THE ISOMORPHISM PROBLEM FOR TWO-GENERATOR ONE-RELATOR GROUPS WITH TORSION IS SOLVABLE

BY

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ABSTRACT. The theorem stated in the title is obtained by determining (in a sense to be made precise) all the generating pairs of an arbitrary two-generator one-relator group with torsion. As a consequence of this determination it is also deduced that every two-generator one-relator group $G$ with torsion is Hopfian, and that the automorphism group of $G$ is finitely generated.

1. Introduction. The main aim of this paper is to establish

THEOREM 1. There is an algorithm to decide for any two presentations $\langle x_1, x_2; P^m \rangle$, $\langle x_1, x_2; Q^n \rangle$, where $m, n > 1$, whether or not the presentations define isomorphic groups.

This theorem is obtained as a consequence of the following lemma.

Let $G$ be a two-generator group. Recall [9] that two generating pairs $(g_1, g_2), (g'_1, g'_2)$ of $G$ are said to be Nielsen equivalent if there is an automorphism $x_1 \mapsto Y_1(x_1, x_2), x_2 \mapsto Y_2(x_1, x_2)$ of the free group $F_2$ on $x_1, x_2$ such that $g_i' = Y_i(g_1, g_2)$ for $i = 1, 2$. Also, the pairs $(g_1, g_2), (g'_1, g'_2)$ are said to lie in the same $T$-system if there is an automorphism $\xi$ of $G$ such that $(\xi(g_1), \xi(g_2))$.

PRINCIPAL LEMMA. Let $G = \langle a, t; R^n \rangle$ where $R$ is not a true power, and where $n > 1$. If $R$ is a primitive in the free group on $a, t$ then $G$ has one Nielsen equivalence class when $n = 2$, or $\varphi(n)$ Nielsen equivalence classes and one $T$-system when $n > 2$. If $R$ is not a primitive then $G$ has one Nielsen equivalence class.

Here $\varphi$ denotes the Euler totient function.

To see how Theorem 1 follows from the Principal Lemma observe that by Lemma 1 of [9] and the Principal Lemma above, two presentations $\langle x_1, x_2; P^m \rangle$, $\langle x_1, x_2; Q^n \rangle$, where $m, n > 1$, are presentations of isomorphic groups if and only if there is an automorphism of $F_2$ mapping $P^m$ to $Q^n$. Since there

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is an algorithm to decide for any two elements $S$ and $T$ of $F_2$ whether or not $T$ is equal to the image of $S$ under an automorphism of $F_2$ (see Theorem N2 of [3]), Theorem 1 follows.

Apart from its use in proving Theorem 1, the Principal Lemma is also helpful in obtaining other information concerning two-generator one-relator groups with torsion.

Theorem 2. Let $G = \langle a, t; R^n \rangle$ where $n > 1$. Then $G$ is Hopfian.

This is an immediate consequence of the Principal Lemma and Theorem 2 of [9].

Theorem 3. Let $G = \langle a, t; R^n \rangle$ where $n > 1$. Then the automorphism group, $\text{Aut}(G)$, of $G$ is finitely generated$^\text{(!)}$.

This result is easily proved for the case when $R$ is a power of a primitive.

Suppose on the other hand, that $R$ is not a power of a primitive. Then $G$ has one Nielsen equivalence class by the Principal Lemma, so that every automorphism from an automorphism

$$a \mapsto Y_1(a, t), \quad t \mapsto Y_2(a, t),$$

where $(Y_1(a, t), Y_2(a, t))$ is a generating pair of the free group $F$ on $a, t$ and where $R(Y_1(a, t), Y_2(a, t))$ is equal in $F$ to either $R(a, t)$ or $R^{-1}(a, t)$. Now it is shown in [4] that the group of automorphisms of $F$ which map $R$ to $R^\pm 1$ is finitely generated. Since the group of inner automorphisms of $G$ is also finitely generated, it follows that $\text{Aut}(G)$ is finitely generated.

The present paper makes heavy use of results and techniques developed in [9] and [10]. The fact that one-relator groups are HNN groups will be made use of frequently throughout the paper, and the reader may like to consult the expository article [5] by McCool and Schupp to see how theorems concerning one-relator groups can be proved using the theory of HNN groups. The standard reference for notation and background material used throughout will be the book [3] by Magnus, Karrass and Solitar. Unexplained concepts and notation which cannot be found in [3] will be as in [9].

It is worthwhile to give here an outline of the proof of the Principal Lemma. The most difficult case to deal with is when $R$ is neither freely equal to 1 nor a primitive. To handle this case it is no loss of generality to assume that $R$ is cyclically reduced and involves $a, t$, and that the exponent sum of $R$ on $t$ is zero. Let $a_i (i = 0, \pm 1, \pm 2, \ldots)$ denote the word $t^{-i}at^i$, and let $P$ be the word obtained from $R$ by rewriting it in terms of the $a_i$. Let $m$ and $M$ be, respectively, the least and greatest integers $i$ for which $a_i$ occurs in $P$. Then, as

$^\text{(!)}$ J. McCool and I have recently established that $\text{Aut}(G)$ is finitely presented.
observed by Moldavanskiï [6], \(G\) can be presented as an HNN group as follows:

\[
G = \langle a_m, \ldots, a_M, t; P^n, t^{-1}a_it = a_{i+1} (i = m, \ldots, M - 1) \rangle.
\]

Now the associated subgroups \(K_{-1} = \text{sgp} \{a_m, \ldots, a_{M-1}\}\) and \(K_1 = \text{sgp} \{a_{m+1}, \ldots, a_M\}\) are malnormal in the base \(H = \langle a_m, \ldots, a_M; P^n \rangle\), and so it follows from Theorem 6 of [9] that every generating pair of \(G\) is Nielsen equivalent to a pair of the form \((th, k)\) where \(h\) and \(k\) belong to \(H\), and where \(k\) is a nonempty cyclically reduced word in the generators of \(K_{-1}\). Moreover \(hkh^{-1} \not\in K_1\).

Let \(k^{(i)} (i = 0, 1, \ldots)\) denote the element \((th)^{-i}k(th)^i\). Then there is an integer \(\lambda\) with \(0 < \lambda < M - m + 1\) such that \(k^{(i)} \in H\) if and only if \(0 \leq i \leq \lambda\). Moreover \(\lambda = M - m + 1\) only if \(k\) is a power of \(a_m\). The main part of the proof is involved with showing that if \((th, k)\) generates \(G\) then \(k^{(0)}, \ldots, k^{(\lambda)}\) generate \(H\). For then, since \(H\) cannot be generated by less than \(M - m + 1\) elements, it follows that \(k = a_m^i\) for some integer \(i\). It can then be established without too much difficulty that \((th, a_m)\) generates \(G\) if and only if \(|l| = 1\) and \(th = a_m^a, a_m^\beta\) for suitable integers \(\alpha, \beta\). Thus \((th, k)\) is Nielsen equivalent to \((t, a_m^i)\), so that \(G\) has one Nielsen equivalence class.

In order to show that \(k^{(0)}, \ldots, k^{(\lambda)}\) generate \(H\) whenever \(th, k\) generate \(G\), it must be established that a word \(W\) in \(th, k^{(0)}, \ldots, k^{(\lambda)}\) which defines an element of \(H\) is equal to a word in \(k^{(0)}, \ldots, k^{(\lambda)}\) alone. This is easily proved using Britton's lemma and induction on the \(\tau\)-length of \(W\) once the following formulae have been established:

\[
\text{sgp} \{k^{(0)}, \ldots, k^{(\lambda)}\} \cap K_{-1} = \text{sgp} \{k^{(0)}, \ldots, k^{(\lambda-1)}\},
\]

\[
h \text{sgp} \{k^{(0)}, \ldots, k^{(\lambda)}\} h^{-1} \cap K_1 = h \text{sgp} \{k^{(1)}, \ldots, k^{(\lambda)}\} h^{-1}.
\]

These formulae follow from Theorem 3 of [10] when \(\lambda > 1\). For if \(\lambda > 1\) then \(h \in K_{-1}\) and \((hk^{(0)}h^{-1}, \ldots, hk^{(\lambda)}h^{-1})\) is \((a_m, a_M)\)-admissible. However, when \(\lambda = 1\), Theorem 3 of [10] is not necessarily applicable. All one knows in general in this case is that \(k^{(0)} \in K_{-1}, k^{(1)} \not\in K_{-1}, hk^{(0)}h^{-1} \not\in K_1, hk^{(1)}h^{-1} \in K_1\). Consequently it is necessary to establish that if \(u \in K_e (|x| = 1)\) and \(v \not\in K_e\) then \(\text{sgp} \{u, v\} \cap K_e = \text{sgp} \{u\}\). In actual fact, it becomes necessary to prove a more general result than this so that the usual inductive techniques for dealing with one-relator groups can be used.

Let \(B = \langle x_j (j \in J) ; S, T, \ldots \rangle\) and for \(j \in J\) define \(L_j\) to be the subgroup of \(B\) generated by those generators of \(B\) other than \(x_j\). Then \(B\) (or more precisely this presentation of \(B\)) will be said to have property-I provided the following holds: for each \(j\) in \(J\), if \(u \in L_j\) and \(v \not\in L_j\) then \(\text{sgp} \{u, v\} \cap L_j = \text{sgp} \{u\}\). It will be shown below that
every one-relator group with torsion has property-I.

The remainder of the paper is divided into three sections. In §2 various concepts and definitions are introduced and several useful lemmas, mainly concerning HNN groups, are obtained. In §3 a proof of (•) is given. §4 investigates the generating pairs of an arbitrary two-generator one-relator group with torsion, culminating in a proof of the Principal Lemma. Each of §§2, 3, 4 is subdivided and has an introduction explaining its contents more fully.

The techniques developed in this paper will be used in a future article to describe the two-generator subgroups of an arbitrary one-relator group with torsion.

2. Preliminaries. In §2.1 the basic notation and definitions needed for the rest of the paper are introduced. It is shown how to present a one-relator group, whose defining relator has zero sum-exponent on some generator, as an HNN group, the base H of which is another one-relator group. Several lemmas concerning such an HNN group are then obtained. In §2.2 the definition is given of standard H-elements (of which the \( k^{(i)} \) in the previous section are examples). These elements are then analysed in some detail.

2.1. Definitions, notation, and some lemmas. Throughout the paper \( e \) (or some variation such as \( e', e_1, e_2 \)) will denote an integer of modulus \( 1 \). The set of integers will be denoted by \( \mathbb{Z} \). If \( v \) is a real number \([v]\) will denote the greatest integer less than or equal to \( v \).

If \( G \) is a group and \( u,v \in G \) then the element \( u^{-1}vu \) of \( G \) will be denoted by \( u^v \), and will be called the conjugate of \( u \) by \( v \). If \( A \) is a subset of \( G \) then the subgroup of \( G \) generated by \( A \) will be denoted by \( \text{sgp} A \). By convention, if \( A \) is empty then \( \text{sgp} A \) is the trivial subgroup \( 1 \).

