d, swap, "\text{(iv)}\ \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}"] & \begin{pmatrix} 0 & 1 \\ -\alpha\delta & 0 \end{pmatrix} \arrow[d, "\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}"] \\
\begin{pmatrix} 0 & 1 \\ -\alpha & \alpha \end{pmatrix} \arrow[r, "\begin{pmatrix} \delta & 1-\delta \\ 0 & 1 \end{pmatrix}"] \arrow[d, swap, "\text{(v)}\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}"] & \begin{pmatrix} 0 & 1 \\ -\alpha\delta & \alpha\delta \end{pmatrix} \arrow[d, "\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}"] \\
\begin{pmatrix} 1 & 1 \\ 0 & \alpha \end{pmatrix} \arrow[r, "\begin{pmatrix} 1 & 1-\delta \\ 0 & \delta \end{pmatrix}"] \arrow[d, swap, "\text{(vi)}\ \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}"] & \begin{pmatrix} 1 & 1 \\ 0 & \alpha\delta \end{pmatrix} \arrow[d, "\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}"] \\
\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \arrow[r, "\begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix}"] & \begin{pmatrix} 1 & 0 \\ 0 & \alpha\delta \end{pmatrix}
\end{tikzcd}$$

Step 1. Consider the correspondence

$$\alpha \cdot U_F \to \begin{pmatrix} \alpha & 1 \\ 0 & 1 \end{pmatrix} \cdot U_{F\oplus\Delta}. \tag{1}$$

The commutative square (i) shows this is order preserving and hence defines a simplicial map $\Phi_1: E_n \to E_{2n}$. But

$$\begin{pmatrix} 1 & \alpha^{-1} \\ 0 & 1 \end{pmatrix} \in U_{F\oplus\Delta}$$

so therefore

$$\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \cdot U_{F\oplus\Delta} = \begin{pmatrix} \alpha & 1 \\ 0 & 1 \end{pmatrix} \cdot U_{F\oplus\Delta}.$$

Hence $\Phi_1 = \Phi$.

Step 2. Before applying (ii) a preliminary homotopy of Φ_1 must be made. For each $F = X_1|X_2|\cdots|X_r$ in P^n let $\Delta \propto F = \{e_{n+1}, \ldots, e_{2n}\}|X_1|X_2|\cdots|X_r$. If $F < G$ then, $\Delta \propto F < \Delta \propto G$; and if $\delta \in U_F$, then $\delta \oplus 1 \in U_{\Delta \propto F}$. Hence the correspondence

$$\alpha \cdot U_F \to \begin{pmatrix} \alpha & 1 \\ 0 & 1 \end{pmatrix} \cdot U_{\Delta \propto F} \tag{1.5}$$

preserves order and defines a simplicial map $\Phi_{1.5}: E_n \to E_{2n}$. We claim that Φ_1 and $\Phi_{1.5}$ are homotopic by a base point preserving homotopy: Let g_1 and g_2 be the two order preserving maps from P^n to P^{2n} defined respectively by $g_1(F) = F \oplus \Delta$ and $g_2(F) = \Delta \propto F$. Apply Lemma of §1 to find a subdivision $(P^n \times I)'$ of $P^n \times I$ and an order preserving map $w: (P^n \times I)' \to P^{2n}$ satisfying conditions (a) through (d). Now consider the simplicial map $\pi: E_n \to P^n$ which takes the vertex $\alpha \cdot U_F$ to the vertex F. This is nondegenerate on simplices. Similarly, if we give $E_n \times I$ and $P^n \times I$ the standard triangulations, then the natural simplicial map $\pi \times 1: E_n \times I \to P^n \times I$ is also nondegenerate on each simplex. Hence the subdivision $(P^n \times I)'$ of $P^n \times I$ induces a subdivision $(E_n \times I)'$ of $E_n \times I$. Now let $\sigma = (\alpha_0 \cdot U_{F_0} < \cdots < \alpha_k \cdot U_{F_k})$ be a simplex of E_n and let v be a vertex in the standard triangulation of $\pi(\sigma) \times I$ in $P^n \times I$. Then $w(v) = F \oplus \Delta$ or $w(v) = \Delta \propto F$ where F is one of the F_i. Since $\delta \oplus 1 \in U_{F\oplus\Delta}$ and $\delta \oplus 1 \in U_{\Delta \propto F}$ for each $\delta \in U_{F_0}$, condition (d) of the Lemma shows that $\delta \oplus 1 \in U_{w(v)}$ for any vertex v of the new triangulation of $\pi(\sigma) \times I$. Now let u be any vertex in the new triangulation of $\sigma \times I$ and let $v = (\pi \times 1)(u)$. Define

