HIGHER ALGEBRAIC $K$-THEORIES

BY

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Abstract. A homotopy fibration is established relating the Volodin or BN-pair definition of algebraic $K$-theory to the theory defined by Quillen.

In [2] we outlined the construction of natural homomorphisms

$$K^Q_0 \to K^BN_0 \to K^V_0 \to K^{KV}_0$$

between higher algebraic $K$-theories $K^Q_0$ of [10] and [11], $K^BN_0$ of [17], $K^V_0$ of [16], and $K^{KV}_0$ of [7] and [8]. This was one of the steps in proving the various definitions of higher $K$-theory are equivalent. It turns out they all agree—including the theory $K^Q_0$ of [14], [5], and [8]—provided one restricts to the category of regular rings when using $K^{KV}_0$. See [1], [2], [5], [8] and [18]. The purpose of this paper is to prove the following theorem, announced in [2], which yields the construction of $K^Q_0 \to K^BN_0$.

Theorem. For any associative ring with identity $A$

$$GL^BN(A) \to B(U_1)^+ \to BGL(A)^+$$

is a homotopy fibration.

For the reader’s convenience and because the presentation of the BN-pair $K$-theory $K^BN_0$ used here is slightly different from that of [17], we shall briefly recall the definition of $GL^BN$ and $B(U_1)$ in the first section.

1. Preliminaries. Let $\{H_\alpha\}$ be the collection of hyperplanes in $n$-dimensional euclidean space $R^n$ given by the condition $\alpha = 0$ where $\alpha = e_i - e_j$, $i \neq j$, is a linear root. Here $e_i$ is the $i$th coordinate function. This determines a stratification of $R^n$ whose strata $F$ we call facettes as in [3]. By definition a facette of codimension $k$ is a component of the complement in the union of the $k$-fold intersections of the $H_\alpha$ of the subset consisting of the union of the $(k + 1)$-fold intersections. Let $P^n$ be the set of facettes of $R^n$ partially ordered by the condition that $F < G$ iff $F \subset G$. We shall also let $P^n$ denote the simplicial complex whose $k$-simplices are $(k + 1)$-tuples $(F_0 < \cdots < F_k)$ where $F_\ell \in P^n$. $P^n$ is a piecewise linear triangulation of the standard $(n - 1)$-simplex. The stabilization map $R^n \to R^{n+1}$ defined by

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(*) \((x_1, \ldots, x_n) \rightarrow (x_1, \ldots, x_n, x_n)\)

takes each facette \(F\) to a facette \(F'\) and preserves the relation "\(<". Thus we can consider \(P^n\) as a subset (or subcomplex) of \(P^{n+1}\) and we let \(P^\infty = \bigcup_n P^n\).

If \(F \in P^n\), let \(U_F \subset GL(n, A)\) be the subgroup generated by the elementary matrices \(e_{ij}(\lambda)\) where \(e_i - e_j > 0\) on \(F\) and \(\lambda \in A\). Note that if \(F \in P^\infty\) lies in \(P^n\), then \(U_F \subset GL(\infty, A)\) is the direct limit \(U_F \rightarrow U_F^* \rightarrow U_F^{**} \rightarrow \cdots\).

Now for \(1 \leq n \leq \infty\) let \(B_n\) be the realization of the simplicial space which in dimension \(k \geq 0\) is the disjoint union of the spaces \((F_0 < \cdots < F_k) \times BU_{F_0}\) where \(F_i \in P^n\). Then \(B_\infty = \lim_{n \to \infty} B_n\) and by definition we let \(B(U_F) = B_\infty\). The inclusions \(BU_F \subset BGL(n, A)\) induce a map

\[B_n \rightarrow BGL(n, A)\]

for \(1 \leq n \leq \infty\). Recall from [2] that \(\pi_1 B(U_F) = St(A)\) and \(\pi_1 B(U_F) \rightarrow \pi_1 BE(A)\) is just \(St(A) \rightarrow E(A)\).

If \(\alpha \cdot U_F\) and \(\beta \cdot U_G\) are two left cosets in \(GL(n, A)\) define

\[\alpha \cdot U_F < \beta \cdot U_G\]

to mean \(F < G\) and \(\alpha \cdot U_F \subset \beta \cdot U_G\). For \(2 \leq n \leq \infty\) define \(G_n\) (resp. \(E_n\)) to be the simplicial complex where \(k\)-simplices are \((k + 1)\)-tuples

\[(\alpha_0 \cdot U_{F_0} < \cdots < \alpha_k \cdot U_{F_k})\]

where \(F_i \in P^n\) and \(\alpha_i \in GL(n, A)\), respectively \(\alpha_i \in E(n, A)\) = the subgroup of elementary metrics. We have \(G_\infty = \operatorname{indlim}_n G_n\) and \(E_\infty = \operatorname{indlim}_n E_n\) and by definition we set

\[GL^{BN}(A) = G_\infty\] and \(E^{BN}(A) = E_\infty\).

The group \(GL(n, A)\) acts on \(G_n\) by left multiplication: \(\alpha \cdot (\alpha_0 \cdot U_{F_0} < \cdots < \alpha_k \cdot U_{F_k}) = (\alpha \alpha_0 \cdot U_{F_0} < \cdots < \alpha \alpha_k \cdot U_{F_k})\) and this restricts to an action of \(E(n, A)\) on \(E_n\). Moreover, \(\pi_0 GL^{BN}(A) = K_1(A)\) and \(GL^{BN}(A) = K_1(A) \times E^{BN}(A)\). See [17].