Let \( G = \langle a,c,d,\ldots,t; R^n \rangle \) where \( n > 1 \), \( R \) is a cyclically reduced word which involves \( a, Rt^a \). For \( i \in \mathbb{Z} \) let \( a_i, c_i, d_i, \ldots \) denote the words \( t^{-i}at^i, t^{-i}ct^i, t^{-i}dt^i, \ldots \) respectively. Then \( t^{-a}Rt^a \) can be rewritten as a cyclically reduced word \( P \) in the \( a_i, c_i, d_i, \ldots \) as follows. Replace a symbol \( x \), where \( x \) is one of \( a, c, d, \ldots \), which appears in \( t^{-a}Rt^a \) by \( x^{a_i} \), where \( i \) is the exponent sum on \( t \) of the initial segment of \( t^{-a}Rt^a \) preceding \( x \). Then clearly \( a_0 \) appears in \( P \). The largest integer \( i \) for which \( a_i \) appears in \( P \) will be denoted by \( M \). Notice that \( P \) involves at least one generator having a nonzero subscript if and only if \( R \) involves \( t \). Notice
also that if $R$ involves $t$ then the length of $P$ is less than the length of $R$.

Now it is not difficult to show using Tietze transformations that

$$G = \langle a_0, \ldots, a_M, c, (i \in \mathbb{Z}), d, (i \in \mathbb{Z}), \ldots, t; P^n, \rangle$$

$$t^{-1}a_0 t = a_1, \ldots, t^{-1}a_{M-1} t = a_M,$$

$$t^{-1}c_i t = c_{i+1} (i \in \mathbb{Z}), t^{-1}d_i t = d_{i+1} (i \in \mathbb{Z}), \ldots \rangle.$$  

Let

$$H = \langle a_0, \ldots, a_M, c, (i \in \mathbb{Z}), d, (i \in \mathbb{Z}), \ldots; P^n \rangle,$$

$$K_{-1} = \langle a_0, \ldots, a_{M-1}, c, (i \in \mathbb{Z}), d, (i \in \mathbb{Z}), \ldots \rangle$$

and

$$K_1 = \langle a_1, \ldots, a_M, c, (i \in \mathbb{Z}), d, (i \in \mathbb{Z}), \ldots \rangle.$$  

Then $K_{-1}$ and $K_1$ are free on the given generators (by the Freiheitssatz), and so $G$ is presented above as an HNN group with base $H$, stable letter $t$, and associated subgroups $K_{-1}$ and $K_1$. This HNN presentation of $G$ will be called the HNN presentation of $G$ with stable letter $t$ and fixed generator $a$.

It should be noted that $P$, $M$, $H$, $K_{-1}$, $K_1$ are all dependent on $R$, $a$, $t$, but to avoid cumbersome notation (such as $P(R, a, t)$, $M(R, a, t)$, etc.) this dependence will not be made explicit. This should cause no confusion.

It was first observed by Moldavanskii [6] that if $A$ is a one-relator group whose defining relator $Q$ is cyclically reduced and has exponent sum zero on some generator occurring in it, then $A$ is an HNN extension of another one-relator group whose defining relator is shorter than $Q$. This observation was taken up by McCool and Schupp [5] and others to give rather elegant induction proofs of the basic results on one-relator groups. Such induction techniques will be employed here. However, it is not always necessary to use induction to obtain results about one-relator groups. In some cases it suffices to know that the group is a nontrivial HNN extension of another one-relator group $B$ and that the associated subgroups lie "suitably" in $B$. This was the approach adopted in [11] for example, and such an approach will also be used here.

Basic facts concerning HNN groups which will be needed in the sequel can be found in [9, §§1.2, 2.1]. Additional results will be obtained below.

It is worthwhile to make some comments concerning $t$-reducing in the HNN group $G$ above. Suppose $w$ is a word in the generators of $K_{-e}$. Then the $t$-reduced form $w'$ of $t^{-e}w^e$ is obtained from $w$ by replacing each occurrence of a generator $x_i$, where $x$ is one of $a$, $c$, $d$, $\ldots$, by $x_{i\pm e}$ (such a procedure is called
"shifting subscripts" in [5]. Now clearly \( w^t = t^r w' \). It follows that if \( W \) is a word \( w_0 r_1 w_1 \cdots r_n w_n \) where the \( w_j \) are words in \( c_i (i \in \mathbb{Z}), d_i (i \in \mathbb{Z}), \ldots \), then the \( r \)-symbols can be "pulled through" either all to the left or all to the right, so that there are words \( u \) and \( v \) in \( c_i (i \in \mathbb{Z}), d_i (i \in \mathbb{Z}), \ldots \) such that

\[
W = t^t u = v t^t,
\]

where \( s = \sum_{j=1}^{n} e_j \).

Several results relevant to the HNN group \( G \) above will now be obtained.

The first lemma is required since the associated subgroups \( K_{-1} \) and \( K_1 \) are free. The lemma is easily proved.

**Lemma 1.** Let \( \mathcal{X} \) be a set and let \( \mathcal{X}' \) be a subset of \( \mathcal{X} \). Let \( F \) and \( F' \) be the free groups on \( \mathcal{X} \) and \( \mathcal{X}' \) respectively. Suppose \( A \subseteq F \) and \( v \in F \setminus F' \). Then:

(i) \( F' \cap \text{sgp} \ A \cup \{v\} = \text{sgp} \ A \);
(ii) if \( \text{sgp} \ A \) is free on \( A \) then \( \text{sgp} \ A \cup \{v\} \) is free on \( A \cup \{v\} \).

**Example.** Let \( k \) be a nonempty freely reduced word in \( c_i (i \in \mathbb{Z}), d_i (i \in \mathbb{Z}), \ldots \), and for \( i \in \mathbb{Z} \) let \( k^{(i)} \) denote the \( r \)-reduced form of \( r^{-1} r_k^i \). Let \( q \) and \( s \) be respectively the lowest and highest integers \( j \) for which \( x_j \) (where \( x \) is one of \( c, d, \ldots \) occurs in \( k \), and for \( l, m \geq 0 \), let \( \mathcal{X}_{l,m} = \{c_i, d_i, \ldots : q - l \leq i \leq s + m\} \) and let \( F_{l,m} \) be the free group on \( \mathcal{X}_{l,m} \). Then

\[
F_{0,0} \subset F_{0,1} \subset F_{1,1} \subset F_{1,2} \subset F_{2,2} \subset \cdots .
\]

Moreover, if \( \mu > 0 \) then

\[
\{k^{(0)}, k^{(1)}, k^{(-1)}, \ldots, k^{(\mu-1)}, k^{(-\mu+1)}\} \subseteq F_{-1,\mu-1}
\]

whereas \( k^{(\mu)} \in F_{-1,\mu} \setminus F_{-1,\mu-1} \), and

\[
\{k^{(0)}, k^{(1)}, k^{(-1)}, \ldots, k^{(\mu-1)}, k^{(-\mu+1)}, k^{(\mu)}\} \subseteq F_{-1,\mu}
\]

whereas \( k^{(-\mu)} \in F_{\mu} \setminus F_{\mu-1,\mu} \). Thus by repeated use of Lemma 1(ii) it is deduced that the \( k^{(i)} (i \in \mathbb{Z}) \) freely generate a subgroup of \( \text{sgp} \{c_i (i \in \mathbb{Z}), d_i (i \in \mathbb{Z}), \ldots\} \).

As well as being free, the groups \( K_{-1} \) and \( K_1 \) have several other useful properties, which for convenience are listed here.

(2.1) \( K_{-1} \) and \( K_1 \) are malnormal in \( H \).

See Lemma 2.1 of [8]. Recall that a subgroup \( B \) of a group \( A \) is said to be malnormal in \( A \) if, for all \( g \in A \), \( g^{-1} B g \cap B \neq 1 \) implies \( g \in B \).

(2.2) A freely reduced (respectively, cyclically reduced) word in the generators of \( K_{-1} \) which involves \( a_0 \) is not equal (resp. conjugate in \( H \)) to an element of \( K_1 \), and a freely reduced (resp. cyclically reduced) word in the generators of \( K_1 \) which involves \( a_M \) is not equal (resp. conjugate in \( H \)) to an element of \( K_{-1} \).

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The result for freely reduced words follows from Newman’s Spelling Theorem (Theorem 3 of [7]). The result for cyclically reduced words is obtained as follows. Suppose for definiteness that $k$ is a cyclically reduced word in the generators of $K_{-1}$ and that $k$ involves $a_0$ (notice that this implies $M > 0$). It must be established that an equation $h^{-1}kh = u$, where $u$ is a cyclically reduced word in the generators of $K_1$ and where $h \in H$, is impossible. This follows from Lemma 2.1 of [8] if $u$ involves $a_M$. On the other hand, if $u$ does not involve $a_M$ then $u \in K_{-1}$, so that $h \in K_{-1}$ by (2.1). Thus the equation takes place in the free group $K_{-1}$, which once again is impossible.

Let $u, w_i (i \in I), v$ be elements of $H$ and let $(u, w_i (i \in I), v)$ denote an $(|I| + 2)$-tuple with $u$ in the first position, $v$ in the last position, and the $w_i$ listed in some order. The tuple will be called weakly $(a_0, a_M)$-admissible if $u \in K_{-1} \setminus K_1, v \in K_1 \setminus K_{-1}$ and $w_i \in K_{-1} \cap K_1$ for $i \in I.$ If in addition the $w_i$ freely generate a subgroup of $K_{-1} \cap K_1$ then the tuple will be called $(a_0, a_M)$-admissible. The concept of an $(a_0, a_M)$-admissible tuple of words in the generators of $H$ was introduced in [10]. It is easily seen using (2.2) that if the tuple $(u, w_i (i \in I), v)$ is $(a_0, a_M)$-admissible in the sense just defined, and if $u, w_i (i \in I), v$ are expressed as words in the generators of $K_{-1}$ or $K_1$ (whichever is appropriate) then the resulting tuple is $(a_0, a_M)$-admissible in the sense of [10]. Conversely an $(a_0, a_M)$-admissible tuple in the sense of [10] is $(a_0, a_M)$-admissible in the sense just defined, by (2.2).

(2.3) If $(u, w_i (i \in I), v)$ is weakly $(a_0, a_M)$-admissible then

$$sgp \{u, w_i (i \in I), v\} \cap K_{-1} = sgp \{u, w_i (i \in I)\}$$

and

$$sgp\{u, w_i (i \in I), v\} \cap K_1 = sgp\{w_i (i \in I), v\}$$

This follows immediately from Theorem 3 of [10] (taking account of the previous discussion) in the case when $(u, w_i (i \in I), v)$ is $(a_0, a_M)$-admissible. But obviously the fact that the $w_i$ freely generate a subgroup of $K_{-1} \cap K_1$ is immaterial.

**Lemma 2.** Let $Z$ be a $t$-reduced word and let $h$ be a $t$-free word. Suppose there is an integer $m_0$ such that $h^{m_0} \neq 1$ and $Z^{-1}h^{m_0}Z$ is not $t$-reduced. Then $Z^{-1}h^{m}Z$ is not $t$-reduced for any integer $m$.

This is simply a special case of Lemma 9 of [9], taking account of (2.1).

**Lemma 3.** Let $k$ be a freely reduced word in the generators of $K_1$ and assume $t^{-r}kt^r$ defines an element of $H$.