$$\Omega(u) = \begin{pmatrix} \alpha_0 & 1 \\ 0 & 1 \end{pmatrix} \cdot U_{w(v)}.$$

The above remarks show $\Omega(u)$ is independent of the representative α_0 of the class $\alpha_0 \cdot U_{F_0}$ and we get a simplicial map $\Omega: (E_n \times I)' \to E_{2n}$ which is the required homotopy between Φ_1 and $\Phi_{1.5}$.

Now consider the correspondence

$$(1.5') \qquad \alpha \cdot U_F \to \begin{pmatrix} 0 & 1 \\ -\alpha & 1 \end{pmatrix} \cdot U_{\Delta \propto F}.$$

The commutative square (ii) shows this is order preserving and induces a simplicial map $\Phi'_{1.5}\colon E_n \to E_{2n}$ such that $\Phi'_{1.5} = \Phi_{1.5}$ because $\left(\begin{smallmatrix} 1 & 0 \\ -\alpha & 1 \end{smallmatrix}\right) \in U_{\Delta \propto F}$ for all $F \in P^n$. Arguing as above, we see that $\Phi'_{1.5}$ is homotopic to the simplicial map $\Phi_2\colon E_n \to E_{2n}$ defined by the correspondence

$$(2) \qquad \alpha \cdot U_F \to \begin{pmatrix} 0 & 1 \\ -\alpha & 1 \end{pmatrix} \cdot U_{F \oplus \Delta}.$$

Step 3. Consider the map

$$(3) \qquad \alpha \cdot U_F \to \begin{pmatrix} 0 & 1 \\ -\alpha & 0 \end{pmatrix} \cdot U_{F \oplus \Delta}.$$

The square (iii) shows this is order preserving and we get a simplicial map $\Phi_3\colon E_n \to E_{2n}$ which agrees with Φ_2 because

$$\begin{pmatrix} 1 & \alpha^{-1} \\ 0 & 1 \end{pmatrix} \in U_{F \oplus \Delta} \quad \text{for all } F \in P^n.$$

Step 4. Square (iv) shows that the correspondence

$$(4) \qquad \alpha \cdot U_F \to \begin{pmatrix} 0 & 1 \\ -\alpha & \alpha \end{pmatrix} \cdot U_{F \oplus \Delta}$$

defines an order preserving simplicial map $\Phi_4\colon E_n \to E_{2n}$ which agrees with Φ_3.

Step 5. As in Step 2 it is first necessary to deform Φ_4 by a homotopy before using (v). Recall [18] that if $F \in P^n$ is of the form $F = X_1|X_2|\cdots|X_r$, then $F \,\square\, F \in P^{2n}$ is defined as