Now let \(G = E(A)\) and define \(E(\alpha \cdot U_F)\) to be the pullback of the diagram

\[\begin{array}{ccc}
E(\alpha \cdot U_F) & \rightarrow & EG \\
\downarrow & & \downarrow \pi \\
B(U_F) & \rightarrow & BG
\end{array}\]
Here we let $EG$ be the realization of the simplicial set whose $k$-simplices are $(k + 1)$-tuples $(g_0, \ldots, g_k)$. The universal principal $G$-bundle $\pi: EG \to BG$ is defined by

$$\pi(g_0, \ldots, g_k) = (g_0^{-1}g_1, \ldots, g_k^{-1}g_k).$$

See [2]. Let $E(\alpha \cdot U_F) \subset EG$ denote the contractible subcomplex whose $k$-simplices are those $(k + 1)$-tuples for which $g_i \in \alpha \cdot U_F$ for $0 \leq i \leq k$. Then $E(\alpha \cdot U_F)$ is the realization of the simplicial space which in dimension $k > 0$ is the disjoint union of the spaces

$$(\alpha_0 \cdot U_{F_0} < \cdots < \alpha_k \cdot U_{F_k}) \times E(\alpha_0 \cdot U_{F_0})$$

where $F_i \in \text{P}^\infty$. Hence by [10, Lemma for Theorem B] the natural map $E(\alpha \cdot U_F) \to E^{BN}(A)$ is a homotopy equivalence. Since $EG$ is contractible the "nine-lemma" [15] implies

$$E(\alpha \cdot U_F) \to B(U_F) \to BG$$

is a homotopy fibration and so

$$E^{BN}(A) \to B(U_F) \to BG$$

is a homotopy fibration. Similarly letting $G = GL(A)$ there is a homotopy fibration

$$GL^{BN}(A) \to B(U_F) \to BGL(A).$$

We must show this remains a fibration when the "plus-construction" is performed on the second and third spaces. Since the universal cover of $BGL(A)^+$ is $BE(A)^+$ and since $\pi_1 BGL(A)^+ = K_1(A)$, to prove the main theorem it suffices to prove

**Theorem 1.** For any associative ring with identity $A$

$$E^{BN}(A) \to B(U_F)^+ \to BE(A)^+$$

is a homotopy fibration.

The idea of the proof is to consider the diagram

$$\begin{align*}
E^{BN}(A) &\twoheadrightarrow B(U_F) &\to & BE(A) \\
\downarrow & &\downarrow & \downarrow \\
X &\to B(U_F)^+ &\to & BE(A)^+
\end{align*}$$
where $X$ is the homotopy theoretic fiber of the map $j$. Suppose we can verify that:

(I) $E^{BN}(A)$ is a connected $H$-space such that $E(A) = \pi_1 BE(A)$ acts trivially on $H_*(E^{BN}(A))$, and

(II) $X$ is a connected $H$-space.

Note that $BE(A)^+$ is simply connected and so its fundamental group acts trivially on $H_*(X)$. Then since the "plus-construction" preserves homology, the Comparison Theorem for the spectral sequence of a fibration [9] implies $E^{BN}(A) \to X$ is a homology equivalence. Hence it is a homotopy equivalence by [4, Lemma 6.2]. Condition (I) will be established in §2 and §3; (II) will be shown in §4 by seeing that $B(U_F)^+ \to BE(A)^+$ is an $H$-map and so by the proof of Theorem 2 of [13] its homotopy fiber is an $H$-space.

For convenience we state the following lemma of [18, Lemma 3.3]. Let $K$ denote a partially ordered set and also the corresponding simplicial complex. Let $f: K \to P^n$ be a map of partially ordered sets. Thus for each vertex $v$ of $K$, $f(v)$ is a facet of $R^n$. Now let $g: K \to P^n$ be any map of sets (not necessarily order preserving). Give $K \times I$ the standard triangulation as a partially ordered set where $I = \{0, 1\}$ with $0 < 1$. Together $f$ and $g$ define a map

\[
\{\text{vertices of } K \times I\} \to P^n
\]

which does not necessarily preserve order except on $K \times 0$.

\textbf{Lemma.} There is a triangulation $(K \times I)'$ of the simplicial complex $K \times I$ as a partially ordered set which refines the standard triangulation of $K \times I$ leaving $K \times 0$ unchanged, and there is an order preserving map $w: (K \times I)' \to P^n$ of the vertices of this new triangulation such that

(a) $w|K \times 0 = f$;

(b) if $v$ is a vertex of $K \times 1$ in the new and also in the old triangulation, then $w(v) = g(v)$;

(c) if $g: K \to P^n$ is order preserving, then $K \times 1$ with the new triangulation is just a copy of $K$;

(d) if $\sigma = (v_0 < \cdots < v_k)$ is a simplex of the standard triangulation of $K \times I$, $v$ is a vertex in $\sigma$ of the new triangulation, and $e_{v_j}(\lambda)$ lies in $U_{w(v)}$ for $0 \leq s \leq k$, then

\[
e_{v_j}(\lambda) \in U_{w(v)}.
\]

2. $H$-space structure on $E^{BN}(A)$. In this section we show the direct sum homomorphism

\[
\alpha \oplus \beta = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}
\]
from $E(m, A) \times E(n, A)$ to $E(m + n, A)$ induces an $H$-space structure on $E_{BN}(A)$. Compare [16].

For this it will be convenient to describe facettes $F \in P^n$ in terms of partitions of the set $\{e_1, \ldots, e_n\}$ of standard dual basis vectors for $R^n$. We write

$$F = X_1|X_2| \cdots |X_r$$

to mean that $F$ is determined by the conditions

$$e_i - e_j = 0 \quad \text{if } e_i, e_j \in X_\alpha,$$

$$e_i - e_j > 0 \quad \text{if } e_i \in X_\alpha, e_j \in X_\beta, \text{ and } \alpha < \beta.$$ 

If $n = \infty$ we require that each $X_\alpha$ is finite for $1 \leq \alpha \leq r$. Let $m, n < \infty$ and let $F = X_1| \cdots |X_r$ and $G = Y_1| \cdots |Y_s$ lie in $P^m$ and $P^n$ respectively. Define

$$F \otimes G = X_1| \cdots |X_r|Y_1| \cdots |Y_s$$

where $Y_j$ is obtained from $Y_j$ by adding $n$ to the indices of the $e_i$ to get a subset of $\{e_{m+1}, \ldots, e_{m+n}\}$. If $F_1 < F_2$ and $G_1 < G_2$, then $F_1 \otimes G_1 < F_2 \otimes G_2$.