(i) Suppose $k$ involves an $a_i$-symbol and let $q$ and $s$ be respectively the least and greatest integers $i$ for which $a_i$ occurs in $k$. Then $-q \leq r \leq M - s$. 

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(ii) The \( t \)-reduced form of \( t^{-r}kt^r \) is obtained from \( k \) by replacing each generator \( x_i \) (where \( x \) is one of \( a, c, d, \ldots \)) appearing in \( k \) by \( x_{i+r} \).

**Proof.** (i) Suppose \( r < -q \). Let \( k^* \) be the word obtained from \( k \) by replacing each generator \( x_i \) (where \( x \) is one of \( a, c, d, \ldots \)) appearing in \( k \) by \( x_{i-q} \). Then \( k^* \) is the \( t \)-reduced form of \( t^qkt^{-q} \) and \( t^{-r}kt^r = t^{-(r+q)}k^*t^{r+q} \). Now \( k^* \) is a word in the generators of \( K_{-1} \) which involves \( a_0 \) and so it follows from (2.2) that \( t^{-(r+q)}k^*t^{r+q} \) is \( t \)-reduced. Consequently \( t^{-r}kt^r \) does not define an element of \( H \), contrary to assumption.

In a similar way, if \( r > M - s \) then the \( t \)-reduced form of \( t^{-r}kt^r \) involves \( t \) and therefore does not define an element of \( H \).

(ii) The result is immediate if \( k \) does not involve an \( a \)-symbol. Otherwise the result follows from (i).

**Lemma 4.** Let \( r \) be an integer.

If \( |r| > M \) then \( t^rH t^{-r} \cap H = \text{sgp} \{c_i (i \in \mathbb{Z}), d_i (i \in \mathbb{Z}), \ldots\} \).

If \( 0 < r < M \) then \( t^rH t^{-r} \cap H = \text{sgp}\{a_0, \ldots, a_{M-r}, c_i (i \in \mathbb{Z}), d_i (i \in \mathbb{Z}), \ldots\} \).

If \( -M < r < 0 \) then \( t^rH t^{-r} \cap H = \text{sgp}\{a_r, \ldots, a_M, c_i (i \in \mathbb{Z}), d_i (i \in \mathbb{Z}), \ldots\} \).

**Proof.** The result is trivial if \( r = 0 \).

Suppose \( r > 0 \), and let \( h \in t^rH t^{-r} \cap H \). Then it follows from Britton's lemma that there is a freely reduced word \( k \) in the generators of \( K_t \) such that \( t^rkt^{-r} = h \). If \( k \) does not involve an \( a \)-symbol then clearly \( h \in \text{sgp}\{c_i (i \in \mathbb{Z}), d_i (i \in \mathbb{Z}), \ldots\} \). Suppose \( k \) involves an \( a \)-symbol and let \( q \) be the least integer \( i \) for which \( a_i \) occurs in \( k \). Then \( r < q \) by Lemma 3(ii), and the \( t \)-reduced form of \( t^rkt^{-r} \) is a word in \( a_0, \ldots, a_{M-r}, c_i (i \in \mathbb{Z}), d_i (i \in \mathbb{Z}), \ldots \). This shows that \( t^rH t^{-r} \cap H \) is contained in \( \text{sgp}\{a_0, \ldots, a_{M-r}, c_i (i \in \mathbb{Z}), d_i (i \in \mathbb{Z}), \ldots\} \) if \( r < q \), and is contained in \( \text{sgp}\{c_i (i \in \mathbb{Z}), d_i (i \in \mathbb{Z}), \ldots\} \) otherwise. The reverse inclusions are obvious.

The case when \( r < 0 \) is handled similarly.

**Lemma 5.** Let \( Z \in G \) and let \( k \) be a nonempty cyclically reduced word in the generators of \( K_t \). If \( Z^{-1}kZ \in H \) then \( Z = trh \) for some integer \( r \) and some element \( h \) of \( H \).

**Proof.** Let \( V \) be an element of minimal \( t \)-length from the set

\[ \{U: \ U \text{ is a } t \text{-reduced word equal to } t^rZ \text{ for some integer } r\} \]

Then \( Z = t^rV \) for some integer \( r \), and \( t^rV \) is \( t \)-reduced. It will be shown that \( V \) is \( t \)-free. Suppose not, and let \( V = \nu^\delta V' \) where \( \nu \) is \( t \)-free and \( \delta = \pm 1 \). It suffices to establish that \( \nu \in K_{-\delta} \). For then \( t^{-\delta}v^\delta \) is equal to a \( t \)-free word \( u \)

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and $t^{-(r+s)}Z = uV'$, which contradicts the minimality of $V$. Now $t^{-r}kr'$ defines an element of $H$, and so it follows from Lemma 3(ii) that the $t$-reduced form $k^*$ of $t^{-r}kr'$ is a nonempty cyclically reduced word in the generators of one of $K_{-r}, K_r$. Moreover since $Z^{-1}kZ \in H, v^{-1}k^*v \in K_{-\delta}$. Thus $k^* \in K_{-\delta}$ by (2.2), and so $v \in K_{-\delta}$ by (2.1).

**Lemma 6.** Suppose $R$ involves $t$ and $M = 0$. Let $k$ be a cyclically reduced word in $a_0, c_0, d_0, \ldots$ which involves $a_0$, and let $Z$ be a $t$-reduced word which involves $t$. Then $Z^{-1}k^mZ$ ($m \neq 0$) is $t$-reduced.

**Proof.** It suffices, by Lemma 2, to show that $Z^{-1}kZ$ is $t$-reduced. Suppose $Z$ has initial segment $zt^r$, where $z$ is $t$-free, and assume by way of contradiction that

$$z^{-1}kz \in \text{sgp}\{c_i (i \in \mathbb{Z}), d_j (i \in \mathbb{Z}), \ldots\} \quad (= K_{-1} = K_1).$$

Then $t^{-r}z^{-1}kzt^r$ is equal to a word in $c_i (i \in \mathbb{Z}), d_j (i \in \mathbb{Z}), \ldots$. Passing back to the one-relator presentation of $G$ it is thus concluded that there is a cyclically reduced word in $a, c, d, \ldots$ involving $a$ which is conjugate to a word which does not involve $a$. But an argument similar to that used to establish (2.2) shows that this is impossible.

2.2. **Standard $H$-elements.** Throughout this subsection $G, H, K_{-1}, K_1$ etc., will be as in §2.1.

Let $p$ be a positive integer and let $u, v \in H$. For $i \in \mathbb{Z}$ let $\nu(i)$ denote the element $(t^pu)^{-1}v(t^pu)^i$ of $G$. Those elements $\nu(i)$ which belong to $H$ will be called the **standard $H$-elements associated with** $(t^pu,v)$ (or simply the standard $H$-elements if $(t^pu,v)$ is understood). Where necessary (for instance when using Britton’s lemma) it will be assumed that the standard $H$-elements are written in terms of the generators of $H$.

It is clear from Britton’s lemma that if $\nu(i) \in H$ for some $i > 0$ ($i < 0$) then $\nu(j) \in H$ whenever $0 < j < i$ ($i < j < 0$).

The standard $H$-elements can be thought of as the “obvious” elements of $H$ which can be obtained from $t^pu, v$. The reason for considering these elements stems from their importance in calculating the intersection of $\text{sgp}\{t^pu,v\}$ with $H$. The determination of such an intersection is a key step in the proof of the Principal Lemma.

There are two main situations where standard $H$-elements arise in the sequel.

(A) Let $v$ be a nonempty cyclically reduced word in the generators of $K_{-1}$ and let $u$ be a $t$-free word. Suppose that not both of $u, v$ belong to $\text{sgp}\{c_i (i \in \mathbb{Z}), d_j (i \in \mathbb{Z}), \ldots\}$ and consider the pair $(t^pu,v), p > 0$.

Now up to conjugation by a power of $t$ it can be assumed that $wvu^{-1} \notin K_1$. Indeed, suppose $wvu^{-1} \in K_1$. Then it follows from (2.2) that $v$ does not
involve \( a_0 \), so that \( v \in K_1 \). Thus \( u \in K_1 \) by (2.1). Assume that \( u \) is written as a freely reduced word in the generators of \( K_1 \) and let \( q \) be the least integer \( i \) for which \( a_i \) occurs in one of \( u, v \). Let \( \bar{u}, \bar{v} \) be the \( t \)-reduced forms of \( t^q u t^{-q}, t^q v t^{-q} \) respectively. Then \( t^q t^p u t^{-q} = t^q u \). Moreover \( \bar{v} \) is cyclically reduced. Now \( \bar{v} \bar{u}^{-1} \notin K_1 \). For since \( \bar{v} \) is cyclically reduced and \( u, \bar{v} \) are freely reduced words in the generators of \( K_{-1} \), it follows that the freely reduced form of \( \bar{v} \bar{u}^{-1} \) involves \( a_0 \), and therefore does not define an element of \( K_1 \) by (2.2).

Assume from now on that \( \bar{w} \bar{u}^{-1} \notin K_1 \).

Suppose \( \nu(t) \in H \) for some positive integer \( \mu \). Then it follows from Lemma 5 that \( t^{-\mu p}(t^p u)^\mu \in H \). Consequently \( u, u', \ldots, u'^{(\mu-1)p} \in H \) and

\[
(2.4) \quad \nu(t) = u^{-1} u^{-t p} \cdots u^{-t^{(\mu-1)p}} u^{t \mu p} u^{t^{(\mu-1)p}} \cdots u^{t \mu p}. 
\]

This implies that there is an integer \( \lambda \) such that \( \nu(t) \in H \) if and only if \( 0 \leq i \leq \lambda \). For if \( \nu(t) \) and \( \nu'(u) \) belonged to \( H \) for infinitely many values of \( \mu \) then \( u \) and \( v \) would both belong to \( \text{sgp} \{c_i (i \in \mathbb{Z}), d_i (i \in \mathbb{Z}), \ldots\} \) by Lemma 4, contrary to assumption.

Now since \( \nu(t) \in H \), it follows from Lemma 3(i) that \( \lambda p \leq M \), and so for \( j = 0, 1, \ldots, \lambda \) one can consider the subgroup \( \mathcal{F}(j) \) of \( H \) generated by \( a_0, \ldots, a_{M-(\lambda-j)p}, c_i (i \in \mathbb{Z}), d_i (i \in \mathbb{Z}), \ldots \). It will be shown that if \( 1 \leq \mu \leq \lambda \) then \( \nu(t) \in \mathcal{F}(\mu) \setminus \mathcal{F}(\mu-1) \).

Now \( u \in t^{(\lambda-1)p} H t^{-\lambda p} \cap H \), and

\[
t^{(\lambda-1)p} H t^{-\lambda p} \cap H = \text{sgp} \{a_0, \ldots, a_{M-(\lambda-1)p}, c_i (i \in \mathbb{Z}), d_i (i \in \mathbb{Z}), \ldots\}
\]

by Lemma 4. Thus:

\[
(2.5) \quad u^{t^p} \in \mathcal{F}(t+1), \quad l = 0, 1, \ldots, \lambda - 1.
\]

In a similar way:

\[
(2.6) \quad u^{t^p} \in \mathcal{F}(l), \quad l = 0, 1, \ldots, \lambda.
\]

It then follows from (2.4)-(2.6) that \( \nu(t) \in \mathcal{F}(\mu) \).