$$F \,\square\, F = X'_1|X_1|X'_2|X_2|\cdots|X'_r|X_r$$

where $X'_i \subset \{e_{n+1}, \ldots, e_{2n}\}$ is obtained from X_i by adding n to the indices of the $e_j \in X_i$. Consider the two maps g_1, g_2: {vertices of P^n} $\to P^{2n}$ defined by $g_1(F) = F \oplus \Delta$ and $g_2(F) = F \,\square\, F$. The map g_1 is order preserving but g_2 is not! Apply the Lemma of §1 to construct a simplicial map $w\colon (P^n \times I)' \to P^{2n}$ satisfying (a) through (d). As in Step 2, let $\sigma = (\alpha_0 \cdot U_{F_0} < \cdots < \alpha_k \cdot U_{F_k})$ be a simplex of E_n and let v be a vertex in the standard triangulation of $\pi(\sigma) \times I$ in $P^n \times I$. Then $w(v) = F \oplus \Delta$ or $w(v) = F \,\square\, F$ where F is one of the F_i. Now for each $\delta \in U_{F_0}$, the matrix $\left(\begin{smallmatrix} \delta & 1-\delta \\ 0 & 1 \end{smallmatrix}\right)$ lies in $U_{F \oplus \Delta}$ and also in $U_{F \,\square\, F}$. Hence condition (d) of the Lemma shows that $\left(\begin{smallmatrix}\delta & 1-\delta\\ 0 & 1\end{smallmatrix}\right) \in U_{w(v)}$ for each vertex v of the new triangulation of $\pi(\sigma) \times I$ in $(P^n \times I)'$. For any vertex u in the new triangulation of $\sigma \times I$ in $(E_n \times I)'$ let $v = (\pi \times 1)(u)$ and define

$$\Omega(u) = \begin{pmatrix} 0 & 1 \\ -\alpha_0 & \alpha_0 \end{pmatrix} \cdot U_{w(v)}.$$

Then the preceding remarks show $\Omega(u)$ is independent of the representative α_0 of $\alpha_0 \cdot U_{F_0}$ and we get a simplicial map $\Omega\colon (E_n \times I)' \to E_{2n}$. Let $\Phi_4' = \Omega | (E_n \times 1)'$. For each vertex v of $\pi(\sigma) \times 1$ in the standard triangulation, the matrix $\left(\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right)$ belongs to $U_{w(v)}$. Hence by (d) of the Lemma this matrix belongs to $U_{w(v)}$ for any vertex v of $(\pi(\sigma) \times 1)'$. Therefore for any vertex u in $(\sigma \times 1)$ we have

$$\Phi_4'(u) = \begin{pmatrix} 1 & 1 \\ 0 & \alpha_0 \end{pmatrix} \cdot U_{w(v)}$$

where $v = (\pi \times 1)(u)$. See (v). Since $\left(\begin{smallmatrix}1 & 1-\delta\\ 0 & \delta\end{smallmatrix}\right) \in U_{w(v)}$ for every $\delta \in U_{F_0}$, this new formula for Φ_4' is independent of the choice of representative α_0 of $\alpha_0 \cdot U_{F_0}$ by (v). Since $\left(\begin{smallmatrix}1 & 1-\delta\\ 0 & \delta\end{smallmatrix}\right)$ belongs to $U_{\Delta \oplus F}$ and to $U_{F \,\square\, F}$ for $\delta \in U_{F_0}$ and $F = F_0, \dots, F_k$, we can construct as above a homotopy between Φ_4' and $\Phi_5\colon E_n \to E_{2n}$ defined by the order preserving correspondence

$$\alpha \cdot U_F \to \begin{pmatrix} 1 & 1 \\ 0 & \alpha \end{pmatrix} \cdot U_{\Delta \oplus F}.$$

Step 6. Finally (vi) shows that Φ_5 is the same as Ψ, which is defined by

$$\alpha \cdot U_F \to \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \cdot U_{\Delta \oplus F},$$

because $\left(\begin{smallmatrix}1 & -1\\ 0 & 1\end{smallmatrix}\right)$ lies in each $U_{\Delta \oplus F}$.

The homotopy between Φ and Ψ in Steps 1 through 6 does not keep the base point fixed. However, it can be deformed to one which does, because $\Delta \propto \Delta = \Delta \,\square\, \Delta$ and hence U_Δ is deformed along a path of the form $\gamma * \gamma^{-1}$ where γ is the path traced out by U_Δ during the first three steps of the argument. Q.E.D.

3. **Action on the fiber.** In §1 we saw that there is a homotopy equivalence $\theta\colon X \to E\{\alpha \cdot U_F\} \cong E^{BN}(A)$ where X is the homotopy fiber of $B\{U_F\} \to BE(A)$.