We shall let $\Delta \in P^n$ be the diagonal facette defined by setting all $e_i - e_j = 0$. There is another stabilization map $F \rightarrow F \otimes \Delta$ from $P^m$ to $P^{m+n}$ which is not quite the same as $n$ repetitions $F \rightarrow F^{(n)}$ of (*) of §1. However, $F^{(n)} < F \otimes \Delta$ for all $F \in P^m$ and if $\alpha \cdot U_F < \beta \cdot U_G$, then there is a commutative square

$$(\alpha \oplus 1) \cdot U_{F^{(n)}} < (\alpha \oplus 1) \cdot U_{G^{(n)}}$$

$$\wedge \quad \wedge$$

$$(\alpha \oplus 1) \cdot U_{F \otimes \Delta} < (\alpha \oplus 1) \cdot U_{G \otimes \Delta}$$

which shows [12] that the two stabilization maps $E_m \rightarrow E_{m+n}$ defined respectively by

$$\alpha \cdot U_F \rightarrow (\alpha \oplus 1) \cdot U_{F^{(n)}} \quad \text{and} \quad \alpha \cdot U_F \rightarrow (\alpha \oplus 1) \cdot U_{G \otimes \Delta}$$

are homotopic. Note that the second stabilization does not take the base point $U_\Delta$ of $E_m$ to the base point $U_\Delta$ of $E_{m+n}$. However, consider the contractible complex $P^m$ as embedded in $E_m$ by the correspondence $F \rightarrow U_F$. Then stabilization $E_m \rightarrow E_{m+n}$ via $F \rightarrow F \otimes \Delta$ takes $P^m$ to $P^{m+n}$ and hence determines a base point preserving map well defined up to base point preserving homotopy. From now on in this section we use the second stabilization.

If $\alpha_0 \cdot U_{F_0} < \alpha_1 \cdot U_{F_1}$ in $E_m$ and $\beta_0 \cdot U_{G_0} < \beta_1 \cdot U_{G_1}$ in $E_n$, then $(\alpha_0 \oplus \beta_0) \cdot U_{F_0 \otimes G_0} < (\alpha_1 \oplus \beta_1) \cdot U_{F_1 \otimes G_1}$ in $E_{m+n}$. This gives a map
\[ \gamma_{m,n} : E_m \times E_n \to E_{m+n} \]

which does not preserve base point; but since \( \gamma_{m,n}(P^n \times P^n) \subset P^{m+n} \), it does determine a base point preserving map well defined up to base point preserving homotopy.

**Proposition 2.** The diagrams \((n \geq 2)\)

\[
\begin{array}{ccc}
E_n \times E_n & \xrightarrow{\gamma_{n,n}} & E_{2n} \times E_{2n} \\
\downarrow & & \downarrow \\
E_{2n} & \xrightarrow{\gamma_{2n,2n}} & E_{4n}
\end{array}
\]

are commutative up to base point preserving homotopy and give rise to an H-space structure on \( E^{BN}(A) \).

**Proof of Proposition 2.** The left-hand restriction map \( \Phi = \gamma_{n,n} : E_n \times U_{\Delta} \to E_{2n} \) is defined by \( \alpha \cdot U_F \to (\alpha \oplus 1) \cdot U_{F \oplus F} \). The main step will be to show that \( \Phi \) is homotopic by a base point preserving homotopy to the right-hand restriction \( \Psi = \gamma_{n,n} : U_{\Delta} \times E_n \to E_{2n} \), which is given by the correspondence \( \alpha \cdot U_F \to (1 \oplus \alpha) \cdot U_{\Delta \oplus F} \). The two maps \( E_n \times E_n \to E_{4n} \) of (**) are induced by the homomorphisms

(a) \[
(\alpha, \beta) \to \begin{pmatrix}
\alpha & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \beta & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

and

(b) \[
(\alpha, \beta) \to \begin{pmatrix}
\alpha & 0 & 0 & 0 \\
0 & \beta & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

The homotopy commutativity of (**) is obtained by applying essentially the same argument for \( \Phi \sim \Psi \) to the "second and third rows and columns." Finally, the \( \gamma_{n,n} \) are telescoped together to give the H-space structure on \( E^{HN}(A) \).
The proof that \( \Phi \sim \Psi \) will be based on the matrix identities
\[
\begin{pmatrix}
\alpha & 0 \\
0 & 1
\end{pmatrix}
\cdot
\begin{pmatrix}
\alpha^{-1} & 0 \\
0 & \alpha
\end{pmatrix}
=
\begin{pmatrix}
1 & 0 \\
0 & \alpha
\end{pmatrix},
\]
\[
\begin{pmatrix}
\alpha^{-1} & 0 \\
0 & \alpha
\end{pmatrix}
=
\begin{pmatrix}
1 & \alpha^{-1} \\
0 & -\alpha
\end{pmatrix}
\cdot
\begin{pmatrix}
\alpha & 0 \\
0 & 1
\end{pmatrix}
\cdot
\begin{pmatrix}
1 & \alpha^{-1} \\
0 & 0
\end{pmatrix}
\cdot
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\cdot
\begin{pmatrix}
1 & -1 \\
0 & 0
\end{pmatrix}
\cdot
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}
\]
together with the following six commutative squares wherein if \( x, y, \) and \( z \) are matrices, then \( x \rightarrow^2 y \) means \( y = x \cdot z \):

\[
\begin{array}{ccc}
\text{(i)} & \frac{(a \, 0)}{(0 \, 1)} & \Rightarrow \frac{(a \delta \, 0)}{(0 \, 1)} \\
\begin{pmatrix} 1 & a^{-1} \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} \delta & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & \delta^{-1}a^{-1} \\ 0 & 1 \end{pmatrix}
\end{array}
\]