It is clear from the definition of \( \lambda \) that \( \nu(t) \notin \mathcal{F}(\lambda-1) \). Suppose by way of contradiction that for some integer \( \mu \), with \( 1 < \mu < \lambda \), \( \nu(t) \in \mathcal{F}(\mu-1) \). Now

\[
\nu(t) = (t^p u)^{-(\lambda-\mu)} \nu(t^p u)^{\lambda-\mu} = u^{-1} \cdots u^{-t^{(\mu-1)p}} (\nu(t)^{t^{(\mu-1)p}} u^{t^{(\mu-1)p}} \cdots u).
\]

Observe that \( u^{t^{(\mu-1)p}} \cdots u \in \mathcal{F}(\lambda-\mu) \) by (2.5). Moreover, since \( \nu(t) \in \mathcal{F}(\mu-1) \) it follows that \( (\nu(t)^{t^{(\mu-1)p}} \in \mathcal{F}(\lambda-1) \). Thus \( \nu(t) \in \mathcal{F}(\lambda-1) \), which is a contradiction.
(B) Let \( z \) be a freely reduced word in the generators of one of \( K_{-1} \), \( K_1 \) and suppose \( z \) involves an \( a_i \)-symbol. Let \( \{ k_j : j \in J \} \) be a set of elements of the subgroup \( F^{(-1)} \) of \( H \) generated by \( c_i (i \in \mathbb{Z}) \), \( d_i (i \in \mathbb{Z}) \), . . . , and suppose the \( k_j \) freely generate a subgroup of \( F^{(-1)} \). Suppose further that there is a permutation \( \psi \) of \( J \) such that, for each \( j \in J \), \( \iota^{-1} k_j t = k_{\psi(j)} \) (in other words, \( \{ k_j : j \in J \} \) is closed under conjugation by \( t \)). Consider the collection \( t, z, k_j (j \in J) \). As in (A), up to conjugation by a power of \( t \), it can be assumed that \( z \in K_{-1} \setminus K_1 \). Let \( z^{(0)}, \ldots, z^{(\lambda)} \) be the standard \( H \)-elements associated with \( (t, z) \). Then clearly \( (z^{(0)}, z^{(1)}, \ldots, z^{(\lambda - 1)}, k_j (j \in J), z^{(\lambda)}) \) is weakly \( (a_0, a_M) \)-admissible. Moreover, \( z^{(0)}, z^{(1)}, \ldots, z^{(\lambda - 1)}, k_j (j \in J) \) freely generate a subgroup of \( K_{-1} \), so that in particular \( (z^{(0)}, z^{(1)}, \ldots, z^{(\lambda - 1)}, k_j (j \in J), z^{(\lambda)}) \) is \( (a_0, a_M) \)-admissible. To see that \( z^{(0)}, z^{(1)}, \ldots, z^{(\lambda - 1)}, k_j (j \in J) \) freely generate a subgroup of \( K_{-1} \) let

\[
F^{(\mu)} = \text{sgp}\{a_0, \ldots, a_{M-1}, c_i (i \in \mathbb{Z}), d_i (i \in \mathbb{Z}), \ldots\}
\]

for \( \mu = 0, 1, \ldots, \lambda \). Then \( \{ z^{(0)}, \ldots, z^{(\mu-1)}, k_j (j \in J) \} \subseteq F^{(\mu-1)} \), whereas \( z^{(\mu)} \in F^{(\mu)} \setminus F^{(\mu-1)} \) (see (A)). Then the result follows by repeated use of Lemma 1(ii).

Now

\[
\text{sgp}\{z^{(0)}, z^{(1)}, \ldots, z^{(\lambda - 1)}, k_j (j \in J), z^{(\lambda)}\} \cap K_{-1}
\]

\[
= \text{sgp}\{z^{(0)}, z^{(1)}, \ldots, z^{(\lambda - 1)}, k_j (j \in J)\}
\]

and

\[
\text{sgp}\{z^{(0)}, z^{(1)}, \ldots, z^{(\lambda - 1)}, k_j (j \in J), z^{(\lambda)}\} \cap K_1
\]

\[
= \text{sgp}\{z^{(1)}, \ldots, z^{(\lambda - 1)}, k_j (j \in J), z^{(\lambda)}\},
\]

by (2.3). Using these formulae it will be deduced that

\[
\text{sgp}\{t, z, k_j (j \in J)\} \cap H = \text{sgp}\{z^{(0)}, \ldots, z^{(\lambda)}, k_j (j \in J)\}.
\]

To prove (2.9) it suffices to show that a word \( W \) in \( t, z^{(0)}, \ldots, z^{(\lambda)}, k_j (j \in J) \) which defines an element of \( H \) is equal to a word in \( z^{(0)}, \ldots, z^{(\lambda)}, k_j (j \in J) \) alone. The proof is by induction on the number of occurrences of \( t \) in \( W \).

If there are none the result holds.

Suppose then that \( W \) involves \( t \) and that \( W \) defines an element of \( H \). Then it follows from Britton’s lemma that \( W \) has a subword \( t^{-e} Q t^e \), where \( Q \) is a word in \( z^{(0)}, \ldots, z^{(\lambda)}, k_j (j \in J) \) and \( Q \) defines an element of \( K_{-\varepsilon} \). Now by (2.7) and (2.8), \( Q \) is equal to another word \( Q' \) in \( z^{(0)}, \ldots, z^{(\lambda)}, k_j (j \in J) \),

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where $Q'$ does not involve either $z^{(0)}$, $z^{(i)}$ according as $e$ is 1, −1. Thus $t^{-e}Qt^e$
 is equal to a word $S$ in $z^{(0)}, \ldots, z^{(i)}, k_j (j \in J)$, where $S$ does not involve
either $z^{(0)}, z^{(i)}$ according as $e$ is 1, −1. Replacing $t^{-e}Qt^e$ by $S$ then gives a word
$W'$ in $t, z^{(0)}, \ldots, z^{(i)}, k_j (j \in J)$ equal to $W$ in $G$ but having less occurrences
of $t$. The inductive hypothesis can now be applied to give the desired
conclusion. This completes the verification of (2.9).

It is possible to generalize (2.9). Indeed, for $\mu = -1, 0, \ldots, \lambda$:

(2.10) \[ \text{sgp}\{t, z, k_j (j \in J)\} \cap F^{(\mu)} = \text{sgp}\{z^{(0)}, \ldots, z^{(\mu)}, k_j (j \in J)\}. \]

To prove this, note that for $i = 0, \ldots, \lambda - 1, \{z^{(0)}, \ldots, z^{(i-1)}, k_j (j \in J)\} \subseteq F^{(i-1)}$ whereas $z^{(i)} \in F^{(i)} \setminus F^{(i-1)}$ (see (A)), so it follows from Lemma 1(i) that

(2.11) \[ \text{sgp}\{z^{(0)}, \ldots, z^{(i-1)}, z^{(i)}, k_j (j \in J)\} \cap F^{(i-1)} = \text{sgp}\{z^{(0)}, \ldots, z^{(i-1)}, k_j (j \in J)\}. \]

This formula is also valid for $i = \lambda$, being in that case merely a restatement
of (2.7). Combining (2.11) and (2.9) establishes that (2.10) holds.

Finally, a presentation of $\text{sgp}\{t, z, k_j (j \in J)\}$ associated with the generators
$t, z^{(0)}, \ldots, z^{(\lambda)}, k_j (j \in J)$ is obtained as follows. By Theorem 1 of [10] every
relation between $z^{(0)}, \ldots, z^{(\lambda)}, k_j (j \in J)$ is a consequence of a single relation

(2.12) \[ Q^n = 1 \]

say, where $Q$ is either empty or is a cyclically reduced word involving $z^{(0)}$ and
$z^{(\lambda)}$. Then an argument similar to that used to derive (2.9) from (2.7) and (2.8)
can be employed to show that every relation between $t, z^{(0)}, \ldots, z^{(\lambda)}, k_j (j \in J)$ is a consequence of (2.12) and the additional relations:

(3.1) \[ t^{-1}z^{(0)}t = z^{(1)}, \ldots, t^{-1}z^{(\mu)}t = z^{(\mu)}, \quad t^{-1}k_j t = k_j (j \in J). \]

3. Intersections. The main aim of this section is to establish the following
theorem.

THEOREM 4. Let $B = \langle x, y, b, \ldots; R^n \rangle$ where $n > 1$. Then $B$ has property-I.

This theorem will be proved by induction on the length of $R$, making use of
the fact that if the cyclically reduced form of $R$ involves at least two generators
then $B$ can be embedded into an HNN group whose base is a one-relator
group, the relator of which has length less than $L(R)$. The following two results
will therefore be useful.

Let
\( \tag{3.1} L = \langle a_0, \ldots, a_N, c_i (i \in \mathbb{Z}), d_i (i \in \mathbb{Z}), \ldots ; Q^n \rangle, \)

where \( N > 0, n > 1, Q \) is a cyclically reduced word which involves \( a_0 \) and \( a_N \). Let \( G \) be the HNN group given by

\[
G = \langle a_0, \ldots, a_N, c_i (i \in \mathbb{Z}), d_i (i \in \mathbb{Z}), \ldots ; Q^n, \]
\[
\tag{3.2} t^{-1} a_i t = a_{i+1} (i = 0, \ldots, N-1), t^{-1} c_i t = c_{i+1} (i \in \mathbb{Z}),
\]
\[
i^{-1} d_i t = d_{i+1} (i \in \mathbb{Z}), \ldots \rangle.
\]

Suppose that \( L \) has property-L.

\( (\dagger) \) If \( u \) is a cyclically reduced word in \( a_0, c_0, d_0, \ldots \) which involves \( a_0 \), and if \( v \not\in \text{sgp} \{a_0, c_0, d_0, \ldots \} \) then \( \text{sgp} \{u, v\} \cap \text{sgp} \{a_0, c_0, d_0, \ldots \} = \text{sgp} \{u\}. \)

(See Proposition 2.)

\( (\ddagger) \) If \( u \) is a nonempty freely reduced word in \( t, c_0, d_0, \ldots \) and if \( v \not\in \text{sgp} \{t, c_0, d_0, \ldots \} \) then \( \text{sgp} \{u, v\} \cap \text{sgp} \{t, c_0, d_0, \ldots \} = \text{sgp} \{u\}. \)

(See Proposition 3.)

Making use of \((\dagger)\) and \((\ddagger)\) it will now be shown how to prove Theorem 4 by induction on \( L(R) \). It can be assumed without loss of generality that \( R \) is cyclically reduced.

If \( L(R) = 0 \) then \( B \) is freely generated by \( x, y, b, \ldots \) and the result is easily established.

Now suppose that \( L(R) > 0 \). Let \( u \) be a freely reduced word in \( y, b, \ldots \), and suppose \( v \not\in \text{sgp} \{y, b, \ldots \} \). It will be shown that \( \text{sgp} \{u, v\} \cap \text{sgp} \{y, b, \ldots \} = \text{sgp} \{u\} \). It suffices to consider the situation where \( u \) is a cyclically reduced word in \( y, b, \ldots \).