PROPOSITION 3. *Under θ the action of $\pi_1 BE(A)$ on X can be identified up to homotopy with left multiplication of $E(A)$ on $E\{\alpha \cdot U_F\}$ which, moreover, induces the identity on $H_*(E\{\alpha \cdot U_F\})$.*

PROOF. If $f\colon K \to L$ is any map we convert it into an actual fibration $X_f \to E_f \xrightarrow{}_{\pi_f} L$ as usual by letting E_f be the set of pairs (x, ω) where $x \in K$ and ω is a path in L with $\omega(1) = f(x)$. The map π_f takes (x, ω) to $\omega(0)$. The fiber X_f consists of those (x, ω) for which $\omega(0) =$ base point of L. Applying this to the horizontal rows of the pullback square of §1 defining $E\{\alpha \cdot U_F\}$ gives the commutative diagram

$$\begin{tikzcd}
\mathrm{pt} \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & G \arrow[d] \\
X_i \arrow[r] \arrow[d] & E_i \arrow[r, "\pi_i"'] \arrow[d] & EG \arrow[d, "\pi"] \\
X_j \arrow[r] & E_j \arrow[r, "\pi_j"'] & BG
\end{tikzcd}$$

Since G is discrete, π has the unique path lifting property. This implies E_i is homomorphic to the pullback of π_j and π. Hence all the horizontal and vertical rows are fibrations, $X_i \to X_j$ is a homeomorphism, and $X_j \simeq E_i$ because EG is contractible. A specific homotopy equivalence $\theta\colon X_j \to E_i$ is defined by $\theta(x, \nu) = (x, \bar{\nu}(1); \bar{\nu})$ where $\bar{\nu}$ is the unique path in EG starting at the base point and lifting ν. Now let γ be a fixed loop in BG representing $g \in \pi_1 BG$. Then

$$\theta(g \cdot (x, \nu)) = \theta(x, \gamma * \nu) = (x, \overline{\gamma * \nu}(1); \overline{\gamma * \nu})$$

and

$$g \cdot \theta(x, \nu) = g \cdot (x, \bar{\nu}(1); \bar{\nu}) = (x, g \cdot \bar{\nu}(1); g \cdot \bar{\nu}).$$

It follows using the standard construction as in [6] for the universal cover of BG that these two maps are homotopic.

It remains to show the correspondence $\alpha \cdot U_F \to g\alpha \cdot U_F$ induces the identity on $H_*(E\{\alpha \cdot U_F\}) = H_*(E^{BN}(A))$. Any homology class is supported in some E_n and we can choose n large enough to have $g \in E(n, A)$. In §2 it was shown that there is a subdivision $(E_n \times I)'$ of the standard triangulation of $E_n \times I$ such $(E_n \times 0)' = E_n \times 0$ and $(E_n \times 1)' = E_n \times 1$ and there is a simplicial map $h\colon (E_n \times I)' \to E_{2n}$ having the property that $h_0 = h | E_n \times 0$ is

$$\alpha \cdot U_F \to (\alpha \oplus 1) \cdot U_{F \oplus \Delta}$$

and $h_1 = h | E_n \times 1$ is

$$\alpha \cdot U_F \to (1 \oplus \alpha) \cdot U_{\Delta \oplus F}.$$

Hence $g \cdot h_0 \colon E_n \to E_{2n}$ defined by

$$\alpha \cdot U_F \to (g\alpha \oplus 1) \cdot U_{F \oplus \Delta}$$

is homotopic to $g \cdot h_1 \colon E_n \to E_{2n}$ defined by

$$\alpha \cdot U_F \to (g \oplus \alpha) \cdot U_{\Delta \oplus F}.$$

Therefore it suffices to show $g \cdot h_1$ is homotopic to the map $E_n \to E_{2n}$ given by

$$\alpha \cdot U_F \to (1 \oplus \alpha) \cdot U_{\Delta \oplus F}.$$

Since $g \in E(n,A)$ is the product of elements lying in the subgroups U_F, this fact is in turn a consequence of several applications of the following: Let $G < G'$ in P^n and assume $g \in U_{G'}$. Let $x \in E(n,A)$. Then the two maps $E_n \to E_{2n}$ defined respectively by