\[
\begin{array}{ccc}
\text{(ii)} & \frac{(a \, 0)}{(0 \, 1)} & \Rightarrow \frac{(a \delta \, 0)}{(0 \, 1)} \\
\begin{pmatrix} 1 & 0 \\ -\alpha & 1 \end{pmatrix} & \begin{pmatrix} \delta & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ -\alpha \delta & 1 \end{pmatrix}
\end{array}
\]

\[
\begin{array}{ccc}
\text{(iii)} & \frac{(a \, 0)}{(0 \, 1)} & \Rightarrow \frac{(a \delta \, 0)}{(0 \, 1)} \\
\begin{pmatrix} 1 & a^{-1} \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} \delta & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & \delta^{-1}a^{-1} \\ 0 & 1 \end{pmatrix}
\end{array}
\]

\[
\begin{array}{ccc}
\text{(iv)} & \frac{(a \, 0)}{(0 \, 1)} & \Rightarrow \frac{(a \delta \, 0)}{(0 \, 1)} \\
\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} \delta & 1-\delta \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}
\end{array}
\]

\[
\begin{array}{ccc}
\text{(v)} & \frac{(a \, 0)}{(0 \, 1)} & \Rightarrow \frac{(a \delta \, 0)}{(0 \, 1)} \\
\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 1-\delta \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}
\end{array}
\]

\[
\begin{array}{ccc}
\text{(vi)} & \frac{(a \, 0)}{(0 \, 1)} & \Rightarrow \frac{(a \delta \, 0)}{(0 \, 1)} \\
\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & \alpha \delta \end{pmatrix} & \begin{pmatrix} 1 & -1 \\ 0 & \alpha \delta \end{pmatrix}
\end{array}
\]

\text{Step 1. Consider the correspondence}
The commutative square (i) shows this is order preserving and hence defines a simplicial map $\Phi_1: E_n \to E_{2n}$. But
\[
\left( \begin{array}{c} 1 \\ \alpha^{-1} \\ 0 \\ 1 \end{array} \right) \in U_{F \Delta}
\]
so therefore
\[
\left( \begin{array}{c} \alpha \\ 0 \\ 0 \\ 1 \end{array} \right) \cdot U_{F \Delta} = \left( \begin{array}{c} \alpha \\ 0 \\ 0 \\ 1 \end{array} \right) \cdot U_{F \Delta}.
\]
Hence $\Phi_1 = \Phi$.

Step 2. Before applying (ii) a preliminary homotopy of $\Phi_1$ must be made. For each $F = X_1|X_2|\cdots|X_r$ in $P^n$ let $\Delta \subset F = \{e_{n+1}, \ldots, e_{2n}\}|X_1|X_2|\cdots|X_r$. If $F < G$ then, $\Delta \subset F < \Delta \subset G$; and if $\delta \in U_F$, then $\delta \oplus 1 \in U_{\Delta \subset F}$. Hence the correspondence
\[
\left( \begin{array}{c} \alpha \\ 0 \\ 0 \\ 1 \end{array} \right) \cdot U_{F \Delta} \rightarrow \left( \begin{array}{c} \alpha \\ 0 \\ 0 \\ 1 \end{array} \right) \cdot U_{\Delta \subset F}
\]
preserves order and defines a simplicial map $\Phi_{1.5}: E_n \to E_{2n}$. We claim that $\Phi_1$ and $\Phi_{1.5}$ are homotopic by a base point preserving homotopy: Let $g_1$ and $g_2$ be the two order preserving maps from $P^n$ to $P^{2n}$ defined respectively by $g_1(F) = F \oplus \Delta$ and $g_2(F) = \Delta \subset F$. Apply Lemma of §1 to find a subdivision $(P^n \times I)'$ of $P^n \times I$ and an order preserving map $w: (P^n \times I)' \rightarrow P^{2n}$ satisfying conditions (a) through (d). Now consider the simplicial map $\pi: E_n \rightarrow P^n$ which takes the vertex $\alpha \cdot U_F$ to the vertex $F$. This is nondegenerate on simplices. Similarly, if we give $E_n \times I$ and $P^n \times I$ the standard triangulations, then the natural simplicial map $\pi \times 1: E_n \times I \rightarrow P^n \times I$ is also nondegenerate on each simplex. Hence the subdivision $(P^n \times I)'$ of $P^n \times I$ induces a subdivision $(E_n \times I)'$ of $E_n \times I$. Now let $\sigma = (\alpha_0 \cdot U_{F_0} < \cdots < \alpha_k \cdot U_{F_k})$ be a simplex of $E_n$ and let $v$ be a vertex in the standard triangulation of $\pi(\sigma) \times I$ in $P^n \times I$. Then $w(v) = F \oplus \Delta$ or $w(v) = \Delta \subset F$ where $F$ is one of the $F_i$. Since $\delta \oplus 1 \in U_{F \Delta}$ and $\delta \oplus 1 \in U_{\Delta \subset F}$ for each $\delta \in U_{F_0}$, condition (d) of the Lemma shows that $\delta \oplus 1 \in U_{\delta(v)}$ for any vertex $v$ of the new triangulation of $\pi(\sigma) \times I$. Now let $u$ be any vertex in the new triangulation of $\pi \times I$ and let $v = (\pi \times 1)(u)$. Define
\[
\Omega(u) = \left( \begin{array}{c} \alpha_0 \\ 0 \\ 1 \end{array} \right) \cdot U_{w(v)}.
\]

The above remarks show $\Omega(u)$ is independent of the representative $\alpha_0$ of the class $\alpha_0 \cdot U_{F_0}$ and we get a simplicial map $\Omega: (E_n \times I)' \rightarrow E_{2n}$ which is the
required homotopy between $\Phi_1$ and $\Phi_{1.5}$.