Case 1: \( x \) does not occur in \( R \).

Then \( B \) is the free product of the free group on \( x \) and the one-relator group generated by the remaining generators. The result thus follows easily using the theory of free products.

Case 2: No generator occurring in \( u \) also occurs in \( R \).

Let \( F \) denote the free group on those generators which occur in \( u \), and let \( B' \) be the one-relator group generated by the remaining generators of \( B \). Then \( B \) is the free product of \( F \) and \( B' \). Suppose \( v = f_0 g_1 f_1 \cdots g_l f_l \) where \( l > 0 \), the \( g_i \) are nontrivial elements of \( B' \), the \( f_i \) are elements of \( F \), nontrivial except possibly for \( f_0 \) and \( f_l \). By assumption, at least one of the \( g_i \) is equal to an element \( g \) not belonging to \( \text{sgp} \{y, b, \ldots \} \).

Now the result is easily established if \( f_1 u^p f_0 \neq 1 \) for all integers \( p \). Suppose on the other hand that \( f_1 u^p f_0 = 1 \) for some integer \( p \). Then it will be shown...
that \( f_0^{-1} \operatorname{sgp} \{u, v^p\} f_0 \cap \operatorname{sgp} \{y, b, \ldots\} = f_0^{-1} \operatorname{sgp} \{u\} f_0 \), from which it follows immediately that \( \operatorname{sgp} \{u, v\} \cap \operatorname{sgp} \{y, b, \ldots\} = \operatorname{sgp} \{u\} \).

Now there is an integer \( j \) with \( 0 \leq j \leq l - 1 \) such that if \( 1 \leq i \leq j \) then the \( i \)th term of \( g_1 f_1 \cdots g_l \) is the inverse of the \((2l - i)\)th term, but the \((j + 1)\)st term is not the inverse of the \((2l - (j + 1))\)st term if \( j < l - 1 \). Let \( T \) be the product of the first \( j \) terms of \( g_1 f_1 \cdots g_l \) (taken in order) and let \( S \) be the product of the next \( 2(l - j) - 1 \) terms, so that \( g_1 f_1 \cdots g_l \) and \( TST^{-1} \) are the same normal form. Now it is clear that the normal form of a product

\[
TS^{q_0} T^{-1} (f_0^{-1} u_0)^{p_0} TS^{q_1} T^{-1} \cdots (f_0^{-1} u_0)^{p_1} TS^{q_l} T^{-1}
\]

where \( r > 0 \), the \(|q_i|\) are nonzero and less than the order of \( S \), the \(|p_i|\) are nonzero and less than the order of \( u \)-has \( g \) as one of its terms (and therefore does not define an element of \( \operatorname{sgp} \{y, b, \ldots\} \)) except possibly if \( g \) is not one of the terms of \( T \) and \( S = g \). To see that the product does not define an element of \( \operatorname{sgp} \{y, b, \ldots\} \) in this case, observe that since \( \operatorname{sgp} \{y, b, \ldots\} \cap B' \) is malnormal in \( B' \) (see [8, Lemma 2.1]), if \( g^q \neq 1 \) for some integer \( q \) then \( g^q \notin \operatorname{sgp} \{y, b, \ldots\} \). Thus the above product is in normal form and each of its terms \( S^{q_i} \) lies outside \( \operatorname{sgp} \{y, b, \ldots\} \).

**Case 3:** \( x \) occurs in \( R \) with zero-sum exponent; one of the generators occurs in both \( u \) and \( R \).

Suppose for definiteness that \( y \) occurs in \( u \) and \( R \). Consider the HNN presentation of \( B \) with stable letter \( x \) and fixed generator \( y \). By the inductive hypothesis the base of \( B \) has property-I, so it follows from (i) that \( \operatorname{sgp} \{u, v\} \cap \operatorname{sgp} \{y, b, \ldots\} = \operatorname{sgp} \{u\} \).

**Case 4:** \( x \) occurs in \( R \); one of the generators which occurs in \( u \) occurs in \( R \) with zero-sum exponent.

Suppose \( y \) occurs in \( u \) and \( R \), and \( \sigma_y(R) = 0 \). Consider the HNN presentation of \( B \) with stable letter \( y \) and fixed generator \( x \). Then the base has property-I by the inductive hypothesis, so that the result follows from (i).

**Case 5:** \( x \) occurs in \( R \); \( \sigma_x(R) \neq 0 \); one of the generators which occurs in \( u \) occurs in \( R \) with non zero-sum exponent.

Suppose for definiteness that \( y \) occurs in \( u \) and \( R \), and \( \sigma_y(R) \neq 0 \). Let \( \alpha = \sigma_x(R), \beta = \sigma_y(R) \). Let \( B_1 = \langle t, a, b, \ldots; R_1^\alpha \rangle \), where \( R_1 \) is the word obtained from \( R \) by replacing each occurrence of \( x \) by \( at^{-\beta} \) and each occurrence of \( y \) by \( t^\alpha \), and then cyclically reducing. Then \( B \) is embedded into \( B_1 \) by the homomorphism \( \Psi \) defined by

\[
x \mapsto at^{-\beta}, \quad y \mapsto t^\alpha, \quad b \mapsto b, \ldots.
\]

Moreover:

\[
(3.3) \quad \operatorname{sgp} \{t, b, \ldots\} \cap \Psi(B) = \operatorname{sgp} \{t^\alpha, b, \ldots\}.
\]
Now $R_t$ certainly involves $a$, and moreover $\sigma_t(R_t) = 0$. Thus one can consider the HNN presentation of $B_t$ with stable letter $t$ and fixed generator $a$. The base of $B_t$ is another one-relator group, the relator of which has length less than $L(R)$. Consequently the base has property I by the inductive hypothesis. Now $\Psi(v) \notin sgp \{t, b, \ldots\}$ by (3.3), and so it follows from (7) that $sgp \{\Psi(u), \Psi(v)\} \cap sgp \{t, b, \ldots\} = sgp \{\Psi(u)\}$. Thus

$$sgp \{u, v\} \cap sgp \{y, b, \ldots\} = sgp \{u\}.$$

The above cases cover all possibilities and so the induction step is proved.

In the following subsections statements (†) and (‡) will be verified, and other results of a similar nature will also be obtained. For the remainder of this section $L$ and $G$ will be as in (3.1), (3.2). The associated subgroups $sgp \{a_0, \ldots, a_{N-1}, c, (i \in \mathbb{Z}), \delta(i \in \mathbb{Z}), \ldots\}$ and $sgp \{a_1, \ldots, a_N, c, (i \in \mathbb{Z}), \delta(i \in \mathbb{Z}), \ldots\}$ of $G$ will be denoted by $A_{-1}$ and $A_1$ respectively. It will be assumed throughout that $L$ has property-I.

### 3.1 Intersections of certain subgroups (1).

**Proposition 1.** Let $p$ be a positive integer, let $k$ be a nonempty cyclically reduced word in the generators of $A_{-1}$, and let $h$ be a t-free word. Assume that $hh^{-1} \notin A_1$, and let $k(0), \ldots, k(\lambda)$ be the standard $L$-elements. Then:

(i) $sgp \{k(0), \ldots, k(\lambda)\} \cap t^pLt^{-p} = sgp \{k(0), \ldots, k(\lambda-1)\}$;

(ii) $h sgp \{k(0), \ldots, k(\lambda)\}h^{-1} \cap A_1 = h sgp \{k(0), \ldots, k(\lambda)\}h^{-1};$

(iii) $sgp \{t^ph, k\} \cap L = sgp \{k(0), \ldots, k(\lambda)\}$.

Consider (i). Suppose first that $k(0), \ldots, k(\lambda)$ all belong to $A_{-1}$ and let $F = t^pLt^{-p} \cap L$. Then it follows from Lemma 4 that $F$ is freely generated by a subset of the generators of $A_{-1}$. Now $k(0), \ldots, k(\lambda-1) \in F$ whereas $k(\lambda) \in A_{-1}\backslash F$, so it follows from Lemma 1(i) that $sgp \{k(0), \ldots, k(\lambda)\} \cap F = sgp \{k(0), \ldots, k(\lambda-1)\}$, as required.

Now suppose that $k(\lambda) \in A_{-1}$ (note then that $\lambda > 0$). If $\lambda > 1$ then $h \in A_{-1}$ (see §2.2(A)) so that $(hk(0)h^{-1}, \ldots, hk(\lambda)h^{-1})$ is weakly $(a_0, a_N)$-admissible. Consequently

$$h sgp \{k(0), \ldots, k(\lambda)\}h^{-1} \cap A_{-1} = h sgp \{k(0), \ldots, k(\lambda-1)\}h^{-1},$$

by (2.3). Conjugating this equation by $h$ then gives

$$sgp \{k(0), \ldots, k(\lambda)\} \cap A_{-1} = sgp \{k(0), \ldots, k(\lambda-1)\}.$$

Observe that (3.4) is also valid if $\lambda = 1$, since $L$ has property-I. Now $t^pLt^{-p} \cap L \subseteq A_{-1}$ and $sgp \{k(0), \ldots, k(\lambda-1)\} \subseteq t^pLt^{-p} \cap L$ so that (i) follows by intersecting both sides of (3.4) with $t^pLt^{-p} \cap L$.

Now consider (ii). If $\lambda = 0$ the result follows from (2.1). If $\lambda = 1$ the result
follows from the fact that $L$ has property-I, for $hk^{(0)}h^{-1} \in A_1$ whereas $hk^{(1)}h^{-1} \in A_1$. Suppose $\lambda > 1$. Then $h \in A_{-1}$ so that $hk^{(0)}h^{-1} \in A_{-1}\setminus A_1$ and $hk^{(i)}h^{-1} \in A_{-1} \cap A_1$ for $i = 1, \ldots, \lambda - 1$. Thus if $hk^{(\lambda)}h^{-1} \in A_{-1}\setminus A_1$ the result follows from (2.3). Suppose on the other hand that $hk^{(\lambda)}h^{-1} \in A_{-1} \cap A_1$. Then no element of $h \text{ sgp } \{k^{(0)}, \ldots, k^{(\lambda)}\}h^{-1}$ can be equal to a freely reduced word in the generators of $A_1$ which involves $a_{N_i}$ by (2.2). Consequently

$$h \text{ sgp } \{k^{(0)}, \ldots, k^{(\lambda)}\}h^{-1} \cap A_1 = h \text{ sgp } \{k^{(0)}, \ldots, k^{(\lambda)}\}h^{-1}$$

$$\cap \text{ sgp } \{a_1, \ldots, a_{N-1}, c_i (i \in \mathbb{Z}), d_i (i \in \mathbb{Z}), \ldots\}.$$ 

But it follows from Lemma 1(i) that

$$h \text{ sgp } \{k^{(0)}, \ldots, k^{(\lambda)}\}h^{-1} \cap \text{ sgp } \{a_1, \ldots, a_{N-1}, c_i (i \in \mathbb{Z}), d_i (i \in \mathbb{Z}), \ldots\} = h \text{ sgp } \{k^{(1)}, \ldots, k^{(\lambda)}\}h^{-1}.$$ 

To prove (iii) it must be established that if $W$ is a word in $t^ph, k^{(0)}, \ldots, k^{(\lambda)}$ which defines an element of $L$ then $W$ is equal to a word in $k^{(0)}, \ldots, k^{(\lambda)}$ alone. The proof is by induction on the number of occurrences of $t^ph$ in $W$.