$$\alpha \cdot U_F \to \begin{pmatrix} x & 0 \\ 0 & \alpha \end{pmatrix} \cdot U_{G \oplus F}$$

and

$$\alpha \cdot U_F \to \begin{pmatrix} xg & 0 \\ 0 & \alpha \end{pmatrix} \cdot U_{G' \oplus F}$$

are homotopic. But this is clear because

$$\begin{pmatrix} x & 0 \\ 0 & \alpha \end{pmatrix} \cdot U_{G \oplus F} < \begin{pmatrix} x & 0 \\ 0 & \alpha \end{pmatrix} \cdot U_{G' \oplus F} = \begin{pmatrix} xg & 0 \\ 0 & \alpha \end{pmatrix} \cdot U_{G' \oplus F}.$$

4. **H-space structure on $B\{U_F\}^+$.** In this section we show how direct sum of matrices gives an H-space structure on $B\{U_F\}^+$.

By a sheaf of spaces over a simplicial complex K we mean a collection $X = \{X_\sigma\}$, σ = simplex of K, together with connecting maps $i_{\sigma\tau} \colon X_\tau \to X_\sigma$ for $\sigma < \tau$ such that $i_{\sigma\tau} \circ i_{\tau\gamma} = i_{\sigma\gamma}$ whenever $\sigma < \tau < \gamma$. The realization $|X|$ of X is the disjoint union $\coprod_{\sigma \subset K} \sigma \times X_\sigma$ modulo the identification setting $(x,y) = (x',y')$ iff $x = x'$ and $y' = i_{\sigma\tau}(y)$ for $y \in X_\tau$, $y' \in X_\sigma$, and $\sigma < \tau$. Any simplicial subdivision K' of K induces a subdivision X' of X as follows: For τ a simplex of K' let $X'_\tau = X_\sigma$ where σ is the smallest simplex of K containing τ. If $\tau < \tau'$ and σ, σ' are the smallest simplices of K containing τ, τ' respectively, then $\sigma < \sigma'$ and we let $i_{\tau\tau'} = i_{\sigma\sigma'}$. The natural map $|X'| \to |X|$ is a homeomorphism. Define $X \times I$ to be the sheaf of spaces over $K \times I$ by setting $(X \times I)_\tau = X_\sigma$ where σ is the smallest simplex of K such that $\tau \subset \sigma \times I$. The space B_n is clearly the realization of a sheaf of spaces of the form BU_F over P^n.

The direct sum homomorphism $\oplus$ from $E(m,A) \times E(n,A) \to E(m+n,A)$ gives a family of compatible homomorphisms

$$\oplus\colon U_F \times U_G \to U_{F\oplus G}$$

and hence a family of compatible maps

$$\oplus\colon BU_F \times BU_G \to BU_{F\oplus G}$$

which fit together to give maps

$$\rho_{m,n}\colon B_m \times B_n \to B_{m+n}.$$

By the universal property of the "plus construction" we get base point preserving maps

$$\rho^+_{m,n}\colon B^+_m \times B^+_n \to B^+_{m+n}.$$

A word about base points: The complex P^n can be embedded in B_n by sending $x \in \sigma = (F_0 < \cdots < F_k)$ to $(x$, base point of $BU_{F_0})$. The base point $*$ of B_n is $(\Delta$, base point of $BU_\Delta)$. The map $\rho_{m,n}$ does not preserve the base point but $\rho_{m,n}(P^m \times P^n) \subset P^{m+n}$. Hence $\rho_{m,n}$ determines a base point preserving map $\rho^+_{m,n}$ defined up to base point preserving homotopy. Also, the stabilization map $B_n \to B_{2n}$ given by $BU_F \to BU_{F\oplus\Delta}$ is not the standard inclusion $B_n \hookrightarrow B_{2n}$ which is given by $BU_F \to BU_{F^{(n)}}$. However, $F^{(n)} < F \oplus \Delta$ for all $F \in P^n$ implies these two maps are homotopic up to a base point preserving homotopy; so from now on we use the stabilization induced by $F \to F \oplus \Delta$.