Now consider the correspondence

$$
1.5' \quad \alpha \cdot U_F \to \begin{pmatrix} 0 & 1 \\ -\alpha & 1 \end{pmatrix} \cdot U_{\Delta} \alpha F.
$$

The commutative square (ii) shows this is order preserving and induces a simplicial map $\Phi_{1.5'}: E_n \to E_{2n}$ such that $\Phi_{1.5'} = \Phi_{1.5}$ because $(-\alpha 0) \in U_{\Delta} \alpha F$ for all $F \in P^n$. Arguing as above, we see that $\Phi_{1.5}$ is homotopic to the simplicial map $\Phi_2: E_n \to E_{2n}$ defined by the correspondence

$$
2 \quad \alpha \cdot U_F \to \begin{pmatrix} 0 & 1 \\ -\alpha & 1 \end{pmatrix} \cdot U_{F \oplus \Delta}.
$$

**Step 3.** Consider the map

$$
3 \quad \alpha \cdot U_F \to \begin{pmatrix} 0 & 1 \\ -\alpha & 0 \end{pmatrix} \cdot U_{F \oplus \Delta}.
$$

The square (iii) shows this is order preserving and we get a simplicial map $\Phi_3: E_n \to E_{2n}$ which agrees with $\Phi_2$ because

$$
\begin{pmatrix} 1 & \alpha^{-1} \\ 0 & 1 \end{pmatrix} \in U_{F \oplus \Delta} \quad \text{for all } F \in P^n.
$$

**Step 4.** Square (iv) shows that the correspondence

$$
4 \quad \alpha \cdot U_F \to \begin{pmatrix} 0 & 1 \\ -\alpha & \alpha \end{pmatrix} \cdot U_{F \oplus \Delta}
$$

defines an order preserving simplicial map $\Phi_4: E_n \to E_{2n}$ which agrees with $\Phi_3$.

**Step 5.** As in Step 2 it is first necessary to deform $\Phi_4$ by a homotopy before using (v). Recall [18] that if $F \in P^n$ is of the form $F = X_1 | X_2 | \cdots | X_r$, then $F \Box F \in P^{2n}$ is defined as

$$
F \Box F = X_1' | X_1' | X_2' | X_2' | \cdots | X_r' | X_r
$$

where $X_i' \subset \{ e_{n+1}, \ldots, e_{2n} \}$ is obtained from $X_i$ by adding $n$ to the indices of the $e_i \in X_i$. Consider the two maps $g_1, g_2: \{ \text{vertices of } P^n \} \to P^{2n}$ defined by $g_1(F) = F \oplus \Delta$ and $g_2(F) = F \Box F$. The map $g_1$ is order preserving but $g_2$ is not! Apply the Lemma of §1 to construct a simplicial map $w: (P^n \times I)' \to P^{2n}$ satisfying (a) through (d). As in Step 2, let $\sigma = (\alpha_0 \cdot U_{F_0} < \cdots < \alpha_k \cdot U_{F_k})$ be a simplex of $E_n$ and let $v$ be a vertex in the standard triangulation of $\pi(\sigma) \times I$ in $P^n \times I$. Then $w(v) = F \oplus \Delta$ or $w(v) = F \Box F$ where $F$ is one of the $F_i$. Now for each $\delta \in U_{F_0}$, the matrix $\begin{pmatrix} \delta^{-1} & \delta \end{pmatrix}$ lies in $U_{F \oplus \Delta}$ and also in $U_{F \Box F}$. Hence
condition (d) of the Lemma shows that \((\delta, 1, -\delta) \in U_{\omega(\delta)}\) for each vertex \(\nu\) of the new triangulation of \(\pi(\sigma) \times I\) in \((P^n \times I)'\). For any vertex \(u\) in the new triangulation of \(\sigma \times I\) in \((E_n \times I)'\) let \(v = (\sigma \times 1)(u)\) and define

\[
\Omega(u) = \begin{pmatrix} 0 & 1 \\ -\alpha_0 & \alpha_0 \end{pmatrix} \cdot U_{\omega(\delta)}.
\]

Then the preceding remarks show \(\Omega(u)\) is independent of the representative \(\alpha_0\) of \(\alpha_0 \cdot UF\) and we get a simplicial map \(\Omega: (E_n \times I)' \to E_{2n}\). Let \(\Phi_4 = \Omega| (E_n \times I)'\). For each vertex \(\nu\) of \(\pi(\sigma) \times 1\) in the standard triangulation, the matrix \((1, 0, 0)\) belongs to \(U_{\nu(\delta)}\). Hence by (d) of the Lemma this matrix belongs to \(U_{\omega(\delta)}\) for any vertex \(\nu\) of \(\pi(\sigma) \times 1\)' Therefore for any vertex \(u\) in \(\sigma \times I\) we have

\[
\Phi_4(u) = \begin{pmatrix} 1 & 1 \\ 0 & \alpha_0 \end{pmatrix} \cdot U_{\omega(\delta)}
\]

where \(\nu = (\sigma \times 1)(u)\). See (v). Since \((\delta, 1, -\delta) \in U_{\omega(\delta)}\) for every \(\delta \in U_{F_0}\), this new formula for \(\Phi_4\) is independent of the choice of representative \(\alpha_0\) of \(\alpha_0 \cdot U_{F_0}\) by (v). Since \((1, -\delta)\) belongs to \(U_{\Delta F}\) and to \(U_{F \Box F}\) for \(\delta \in U_{F_0}\) and \(F = F_0, \ldots, F_k\), we can construct as above a homotopy between \(\Phi_4\) and \(\Phi_5: E_n \to E_{2n}\) defined by the order preserving correspondence

\[
\alpha \cdot U_F \to \begin{pmatrix} 1 & 1 \\ 0 & \alpha \end{pmatrix} \cdot U_{\Delta F}.
\]

**Step 6.** Finally (vi) shows that \(\Phi_5\) is the same as \(\Psi\), which is defined by

\[
\alpha \cdot U_F \to \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \cdot U_{\Delta F},
\]

because \((1, -1)\) lies in each \(U_{\Delta F}\).