If there are none the result holds.

Suppose that $W$ involves $t^ph$ and that $W$ defines an element of $L$. Now if $W$ has subword $t^phTh^{-1}t^{-p}$ where $T$ is a word in $k^{(0)}, \ldots, k^{(\lambda)}$ and where $hTh^{-1} \in A_1$ then it follows from (ii) that $T$ is equal to a word in $k^{(1)}, \ldots, k^{(\lambda)}$. Consequently the subword $t^phTh^{-1}t^{-p}$ can be replaced by a word in $k^{(0)}, \ldots, k^{(\lambda-1)}$ to give a word $W'$ equal to $W$ in $G$ and where $W'$ has less occurrences of $t^ph$. The inductive hypothesis can then be applied. Suppose on the other hand that $W$ does not have any subword $t^phTh^{-1}t^{-p}$ as above. Then it follows from Britton’s lemma that $W$ must have at least one subword of the form $h^{-1}t^{-p}St^ph$ where $S$ is a word in $k^{(0)}, \ldots, k^{(\lambda)}$ and $S \in A_{-1}$. Moreover, for at least one such subword, $S$ must belong to $t^pL^{-p}$. For if this were not the case then the $t$-reduced form of every subword $h^{-1}t^{-p}St^ph$ would involve $t$, so that the $t$-reduced form of $W$ would involve $t$, contrary to the fact that $W$ defines an element of $L$. Suppose then that $h^{-1}t^{-p}St^ph$ is a subword of $W$, where $S$ is a word in $k^{(0)}, \ldots, k^{(\lambda)}$ which defines an element of $t^pL^{-p}$. Then it follows from (i) that $S$ is equal to a word in $k^{(0)}, \ldots, k^{(\lambda-1)}$ so that $h^{-1}t^{-p}St^ph$ can be replaced by a word in $k^{(1)}, \ldots, k^{(\lambda)}$ and the inductive hypothesis can be applied to the resulting word.

This completes the proof of the proposition.

The following corollary to the proof of (iii) will be needed later.

**Corollary.** Let $W$ be a word in $t^ph, k^{(0)}, \ldots, k^{(\lambda)}$. Then either $W$ is equal to a word $W'$ in $t^ph, k^{(0)}, \ldots, k^{(\lambda)}$ where $W'$ has less occurrences of $t^ph$ than
2. Intersections of certain subgroups (2).

Proposition 2. Let $u$ be a cyclically reduced word in $a_0, c_0, d_0, \ldots$ which involves $a_0$, and let $v$ be an element of $G$ which does not belong to $\text{sgp} \{a_0, c_0, d_0, \ldots\}$. Then $\text{sgp} \{u, v\} \cap \text{sgp} \{a_0, c_0, d_0, \ldots\} = \text{sgp} \{u\}$.

If $Q$ does not involve any generator having a nonzero subscript then $G = \langle a_0, c_0, d_0, \ldots; Q^\prime \rangle$, and the result is easily established using the theory of free products. From now on therefore, it will be assumed that $Q$ involves at least one generator having a nonzero subscript.

Let $Z$ be an element of minimal $t$-length from the set

$$\{V: V \text{ is the cyclically } t\text{-reduced form of } vu^l \text{ for some integer } l\}.$$  

Then there is an integer $m$ and a $t$-reduced word $T$ such that $vu^m = T Z T^{-1}$, and $T Z T^{-1}$ is $t$-reduced. It suffices to show that $\text{sgp} \{T Z T^{-1}, u\} \cap \text{sgp} \{a_0, c_0, d_0, \ldots\} = \text{sgp} \{u\}$.

If for every integer $s$, $Tu^s T^{-1}$ has $t$-reduced form of $t$-length greater than zero then $\text{sgp} \{T Z T^{-1}, u\} \cap L = \text{sgp} \{u\}$, so the result is clear.

Suppose on the other hand that $T^{-1} u^s T$ defines an element of $L$ for some nonzero integer $s$. Then it follows from Lemmas 5, 6 and 3(i) that $T = t^i g$ where $0 \leq r \leq N$ and $g$ is $t$-free. Replacing $Z$ by $g Z g^{-1}$ if necessary it can be supposed that $g$ is empty. It thus suffices to show that $\text{sgp} \{Z, u_r\} \cap \text{sgp} \{a_r, c_r, d_r, \ldots\} = \text{sgp} \{u_r\}$. Here $u_r$ is the $t$-reduced form of $t^r u t^r$ (that is, $u_r$ is the word obtained from $u$ by replacing $a_0$ by $a_r$, $c_0$ by $c_r$, $d_0$ by $d_r$, \ldots).

If $Z$ is $t$-free then the result follows from the fact that $L$ has property-I, for $Z \notin \text{sgp} \{a_r, c_r, d_r, \ldots\}$.

Suppose $Z$ involves $t$. Then it follows from the definition of $Z$ that $Z u_r Z$ is $t$-reduced for all integers $l$. It is necessary to investigate the $t$-reductions of words $Z^{-1} u_r Z$ and $Z u_r Z^{-1}$ where $l, j$ are nonzero integers. By Lemma 2 it is enough to investigate the $t$-reductions of $Z^{-1} u_r Z$ and $Z u_r Z^{-1}$. Suppose that neither of $Z^{-1} u_r Z$, $Z u_r Z^{-1}$ is $t$-reduced. Let $Z$ have initial segment $z t^e$ and terminal segment $t^d w$. Here $\delta = \pm 1$ and $z, w$ are $t$-free. Then it follows from Lemmas 5 and 6 that $N > 0$ and there are $t$-free words $z_i, w_i$ such that $z t^e = t^r z_i$ and $t^d w = w_1 t^\delta$. Consequently $\epsilon = \delta$ since $Z$ is cyclically $t$-reduced. However $\epsilon \neq \delta$. This is clear if $r > 0$ and $t^r Z t^{-r}$ is $t$-reduced. On the other hand, if $r = 0$ then since by assumption $t^\epsilon u_r t^\epsilon$ and $t^\delta u_r t^{-\delta}$ both define elements of $L$, equality of $\epsilon$ and $\delta$ would imply $u_r \in A_1$, contrary to (2.2).

It has now been established that one of $Z^{-1} u_r Z$, $Z u_r Z^{-1}$ is $t$-reduced. By inverting $Z$ if necessary it can be supposed that $Z u_r Z^{-1}$ is $t$-reduced. Then
$Zu_j'Z^{-1}$ is $t$-reduced for all nonzero integers $j$, by Lemma 2. It is thus easy to see that if the $t$-reduced form of $Z^{-1}u_j'Z$ involves $t$ for every nonzero integer $l$ then a freely reduced word in $Z$, $u_r$, which involves $Z$ has $t$-reduced form of $t$-length greater than zero. Consequently $\text{sgp} \{Z, u_r\} \cap L = \text{sgp} \{u_r\}$, so that $\text{sgp} \{Z, u_r\} \cap \text{sgp} \{a_r, c_r, d_r, \ldots\} = \text{sgp} \{u_r\}$ as required.

Now suppose that $Z^{-1}u_j'Z$ defines an element of $L$ for some nonzero integer $l$. Then it follows from Lemmas 5, 6 and 3(i) that $N > 0$ and $Z = t^p h$, where $h$ is $t$-free and $0 < p < N - r$ (if $t$ cannot be negative since $t'Zt^{-r}$ is $t$-reduced). Let $u_r^{(0)}, \ldots, u_r^{(\lambda)}$ be the standard $L$-elements associated with $(t^p h, u_r)$. For $j = 0, \ldots, \lambda$ let $F^{(j)}$ denote the subgroup of $L$ generated by $a_0, \ldots, a_{N-(\lambda-j)p}, c_i (i \in Z), d_i (i \in Z), \ldots$. Then

\[(3.5) \quad \text{sgp} \{t^p h, u_r\} \cap F^{(\lambda)} = \text{sgp} \{u_r^{(0)}, \ldots, u_r^{(\lambda)}\}\]

by Proposition 1(iii). Also:

\[(3.6) \quad \text{sgp} \{u_r^{(0)}, \ldots, u_r^{(j-1)}, u_r^{(j)}\} \cap F^{(j-1)} = \text{sgp} \{u_r^{(0)}, \ldots, u_r^{(j-1)}\} \quad (j = 1, \ldots, \lambda).\]

This follows from Proposition 1(i) if $j = \lambda$ (making use of Lemma 4). On the other hand if $j < \lambda$ then it follows from Lemma 1(i) since $u_r^{(j)} \in F^{(j)} \setminus F^{(j-1)}$ (see §2.2(A)). Now

$\text{sgp} \{a_r, c_r, d_r, \ldots\} \subseteq F^{(0)} \subset F^{(1)} \subset \cdots \subset F^{(\lambda)},$

and this together with (3.5) and (3.6) shows that $\text{sgp} \{t^p h, u_r\} \cap \text{sgp} \{a_r, c_r, d_r, \ldots\} = \text{sgp} \{u_r^{(0)}\}$, as required.

This completes the proof of Proposition 2.

The following corollary of the proof will be needed in §4.

**Corollary.** Suppose that $L$ does not have any generators $c_i (i \in Z), d_i (i \in Z), \ldots$, and that $N > 0$. If $(v, a_0^q)$ generates $G$ then $|q| = 1$ and $v$ or its inverse is expressible in the form $th^*$, where $h^*$ is $t$-free. Moreover, if $N = 1$ then $(a_0, h^{-1}a_1^1 h^*)$ generates $L$, whereas if $N > 1$ there are integers $\alpha, \beta$ such that $h^* = a_1^\alpha a_0^\beta$.

Suppose $(v, a_0^q)$ generates $G$. Then $q \neq 0$ since $G$ is not cyclic. Taking $u = a_0^q$ and following through the proof of Proposition 2 it can be seen that there are integers $m$ and $r$, with $r > 0$, such that $t^{-r}(u_0^{m \alpha})t^r = t^p h$ where $0 < p \leq N - r$, $h$ is $t$-free, $ha_r^{-1} \in A_1$, and the standard $L$-elements $(a_0^q)^{(0)}$, \ldots, $(a_0^q)^{(\lambda)}$ generate $L$. By considering the factor group of $G$ by the normal subgroup generated by $L$ it is easily established that $p = 1$. Also, since $L$ cannot be generated by less than $N + 1$ elements, $\lambda = N, r = 0$ and $h \in \text{sgp} \{a_0, a_1\}$ (see §2.2(A)).
Clearly if \( N = 1 \) then \( a_0 \) and \( h^{-1} a_1 h \) must generate \( L \). Also \( \text{sgp} \{ a_0^\delta, h^{-1} a_0^\delta h \} \cap A_{-1} = \text{sgp} \{ a_0^\delta \} \) since \( L \) has property-I, so that \( |q| = 1 \).