PROPOSITION 4. *For $n \geqslant 3$ the diagrams*

$$\begin{tikzcd} B^+_n \times B^+_n \arrow[r] \arrow[d, "\rho^+_{n,n}"] & B^+_{2n} \times B^+_{2n} \arrow[d, "\rho^+_{2n,2n}"] \\ B^+_{2n} \arrow[r] & B^+_{4n} \end{tikzcd}$$

are homotopy commutative and give rise to an H-space structure on $B\{U_F\}^+$ in such a way that $B\{U_F\}^+ \to BE(A)^+$ is an H-map.

REMARK. It follows immediately from [13] that the homotopy theoretic fiber X of $B\{U_F\}^+ \to BE(A)^+$ is a connected H-space as required by (II) of §1.

PROOF OF PROPOSITION 4. The restriction map $\Phi = \rho_{n,n}\colon B^+_n \times * \to B^+_{2n}$ comes from the inclusion $B_n \to B_{2n}$ induced by the stabilization homomorphism

$$\phi(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \in U_{F\oplus\Delta}.$$

The essential step in the proof is to show that the restriction map

$$\Psi = \rho_{n,n}: * \times B_n^+ \to B_{2n}^+$$

is homotopic to Φ by a base point preserving homotopy. Note that Ψ is induced by the homomorphism

$$\psi(\alpha) = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$$

satisfying $\psi(U_F) \subset U_{\Delta\oplus F}$ for each $F \in P^n$. By the universal property of the "plus construction" it suffices to show

(†) For $n \geqslant 3$ the maps Φ, Ψ: $B_n \to B_{2n}^+$ are homotopic by a base point preserving homotopy.

In fact B_{2n}^+ is simply connected for $2 \leqslant n$ so it suffices to show Φ and Ψ are homotopic as maps into B_{2n}. To prove this we shall use the following matrix identities:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}.$$

Step 1. Consider the homomorphism $\phi_1: E(n,A) \to E(2n,A)$ given by

$$\phi_1(\alpha) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \alpha - 1 \\ 0 & 1 \end{pmatrix}.$$

For each $F \in P^n$ the matrix $\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right) \in U_{F\oplus\Delta}$; so $\phi_1(U_F) \subset U_{F\oplus\Delta}$. The induced maps $BU_F \to BU_{F\oplus\Delta}$ fit together to give a map $\Phi_1: B_n \to B_{2n}$. The construction of [12] gives a specific homotopy H_F: $BU_F \times I \to BU_{F\oplus\Delta}$ for each facette $F \in P^n$ between the maps induced by the conjugate homomorphisms ϕ and ϕ_1 such that $H_{F'}|BU_F \times I = H_F$ whenever $F < F'$. Hence these homotopies fit together to give a homotopy H: $B_n \times I \to B_{2n}$ from Φ to Φ_1.

Step 2. Let f, g: $P^n \to P^{2n}$ be defined by $f(F) = F \oplus \Delta$ and $g(F) = F \,\square\, F$. The map f preserves order but g does not. Apply the lemma of §1 to find an order preserving map w: $(P^n \times I)' \to P^{2n}$ of some subdivision of the standard triangulation of $P^n \times I$ satisfying (a) through (d). This subdivision induces a subdivision $(B_n \times I)'$ of $B_n \times I$. Consider a simplex $\sigma = (F_0 < \cdots < F_k)$ in P^n and let v be any vertex of a simplex in the standard triangulation of $\sigma \times I$. Then $w(v) = F \oplus \Delta$ or $w(v) = F \,\square\, F$ where F is one of the facettes