The homotopy between \(\Phi\) and \(\Psi\) in Steps 1 through 6 does not keep the base point fixed. However, it can be deformed to one which does, because \(\Delta \triangleleft \Delta = \Delta \Box \Delta\) and hence \(U_\Delta\) is deformed along a path of the form \(\gamma \cdot \gamma^{-1}\) where \(\gamma\) is the path traced out by \(U_\Delta\) during the first three steps of the argument. Q.E.D.

3. Action on the fiber. In §1 we saw that there is a homotopy equivalence \(\theta: X \to E(\alpha \cdot U_F) \approx E^{BN}(A)\) where \(X\) is the homotopy fiber of \(B(U_F)\) \to \(BE(A)\).

**Proposition 3.** Under \(\theta\) the action of \(\pi_1 BE(A)\) on \(X\) can be identified up to homotopy with left multiplication of \(E(A)\) on \(E(\alpha \cdot U_F)\) which, moreover, induces the identity on \(H_\ast(E(\alpha \cdot U_F))\).
Proof. If \( f: K \to L \) is any map we convert it into an actual fibration \( X_f \to E_f \to \pi_f L \) as usual by letting \( E_f \) be the set of pairs \((x, \omega)\) where \( x \in K \) and \( \omega \) is a path in \( L \) with \( \omega(1) = f(x) \). The map \( \pi_f \) takes \((x, \omega)\) to \( \omega(0) \). The fiber \( X_f \) consists of those \((x, \omega)\) for which \( \omega(0) = \) base point of \( L \). Applying this to the horizontal rows of the pullback square of \( \S 1 \) defining \( E(\alpha \cdot U_F) \) gives the commutative diagram

\[
\begin{array}{ccc}
\text{pt} & \to & G \\
\downarrow & & \downarrow \\
X_i & \to & E_i \\
\downarrow & & \downarrow \pi_i \\
X_f & \to & E_f \\
\downarrow & & \downarrow \pi_f \\
& & \downarrow \pi \\
& & BG
\end{array}
\]

Since \( G \) is discrete, \( \pi \) has the unique path lifting property. This implies \( E_i \) is homomorphic to the pullback of \( \pi_f \) and \( \pi \). Hence all the horizontal and vertical rows are fibrations, \( X_i \to X_f \) is a homeomorphism, and \( X_f \simeq E_f \) because \( EG \) is contractible. A specific homotopy equivalence \( \theta: X_f \to E_f \) is defined by \( \theta(x, \nu) = (x, \nu(1); \nu) \) where \( \nu \) is the unique path in \( EG \) starting at the base point and lifting \( \nu \). Now let \( \gamma \) be a fixed loop in \( BG \) representing \( g \in \pi_1 BG \). Then

\[
\theta(g \cdot (x, \nu)) = \theta(x, \gamma \cdot \nu) = (x, \gamma \cdot \nu(1); \gamma \cdot \nu)
\]

and

\[
g \cdot \theta(x, \nu) = g \cdot (x, \nu(1); \nu) = (x, g \cdot \nu(1); g \cdot \nu).
\]

It follows using the standard construction as in [6] for the universal cover of \( BG \) that these two maps are homotopic.

It remains to show the correspondence \( \alpha \cdot U_F \to g \alpha \cdot U_F \) induces the identity on \( H_n(E(\alpha \cdot U_F)) \) \( = H_n(E^{BN}(A)) \). Any homology class is supported in some \( E_n \) and we can choose \( n \) large enough to have \( g \in E(n, A) \). In \( \S 2 \) it was shown that there is a subdivision \((E_n \times I)' \) of the standard triangulation of \( E_n \times I \) such \((E_n \times 0)' = E_n \times 0 \) and \((E_n \times 1)' = E_n \times 1 \) and there is a simplicial map \( h: (E_n \times I)' \to E_{2n} \) having the property that \( h_0 = h|E_n \times 0 \) is

\[
\alpha \cdot U_F \to (\alpha \oplus 1) \cdot U_{F\oplus \Delta}
\]

and \( h_1 = h|E_n \times 1 \) is
\[ \alpha \cdot U_F \rightarrow (1 \oplus \alpha) \cdot U_{\Delta F}. \]

Hence \( g \cdot h_0 : E_n \rightarrow E_{2n} \) defined by
\[ \alpha \cdot U_F \rightarrow (g\alpha \oplus 1) \cdot U_{\Delta F}, \]
is homotopic to \( g \cdot h_1 : E_n \rightarrow E_{2n} \) defined by
\[ \alpha \cdot U_F \rightarrow (g \oplus \alpha) \cdot U_{\Delta F}. \]
Therefore it suffices to show \( g \cdot h_1 \) is homotopic to the map \( E_n \rightarrow E_{2n} \) given by
\[ \alpha \cdot U_F \rightarrow (1 \oplus \alpha) \cdot U_{\Delta F}. \]

Since \( g \in E(n, A) \) is the product of elements lying in the subgroups \( U_\alpha \), this fact is in turn a consequence of several applications of the following: Let \( G < G' \) in \( P^n \) and assume \( g \in U_{G'} \). Let \( x \in E(n, A) \). Then the two maps \( E_n \rightarrow E_{2n} \) defined respectively by
\[ \alpha \cdot U_F \rightarrow \left( \begin{array}{cc} x & 0 \\ 0 & \alpha \end{array} \right) \cdot U_{G \oplus F} \]
and
\[ \alpha \cdot U_F \rightarrow \left( \begin{array}{cc} xg & 0 \\ 0 & \alpha \end{array} \right) \cdot U_{G' \oplus F} \]
are homotopic. But this is clear because
\[ \left( \begin{array}{cc} x & 0 \\ 0 & \alpha \end{array} \right) \cdot U_{G \oplus F} \leq \left( \begin{array}{cc} x & 0 \\ 0 & \alpha \end{array} \right) \cdot U_{G' \oplus F} = \left( \begin{array}{cc} xg & 0 \\ 0 & \alpha \end{array} \right) \cdot U_{G' \oplus F}. \]

4. H-space structure on \( B\{U_F\}^+ \). In this section we show how direct sum of matrices gives an H-space structure on \( B\{U_F\}^+ \).