Suppose on the other hand that \( N > 1 \). Then \( ha_0^\delta h^{-1}, a_0^\delta, h^{-1} a_0^\delta h, \ldots, h^{-1} a^{N-1}_0 h^{N-1} \cdots h^\delta \) is weakly \((a_0,a_\ast)\)-admissible and generates \( L \). Thus (see [10, Corollary 3.1]) \( ha_0^\delta h^{-1} = wa_0^\delta w' \) where \( w, w' \in \text{sgp} \{ a_1, \ldots, a_{N-1} \} \) and \( \delta = \pm 1 \). Using the fact that \( A_{-1} \) is freely generated by \( a_0, \ldots, a_{N-1} \), it follows easily that \( q = \delta \) and \( h = a_0^\delta \) for suitable integers \( k, l \).

It now suffices to take \( h^\ast = ha_0^{-\delta m} \) if \( \varepsilon = 1 \) and \( h^\ast = a_0^{\delta m} h \) if \( \varepsilon = -1 \).

3.3. A null-intersection lemma. Let \( B = \langle x_j (j \in J); S, T, \ldots \rangle \) and for \( y \in J \) define \( L_j \) to be the subgroup of \( B \) generated by those generators of \( B \) other than \( x_j \). Then \( B \) (or more precisely this presentation of \( B \)) will be said to have property-NI provided the following holds: for each \( j \in J \), if \( u, v \in L_j \) and \( z \in L_j \) then \( z \cap \langle u, z^{-1} v z \rangle \cap L_j \) is empty.

The following lemma is needed for the proof of Proposition 3 in §3.4.

**Lemma 7.** Let \( B = \langle x, y, b, \ldots; R^n \rangle \) where \( R \) is cyclically reduced and \( n > 1 \). Then \( B \) has property-NI.

The proof is by induction on the length of \( R \).

If \( R \) is empty then \( B \) is freely generated by \( x, y, b, \ldots \) and the result is easily established.

Now suppose that \( L(R) > 0 \). Let \( u, v \) be freely reduced words in \( y, b, \ldots \), and suppose \( z \in \text{sgp} \{ y, b, \ldots \} \). It will be shown that \( z \cap \langle u, z^{-1} uv \rangle \cap \text{sgp} \{ y, b, \ldots \} \) is empty. This is trivial if either \( u \) or \( v \) is equal to \( 1 \), so it suffices to consider the case when \( u \neq 1, v \neq 1 \) and show that it is impossible for an equation

\[
zu^m z^{-1} v^n z \cdots u^m z^{-1} v^n z = w
\]

where \( s \geq 0 \), the \( |m_i| (i = 1, 2, \ldots, s) \) are greater than zero and less than the order of \( u \), the \( |n_i| (i = 1, 2, \ldots, s) \) are greater than zero and less than the order of \( v \), \( w \) is a word in \( y, b, \ldots \) — to take place in \( B \). It can be assumed that \( u \) and \( v \) are cyclically reduced. For suppose \( u = gu_1 g^{-1} \) and \( v = h^{-1} v_1 h \), where \( u_1 \) and \( v_1 \) are cyclically reduced. Let \( z_1 = hzg \). Then (3.7) is equivalent to

\[
z_1 u_1^m z_1^{-1} v_1^n z_1 \cdots u_1^m z_1 = hwg.
\]

**Case 1:** \( x \) does not occur in \( R \).

Then \( B \) is the free product of the free group on \( x \) and the one-relator group generated by the remaining generators. The result is thus easily established using the theory of free products.

**Case 2:** No generator occurring in \( u \) or \( v \) also occurs in \( R \).

Then \( B \) is the free product of the free group \( F \) on those generators occurring in one of \( u \), \( v \) with the one-relator group \( B' \) generated by the remaining...
generators. Now \( z = f_0 g_1 f_1 \cdots g_l f_l \) where \( l > 0 \), the \( f_i \) are elements of \( F \), nontrivial except possibly for \( f_0 \) and \( f_l \), the \( g_i \) are nontrivial elements of \( B' \). Moreover, since \( z \notin \text{sgp} \{ y, b, \ldots \} \) at least one of the \( g_i \) does not belong to \( \text{sgp} \{ y, b, \ldots \} \). Now the left-hand side of (3.7) is equal to

\[
 f_0 g_1 f_1 \cdots g_l (f_i u^m f_i^{-1}) g_i^{-1} \cdots f_1^{-1} g_1^{-1} (f_0^{-1} v^n f_0) g_1 f_1 \cdots g_l f_l,
\]

and this latter is a normal form apart from trivial complications caused at the ends if \( f_0 \) or \( f_l \) is equal to 1. Since all terms of the normal form of \( w \) belong to \( \text{sgp} \{ y, b, \ldots \} \) it thus follows that (3.7) is impossible.

Case 3: \( x \) occurs in \( R \) with zero-sum exponent; one of \( u, v \) involves a generator which occurs in \( R \).

Suppose for definiteness that \( y \) occurs in \( u \) and \( R \). Calculations will be done relative to the HNN presentation of \( B \) with stable letter \( x \) and fixed generator \( y \).

Let \( Z \) denote the \( x \)-reduced form of \( z \). Then substituting into (3.7) gives

\[
(Z u^m Z^{-1} v^n Z \cdots u^m Z^{-1} v^n Z = w).
\]

Now in order for this equation to take place, the \( x \)-reduced form of the left-hand side must be \( x \)-free. In particular \( \sigma_x(Z) = 0 \). Now by Lemmas 5 and 6 if \( S \) is an initial segment of \( Z \) such that \( S^{-1} v^n S \) \( x \)-reduces to an \( x \)-free word, then \( S = x^p h_1 \) for some integer \( p \) and some \( x \)-free word \( h_1 \). Also, if \( T \) is a terminal segment of \( Z \) such that \( T u^m T^{-1} \) \( x \)-reduces to an \( x \)-free word then it follows from Lemmas 5, 6 and 3(i) that \( T = h_2 t^{-q} \) for some integer \( q \) with \( 0 \leq q \leq M \), and some \( x \)-free word \( h_2 \). Consequently the only way (3.8) can hold is if \( Z = t^r h t^{-r} \) where \( 0 \leq r \leq M \) and \( h \) is \( x \)-free. But then (3.8) is equivalent to

\[
h u^m h^{-1} v^n h \cdots u^m h^{-1} v^n h = w_r.
\]

Here \( u_r, v_r, w_r \) are the words obtained from \( u, v, w \) respectively by replacing \( y_0 \) by \( y_r \), \( b_0 \) by \( b_r \), ... However since \( h \notin \text{sgp} \{ y_r, b_r, \ldots \} \) this equation is impossible, for the base of \( B \) has property-NI by the inductive hypothesis.

Case 4: \( x \) occurs in \( R \); one of \( u, v \) involves a generator which occurs in \( R \) with zero-sum exponent.

Suppose for definiteness that \( y \) occurs in \( u \) and \( R \), and \( \sigma_y(R) = 0 \). Calculations will be done relative to the HNN presentation of \( B \) with stable letter \( y \) and fixed generator \( x \).

Now \( z \) can be expressed in the form \( y^k y^\theta k \) where \( y^\theta k \) is \( y \)-reduced and where \( k \) is such that \( k y^{\pm 1}, y^{\pm 1} k \) are all \( y \)-reduced. Then there are integers...
p, q, r and words \( u, v, w \) in \( b_i (i \in \mathbb{Z}) \), \ldots such that \( y^p u y^{-p} = y^p u, y^q v y^{-q} = y^q v, y^r w y^{-r} = y^r w \). Clearly (3.7) is equivalent to

\[(3.9) \quad y^{-r} k (y^p u)^m k^{-1} (y^q v)^n k = w.\]

Now in order for (3.9) to hold, the \( y \)-reduced form of the left-hand side must be \( y \)-free. This implies that \( k \) is \( y \)-free. For suppose by way of contradiction that \( k \) involves \( y \). Then \( k (y^p u)^m k^{-1} \) is \( y \)-reduced for each \( i \). This is clear if \( p \neq 0 \). Suppose on the other hand that \( p = 0 \), and let \( k \) have initial segment \( gy^s \) where \( g \) is \( y \)-free. If \( g^{-1} u^m g \in K^{-} \) then \( g \in K^{-} \) by (2.1) so that \( y^{-e} k \) is not \( y \)-reduced contrary to the definition of \( k \). In a similar way \( k^{-1} (y^q v)^n k \) is \( y \)-reduced for each \( i \). Thus the left-hand side of (3.9) is \( y \)-reduced and involves \( y \), which is a contradiction.

Suppose that \( k \in K_{-1} \cup K_1 \). Conjugating (3.9) by a power of \( y \) if necessary, it can be supposed that \( k \in K_{-1} \setminus K_1 \) (note that \( k \not\in \text{sgp} \{ b_i (i \in \mathbb{Z}), \ldots \} \)). Now the set \( \{ b_i (i \in \mathbb{Z}), \ldots \} \) is closed under conjugation by \( y \), so it follows from §2.2(B) that \( \text{sgp} \{ y, k, b, \ldots \} \) has presentation \( (y, k, b, \ldots; T'(y, k, b, \ldots)) \) where \( T' \) is cyclically reduced and is either empty or involves \( y, k \).

Let \( W \) denote an arbitrary word in the symbols \( y, k, b, \ldots \) of the form

\[(3.10) \quad w_0 k w_1 k^{-1} w_2 k \cdots w_{2\mu}^{-1} k^{-1} w_{2\mu},\]

where the \( w_i \) are freely reduced words in \( y, b, \ldots \). In order to show that (3.9) is impossible it suffices to establish that \( W \neq 1 \). The proof is by induction on \( \mu \). If \( \mu = 0 \) the result follows from Newman's Spelling Theorem. Suppose \( \mu > 0 \). The only case requiring attention is when all of \( w_1, w_2, \ldots, w_{2\mu-1}, w_{2\mu} \) are nonempty. Then if \( W = 1 \), \( T \) must be nonempty and \( W \) must have a subword \( (k^* S)^{n-1} k^* \) where \( k^* S \) is a cyclic permutation of \( T^\pm 1 \) (see Statement 1, p. 1439 of [2]). Replacing this subword of \( W \) by \( S^{-1} \) and freely reducing the \( k^* \)-free subwords of the resulting word gives a word \( W' \) of the form (3.10) which is equal to \( W \) and to which the inductive hypothesis applies. Thus \( W' \neq 1 \) so that \( W \neq 1 \).

Now suppose that \( k \notin K_{-1} \cup K_1 \). Then the left-hand side of (3.9) is \( y \)-reduced. This is clear except in the case when one of \( p, q \) is nonzero and the other is zero. To deal with this case it suffices to observe that if \( u^m \neq 1 \) then \( k u^m k^{-1} \in K_{-1} \cup K_1 \), and if \( v^m \neq 1 \) then \( k^{-1} v^m k \in K_{-1} \cup K_1 \) (this follows from (2.1)). Now since the \( y \)-reduced form of the the left-hand side of (3.9) must be \( y \)-free, \( r = p = q = 0 \). But then equation (3.9) takes place in the base of \( B \). However this is impossible since the base of \( B \) has property-NI by the inductive hypothesis.