F_i. The formula for ϕ_1 shows that $\phi_1(U_{F_0}) \subset U_{w(v)}$ and hence by (d) the same inclusion holds for any vertex v of $(\sigma \times I)'$. This yields homotopy Ω: $(B_n \times I)' \to B_{2n}$ between Φ_1 and the map Φ_1': $(B_n \times 1)' \to B_{2n}$ defined as the restriction of Ω to $(B_n \times 1)'$. Now for $F \in P^n$ the matrix $\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$ belongs to $U_{F \square F}$. Applying (d) of the lemma shows that $\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right) \in U_{w(v)}$ for any vertex v of $(P^n \times 1)'$. Let ϕ_2: $E(n,A) \to E(2n,A)$ be defined by

$$\phi_2(\alpha) = \begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}\alpha&\alpha-1\\0&1\end{pmatrix}\begin{pmatrix}1&0\\-1&1\end{pmatrix} = \begin{pmatrix}1&\alpha-1\\0&\alpha\end{pmatrix}$$

and let Φ_2' be the induced map. More precisely, if v is a vertex of $(P^n \times 1)'$, let $\sigma = (F_0 < \cdots < F_k)$ be the smallest simplex of $P^n \times 1$ such that $v \in \sigma$. Then $\phi_2(U_{F_0}) \subset U_{w(v)}$ so there is an induced map $BU_{F_0} \to BU_{w(v)}$. These fit together to give Φ_2'. Since $\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right) \in U_{w(v)}$ for any vertex v of $(P^n \times 1)'$, the conjugate homomorphisms ϕ_1 and ϕ_2 give rise to homotopic maps Φ_1' and Φ_2'.

Now let f, g: $P^n \to P^{2n}$ be defined by $f(F) = F \square F$ and $g(F) = \Delta \oplus F$. The map g preserves order but f does not. Use the lemma to construct a homotopy w: $(P^n \times I)' \to P^{2n}$ satisfying (a) through (d). Again let $\sigma = (F_0 < \cdots < F_k)$ be a simplex of $P^n \times 1 \simeq P^n$ and let v be a vertex in the standard triangulation. Then $w(v) = F \square F$ or $w(v) = \Delta \oplus F$ where F is one of the F_i. The formula for ϕ_2 shows that $\phi_2(U_{F_0}) \subset U_{w(v)}$ and hence by (d) the same is true for any vertex v of $(\sigma \times I)'$. As above there is a homotopy between Φ_2' and Φ_2: $B_n \to B_{2n}$ which, by definition, is obtained by fitting together the maps $BU_F \to BU_{\Delta \oplus F}$ induced by ϕ_2.

Step 3. Let ϕ_3: $E(n,A) \to E(2n,A)$ be defined by

$$\phi_3(\alpha) = \begin{pmatrix}1&-1\\0&1\end{pmatrix}\begin{pmatrix}1&\alpha-1\\0&\alpha\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix} = \begin{pmatrix}1&0\\0&\alpha\end{pmatrix}.$$

Note that $\phi_3 = \psi$. Since $\left(\begin{smallmatrix}1&-1\\0&1\end{smallmatrix}\right) \in U_{\Delta \oplus F}$ for all $F \in P^n$ we get a homotopy between Φ_2 and Ψ as in Step 1.

To show the homotopy commutativity of the diagram in Proposition 4 one applies essentially the same arguments as above to the second and third rows and columns of the homomorphisms (a) and (b) of §2.

Finally, the maps $\rho_{n,n}$: $B_n^+ \times B_n^+ \to B_{2n}^+$ telescope together to give the H-space structure on $B\{U_F\}^+$. That $B\{U_F\}^+ \to BE(A)^+$ is an H-map follows immediately from the fact that the H-space structure on $BE(A)^+$ arises from the direct sum maps

$$BE(n,A) \times BE(n,A) \to BE(2n,A)$$

and that we have a strictly commutative diagram

$$\begin{tikzcd}
B_n \times B_n \arrow[r] \arrow[d] & B_{2n} \arrow[d] \\
BE(n, A) \times BE(n, A) \arrow[r] & BE(2n, A)
\end{tikzcd}$$

which commutes up to base point preserving homotopy when the "plus construction" is performed. Q.E.D.

References

1. D. Anderson, *Relationship among K-theories*, Algebraic K-Theory. I: Higher K-Theories (Proc. Conf., Seattle Res. Cntr., Battelle Mem. Inst., 1972), (H. Bass, editor), Lecture Notes in Math., vol. 341, Springer-Verlag, Berlin and New York, 1973, pp. 57–72. MR **50** #2302.