By a sheaf of spaces over a simplicial complex \( K \) we mean a collection \( X = \{X_\sigma\}, \sigma = \) simplex of \( K \), together with connecting maps \( i_{\sigma \tau} : X_\tau \rightarrow X_\sigma \) for \( \sigma < \tau \) such that \( i_{\sigma \sigma} \circ i_{\tau \tau} = i_{\sigma \tau} \) whenever \( \sigma < \tau < \gamma \). The realization \( |X| \) of \( X \) is the disjoint union \( \coprod_{\sigma \subseteq K} \sigma \times X_\sigma \) modulo the identification setting \( (x,y) = (x',y') \) iff \( x = x' \) and \( y = i_{\sigma \tau}(y) \) for \( y \in X_\tau, y' \in X_\sigma \), and \( \sigma < \tau \). Any simplicial subdivision \( K' \) of \( K \) induces a subdivision \( X' \) of \( X \) as follows: For \( \tau \) a simplex of \( K' \) let \( X' = X_\sigma \) where \( \sigma \) is the smallest simplex of \( K \) containing \( \tau \). If \( \tau < \tau' \) and \( \sigma, \sigma' \) are the smallest simplices of \( K \) containing \( \tau, \tau' \) respectively, then \( \sigma < \sigma' \) and we let \( i_{\sigma \tau'} = i_{\sigma \sigma'} \). The natural map \( |X'| \rightarrow |X| \) is a homeomorphism. Define \( X \times I \) to be the sheaf of spaces over \( K \times I \) by setting \( (X \times I)_\sigma = X_\sigma \) where \( \sigma \) is the smallest simplex of \( K \) such that \( \tau \subseteq \sigma \times I \). The space \( B_n \) is clearly the realization of a sheaf of spaces of the form \( BU_F \) over \( P^n \).
The direct sum homomorphism $\oplus$ from $E(m, A) \times E(n, A) \to E(m + n, A)$ gives a family of compatible homomorphisms

$$\oplus: U_F \times U_G \to U_{F \oplus G}$$

and hence a family of compatible maps

$$\oplus: BU_F \times BU_G \to BU_{F \oplus G}$$

which fit together to give maps

$$\rho_{m,n}: B_m \times B_n \to B_{m+n}.$$  

By the universal property of the "plus construction" we get base point preserving maps

$$\rho^+_{m,n}: B^+_m \times B^+_n \to B^+_{m+n}.$$  

A word about base points: The complex $P^n$ can be embedded in $B_n$ by sending $x \in \sigma = (F_0 < \cdots < F_k)$ to $(x, \text{base point of } BU_F)$. The base point $*$ of $B_n$ is the base point of $BU_{\Delta}$. The map $\rho_{m,n}$ does not preserve the base point but $\rho_{m,n}(P^m \times P^n) \subset P^{m+n}$. Hence $\rho_{m,n}$ determines a base point preserving map $\rho^+_{m,n}$ defined up to base point preserving homotopy. Also, the stabilization map $B_n \to B_{2n}$ given by $B U_F \to B U_{F \oplus \Delta}$ is not the standard inclusion $B_n \to B_{2n}$ which is given by $B U_F \to B U_{F(n)}$. However, $F(\alpha) < F \oplus \Delta$ for all $F \in P^n$ implies these two maps are homotopic up to a base point preserving homotopy; so from now on we use the stabilization induced by $F \to F \oplus \Delta$.

**Proposition 4.** For $n \geq 3$ the diagrams

$$\begin{array}{ccc}
B^+_n \times B^+_n & \longrightarrow & B^+_4 \times B^+_4 \\
\rho^+_{n,n} \downarrow & & \rho^+_{4,4} \downarrow \\
B^+_2 & \longrightarrow & B^+_4 \\
\end{array}$$

are homotopy commutative and give rise to an $H$-space structure on $B(U_F)^+$ in such a way that $B(U_F)^+ \to BE(A)^+$ is an $H$-map.

**Remark.** It follows immediately from [13] that the homotopy theoretic fiber $X$ of $B(U_F)^+ \to BE(A)^+$ is a connected $H$-space as required by (II) of §1.

**Proof of Proposition 4.** The restriction map $\Phi = \rho_{n,n}: B_n^+ \times \bullet \to B_{2n}^+$ comes from the inclusion $B_n \to B_{2n}$ induced by the stabilization homomorphism.
The essential step in the proof is to show that the restriction map
\[ \Psi = \rho_{n,n} : \ast \times B_n^+ \rightarrow B_{2n}^+ \]
is homotopic to \( \Phi \) by a base point preserving homotopy. Note that \( \Psi \) is induced by the homomorphism
\[ \psi(\alpha) = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \]
satisfying \( \psi(U_F) \subseteq U_{\Delta \Theta F} \) for each \( F \in P^n \). By the universal property of the "plus construction" it suffices to show

(†) For \( n \geq 3 \) the maps \( \Phi, \Psi : B_n \rightarrow B_{2n}^+ \) are homotopic by a base point preserving homotopy.

In fact \( B_{2n}^+ \) is simply connected for \( 2 < n \) so it suffices to show \( \Phi \) and \( \Psi \) are homotopic as maps into \( B_{2n}^+ \). To prove this we shall use the following matrix identities:

\[
\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},
\]
\[
\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}.
\]

**Step 1.** Consider the homomorphism \( \phi_1 : E(n,A) \rightarrow E(2n,A) \) given by

\[ \phi_1(\alpha) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \alpha - 1 \\ 0 & 1 \end{pmatrix}. \]

For each \( F \in P^n \) the matrix \( \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \in U_{\Delta \Theta A} \); so \( \phi_1(U_F) \subseteq U_{\Delta \Theta A} \). The induced maps \( BU_F \rightarrow BU_{\Delta \Theta A} \) fit together to give a map \( \Phi_1 : B_n \rightarrow B_{2n} \). The construction of [12] gives a specific homotopy \( H_F : BU \times I \rightarrow BU_{\Delta \Theta A} \) for each facette \( F \in P^n \) between the maps induced by the conjugate homomorphisms \( \phi \) and \( \phi_1 \) such that \( H_F | BU \times I = H_F \) whenever \( F < F' \). Hence these homotopies fit together to give a homotopy \( H : B_n \times I \rightarrow B_{2n} \) from \( \Phi \) to \( \Phi_1 \).

**Step 2.** Let \( f, g : P^n \rightarrow P^{2n} \) be defined by \( f(F) = F \oplus \Delta \) and \( g(F) = F \square F \). The map \( f \) preserves order but \( g \) does not. Apply the lemma of §1 to find an order preserving map \( w : (P^n \times I) \rightarrow P^{2n} \) of some subdivision of the standard triangulation of \( P^n \times I \) satisfying (a) through (d). This subdivision induces a subdivision \( (B_n \times I) \) of \( B_n \times I \). Consider a simplex \( \sigma = (F_0 < \cdots < F_k) \) in \( P^n \) and let \( v \) be any vertex of a simplex in the standard triangulation of \( \sigma \times I \). Then \( w(v) = F \oplus \Delta \) or \( w(v) = F \square F \) where \( F \) is one of the facettes.
The formula for \( \phi_1 \) shows that \( \phi_1(U_{F_0}) \subset U_{w(\psi)} \) and hence by (d) the same inclusion holds for any vertex \( v \) of \((\sigma \times I)'.\) This yields homotopy \( \Omega \colon (B_n \times I)' \to B_{2n} \) between \( \Phi_1 \) and the map \( \Phi'_1 \colon (B_n \times I)' \to B_{2n} \) defined as the restriction of \( \Omega \) to \((B_n \times I)' .\) Now for \( F \in P^n \) the matrix \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) belongs to \( U_F \bigoplus F.\) Applying (d) of the lemma shows that \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in U_{w(v)} \) for any vertex \( v \) of \((P^n \times 1)' .\) Let \( \phi_2 \colon E(n, A) \to E(2n, A) \) be defined by

\[
\phi_2(\alpha) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \alpha - 1 \\ 0 & \alpha \end{pmatrix}
\]

and let \( \Phi'_2 \) be the induced map. More precisely, if \( v \) is a vertex of \((P^n \times 1)', \) let \( \sigma = (F_0 < \cdots < F_k) \) be the smallest simplex of \( P^n \times 1 \) such that \( v \in \sigma .\) Then \( \phi_2(U_{F_0}) \subset U_{w(\psi)} \) so there is an induced map \( BU_{F_0} \to BU_{w(\psi)} \) These fit together to give \( \Phi'_2 .\) Since \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in U_{w(v)} \) for any vertex \( v \) of \((P^n \times 1)', \) the conjugate homomorphisms \( \phi_1 \) and \( \phi_2 \) give rise to homotopic maps \( \Phi_1 \) and \( \Phi_2 .\)

Now let \( f, g \colon P^n \to P^{2n} \) be defined by \( f(F) = F \square F \) and \( g(F) = \Delta \oplus F.\) The map \( g \) preserves order but \( f \) does not. Use the lemma to construct a homotopy \( w \colon (P^n \times I)' \to P^{2n} \) satisfying (a) through (d). Again let \( \sigma = (F_0 < \cdots < F_k) \) be a simplex of \( P^n \times 1 \simeq P^n \) and let \( v \) be a vertex in the standard triangulation. Then \( w(v) = F \square F \) or \( w(v) = \Delta \oplus F \) where \( F \) is one of the \( F_i \) The formula for \( \phi_2 \) shows that \( \phi_2(U_{F_0}) \subset U_{w(\psi)} \) and hence by (d) the same is true for any vertex \( v \) of \((\sigma \times I)'.\) As above there is a homotopy between \( \Phi'_2 \) and \( \Phi_2 \colon B_n \to B_{2n} \) which, by definition, is obtained by fitting together the maps \( BU_F \to BU_{\Delta \oplus F} \) induced by \( \phi_2 .\)

**Step 3.** Let \( \phi_3 \colon E(n, A) \to E(2n, A) \) be defined by

\[
\phi_3(\alpha) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha - 1 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}
\]

Note that \( \phi_3 = \psi .\) Since \( \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \in U_{\Delta \oplus F} \) for all \( F \in P^n \) we get a homotopy between \( \Phi_2 \) and \( \Psi \) as in Step 1.

To show the homotopy commutativity of the diagram in Proposition 4 one applies essentially the same arguments as above to the second and third rows and columns of the homomorphisms (a) and (b) of §2.

Finally, the maps \( \rho_{n,n} \colon B_n^+ \times B_n^+ \to B_{2n}^+ \) telescope together to give the \( H \)-space structure on \( B(U_F)^+ .\) That \( B(U_F)^+ \to BE(A)^+ \) is an \( H \)-map follows immediately from the fact that the \( H \)-space structure on \( BE(A)^+ \) arises from the direct sum maps

\[
BE(n, A) \times BE(n, A) \to BE(2n, A)
\]

and that we have a strictly commutative diagram
\[ B_n \times B_n \longrightarrow B_{2n} \]
\[ \downarrow \quad \downarrow \]
\[ BE(n, A) \times BE(n, A) \longrightarrow BE(2n, A) \]

which commutes up to base point preserving homotopy when the "plus construction" is performed. Q.E.D.

REFERENCES


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