2. D. Anderson, M. Karoubi and J. Wagoner, *Relations between higher algebraic K-theories*, Algebraic K-Theory. I: Higher K-Theories (Proc. Conf., Seattle Res. Cntr., Battelle Mem. Inst., 1972), (H. Bass, editor), Lecture Notes in Math., vol. 341, Springer-Verlag, Berlin and New York, 1973, pp. 73–81. MR **50** #7291.

3. N. Bourbaki, *Éléments de mathématique*, Fasc. 34, *Groupes et algèbres de Lie*, Chaps. 4, 5, 6, Actualités Sci. Indust., no. 1337, Hermann, Paris, 1968, p. 57. MR **39** #1590.

4. W. Browder, *Higher torsion in H-spaces*, Trans. Amer. Math. Soc. **108** (1963), 353–375. MR **27** #5260.

5. S. M. Gersten, *Higher K-theory of rings*, Algebraic K-Theory. I: Higher K-Theories (Proc. Conf., Seattle Res. Cntr., Battelle Mem. Inst., 1972), (H. Bass, editor), Lecture Notes in Math., vol. 341, Springer-Verlag, Berlin and New York, 1973, pp. 3–42. MR **48** #3656a.

6. S.-T. Hu, *Homotopy theory*, Academic Press, New York and London, 1959. MR **21** #5186.

7. M. Karoubi and O. Villamayor, *Foncteurs K^n en algèbre et en topologie*, C. R. Acad. Sci. Paris Sér. A-B **269** (1969), A416-A419. MR **40** #4944.

8. F. Keune, *Derived functors and algebraic K-theory*, Algebraic K-Theory. I: Higher K-Theories (Proc. Conf., Seattle Res. Cntr., Battelle Mem. Inst., 1972), (H. Bass, editor), Lecture Notes in Math., vol. 341, Springer-Verlag, Berlin and New York, 1973, pp. 166–176. MR **48** #3656a.

9. S. Mac Lane, *Homology*, Springer-Verlag, Berlin; Academic Press, New York, 1963, p. 355. MR **28** #122.

10. D. Quillen, *On the cohomology and K-theory of the general linear groups over a finite field*, Ann. of Math. (2) **96** (1972), 552–586. MR **47** #3565.

11. ———, *Higher algebraic K-theory*. I: Algebraic K-Theory. I: Higher K-Theories (Proc. Conf., Seattle Res. Cntr., Battelle Mem. Inst. (1972), (H. Bass, editor), Lecture Notes in Math., vol. 341, Springer-Verlag, Berlin and New York, 1973, pp. 85–147. MR **49** #2895.

12. G. Segal, *Classifying spaces and spectral sequences*, Inst. Hautes Études Sci. Publ. Math., No. **34** (1968), 105–112. MR **38** #718.

13. J. D. Stasheff, *On extensions of H-spaces*, Trans. Amer. Math. Soc. **105** (1962), 126–135. MR **31** #2726.

14. R. G. Swan, *Nonabelian homological algebra and K-theory*, Applications of Categorical Algebra (Proc. Sympos. Pure Math., vol. 17), Amer. Math. Soc., Providence, R.I., 1970, pp. 88–123. MR **41** #1839.

15. E. Thomas, *Fields of tangent k-planes on manifolds*, Invent. Math. **3** (1967), 334–347. MR **36** #903.

16. I. A. Volodin, *Algebraic K-theory as an extraordinary homology theory on the category of associative rings with a unit*, Izv. Akad. Nauk SSSR Ser. Mat. **35** (1971), no. 4, 844–873 = Math. USSR Izv. **5** (1971), no. 4, 859–887. MR **45** #5201.

17. J. Wagoner, *Buildings, stratifications, and higher K-theory*, Algebraic K-Theory. I: Higher K-Theories (Proc. Conf., Seattle Res. Cntr., Battelle Mem. Inst., 1972), (H. Bass, editor), Lecture Notes in Math., vol. 341, Springer-Verlag, Berlin and New York, 1973, pp. 148–165. MR **48** #3656a.

18. ———, *Equivalence of algebraic K-theories*, Univ. of Calif., Berkeley (preprint).

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