Case 5: \( x \) occurs in \( R; \alpha_x (R) \neq 0; \) one of \( u, v \) involves a generator which occurs in \( R \) with non zero-sum exponent.
Suppose for definiteness that \( y \) occurs in \( u \) and \( R \) and \( \sigma_y(R) \neq 0 \). Let \( \alpha = \sigma_x(R) \), \( \beta = \sigma_y(R) \). Let \( B_1 = \langle x, y, b, \ldots ; R^n \rangle \) where \( R_1 \) is obtained from \( R \) by replacing each occurrence of \( x \) by \( xy^{-\beta} \) and each occurrence of \( y \) by \( y^\alpha \), and cyclically reducing. Then \( B \) is embedded into \( B_1 \) by the homomorphism \( \Psi \) defined by

\[
x \mapsto xy^{-\beta}, \quad y \mapsto y^\alpha, \quad b \mapsto b, \ldots
\]

Moreover:

\[
\Psi(B) \cap \text{sgp} \{ y, b, \ldots \} = \text{sgp} \{ y^\alpha, b, \ldots \}.
\]

Consequently \( \Psi(z) \notin \text{sgp} \{ y, b, \ldots \} \).

Now if \( R_1 \) involves both \( x \) and \( y \) then it follows as in Case 4 that \( \Psi(z \text{sgp} \{ u, z^{-1}vz \}) \cap \text{sgp} \{ y, b, \ldots \} \) is empty. On the other hand if \( R_1 \) does not involve \( y \) then \( L(R_1) < L(R) \) so it follows from the inductive hypothesis that \( \Psi(z \text{sgp} \{ u, z^{-1}vz \}) \cap \text{sgp} \{ y, b, \ldots \} \) is empty. Thus in either situation it is easily seen that \( z \text{sgp} \{ u, z^{-1}vz \} \cap \text{sgp} \{ y, b, \ldots \} \) is empty.

The above cases cover all possibilities and the induction step is proved.

3.4. Intersections of certain subgroups (3).

**Proposition 3.** Let \( u \) be a nonempty freely reduced word in \( t, c_0, d_0, \ldots \) and let \( v \) be an element of \( G \) which does not belong to \( \text{sgp} \{ t, c_0, d_0, \ldots \} \). Then \( \text{sgp} \{ u, v \} \cap \text{sgp} \{ t, c_0, d_0, \ldots \} = \text{sgp} \{ u \} \).

In order to prove this proposition it is of course necessary to determine which elements of \( \text{sgp} \{ u, v \} \) are also elements of \( \text{sgp} \{ t, c_0, d_0, \ldots \} \). Now an element of \( \text{sgp} \{ t, c_0, d_0, \ldots \} \) can be expressed in the form \( t^{-s}w \) where \( w \) is a word in \( c_i (i \in \mathbb{Z}), d_i (i \in \mathbb{Z}), \ldots \) (or alternatively in the form \( w't^{-s} \) where \( w' \) is a word in \( c_i (i \in \mathbb{Z}), d_i (i \in \mathbb{Z}), \ldots \)). Consequently, a good deal of the proof of Proposition 3 will be concerned with determining whether for a given element \( W \) of \( \text{sgp} \{ u, v \} \) there is an integer \( s \) such that \( t^sW \) (or \( Wt^s \)) belongs to \( \text{sgp} \{ c_i (i \in \mathbb{Z}), d_i (i \in \mathbb{Z}), \ldots \} \).

Let \( V \) be an element of minimal \( t \)-length from the set

\[
\{ U: U \text{ is a } t \text{-reduced word equal to } t^{-\alpha}u^\gamma v^\eta \text{ for integers } \alpha, \gamma, \eta \}.
\]

Then there are integers \( \kappa, \beta, \omega \) such that \( V = t^{-\kappa}u^{\beta}v^{\omega}t^\rho \). Moreover, it is not difficult to establish that there are integers \( \theta, \rho \) such that \( V = t^\theta zt^\rho \), where \( t^\theta zt^\rho \) is \( t \)-reduced and where each of the words \( t^{\pm 1}z, zt^{\pm 1} \) is \( t \)-reduced.

Now \( t^{-(\kappa-\rho)}ut^{\kappa-\rho} \) is equal in \( G \) to a word \( t^m k \) where \( k \) is a word in \( c_i (i \in \mathbb{Z}), d_i (i \in \mathbb{Z}), \ldots \). Let \( p = \rho + \theta \). Then it suffices to show that \( \text{sgp} \{ t^p z, t^m k \} \cap \text{sgp} \{ t, c_0, d_0, \ldots \} = \text{sgp} \{ t^m k \} \).

**Case 1:** \( z \) involves \( t \), \( p 
eq 0 \).
Let $W$ denote a fixed but arbitrary word of the form

$$W = (t^p z)^q_0 (t^{m_k} k)^i (t^p z)^{q_i} \cdots (t_{m_k} k)^{i_r} (t^p z)^{q_r},$$

where $r \geq 0$, and where the $q_i$, $l_i$ are nonzero integers. It suffices to establish that for no integer $s$ does $t^s W$ define an element of $L$. To prove this it is enough to show that in $t$-reducing $t^s W$ no $t$-symbol from any subword $z^\pm 1$ is removed. This is easily deduced from the following remarks.

First note that the minimality of $V$ implies that in $t$-reducing $(t^{m_k} k)^{z} t^p z$ no more than $\lfloor |m|/2 \rfloor$ $t$-symbols from $(t^{m_k} k)^{z}$ are used up, and the definition of $z$ implies that no $t$-symbols from $z$ are used up. Also, the definition of $z$ implies that $t^p z t^{m_k} k$ and $t^p z (t^{m_k} k)^{-1}$ are both $t$-reduced. It thus follows that if $m \neq 0$ then in $t$-reducing a word of the form $t^p z (t^{m_k} k)^l t^p z$ ($l \neq 0$) no $t$-symbols from either copy of $z$ are used up. This is also easily seen to be true if either $m = 0$ or $l = 0$ by the definition of $z$.

Secondly, observe that a word of the form $t^p z (t^{m_k} k)^l z^{-1} t^{-p}$ ($l \neq 0$) is $t$-reduced. This follows immediately from the definition of $z$ if $m \neq 0$. On the other hand suppose $m = 0$, and let $z$ have terminal segment $t^h$, where $h$ is $t$-free. Now if $hk^{-1} h^{-1} \in A_z$ then $h \in A_z$ by (2.1). Consequently $zt^{-x}$ is not $t$-reduced, which contradicts the definition of $z$.

Finally, consider $z^{-1} t^{-p} (t^{m_k} k)^l t^p z$ ($l \neq 0$). Now $t^{-p} t^{m_k} k t^p = t^{m_k} k'$, where $k'$ is a word in $c_i (i \in \mathbb{Z}), d_i (i \in \mathbb{Z}), \ldots$. Then an argument similar to that in the previous paragraph shows that $z^{-1} (t^{m_k} k)^l z$ is $t$-reduced.

**Case 2:** $z$ involves $t, p = 0$.

It follows as in Case 1 that if $l$ is a nonzero integer then $z^{-1} (t^{m_k} k)^l z$ and $z (t^{m_k} k)^{-1} z$ are $t$-reduced.

Suppose that for every integer $j$, $z (t^{m_k} k)^j z$ is $t$-reduced. If $W$ is a word as in (3.11) and $s$ is an arbitrary integer then it is easily seen that $t^s W$ is $t$-reduced and therefore does not define an element of $\text{sgp} \{ c_i (i \in \mathbb{Z}), d_i (i \in \mathbb{Z}), \ldots \}$. Thus $\text{sgp} \{ z, t^{m_k} k \} \cap \text{sgp} \{ l, c_0 d_0, \ldots \} = \text{sgp} \{ t^{m} \}$, as required.

Suppose on the other hand that for some integer $j$, $z (t^{m_k} k)^j z$ is not $t$-reduced. Notice that, by the definition of $z$, this implies either $j = 0$ or $m = 0$. Let $Y$ be the cyclically $t$-reduced form of $z (t^{m_k} k)^j$. Then there is an initial segment $T$ of $z$, where $T$ has positive $t$-length, such that $z (t^{m_k} k)^j = TY T^{-1}$ and $TY T^{-1}$ is $t$-reduced. Since for every nonzero integer $l$, $z^{-1} (t^{m_k} k)^l z$ is $t$-reduced, it follows that $T^{-1} (t^{m_k} k)^j T$ is $t$-reduced. It is thus easy to see that if $X$ is a word of the form

$$TY \cdot T^{-1} (t^{m_k} k)^j \cdot TY \cdot T^{-1} \cdots (t^{m_k} k)^j \cdot TY \cdot T^{-1},$$

where $r \geq 0$, the $|q_i|$ are nonzero and less than the order of $Y$, the $l_i$ are nonzero, then for every integer $s$, $t^s X$ is $t$-reduced and therefore does not
define an element of \( \text{sgp} \{ c_i (i \in \mathbb{Z}), d_i (i \in \mathbb{Z}), \ldots \} \). Thus \( \text{sgp} \{ z(t^{m_k})^t, t^{m_k} \} \cap \text{sgp} \{ t, c_0, d_0, \ldots \} = \text{sgp} \{ t^{m_k} \} \), as required.

**Case 3: \( z \) is \( t \)-free and defines an element of \( A_{-1} \cup A_1 \).**

Conjugating the pair \( (t^{p_z}, t^{m_k}) \) by a power of \( t \) if necessary, it can be supposed that \( z \in A_{-1} \setminus A_1 \). Let \( W \) denote an element of \( \text{sgp} \{ t^{p_z}, t^{m_k} \} \), let \( s \) be an integer, and let \( w \) be a word in \( c_i (i \in \mathbb{Z}), d_i (i \in \mathbb{Z}), \ldots \). It is required to determine when an equality

\[
(3.12) \quad t^s W = w
\]

can take place in \( G \). To do this it is convenient to analyse \( \text{sgp} \{ t, k, z \} \).

First suppose that \( k \neq 1 \), and let \( \mathcal{K} \) denote the set of standard \( L \)-elements associated with \( (t, k) \). Then \( \mathcal{K} \) is closed under conjugation by \( t \) and the elements of \( \mathcal{K} \) freely generate a subgroup of \( \text{sgp} \{ c_i (i \in \mathbb{Z}), d_i (i \in \mathbb{Z}), \ldots \} \) (see the example of p. ). It therefore follows from (2.10) (with \( \mu = -1 \)) that \( \text{sgp} \{ c_i (i \in \mathbb{Z}), d_i (i \in \mathbb{Z}), \ldots \} \cap \text{sgp} \{ t, z \} \cup \mathcal{K} = \text{sgp} \mathcal{K} \). Consequently if (3.12) holds then \( w \in \text{sgp} \mathcal{K} \). Thus

\[
\text{sgp} \{ t^{p_z}, t^{m_k} \} \cap \text{sgp} \{ t, c_0, d_0, \ldots \} = \text{sgp} \{ t^{p_z}, t^{m_k} \} \cap \text{sgp} \{ t, k \}
\]

\[
= \text{sgp} \{ t^{p_z}, t^{m_k} \} \cap \text{sgp} \{ t, t^{m_k} \}.
\]

Now it follows from §2.2(B) that \( \text{sgp} \{ t, z, k \} \) has presentation \( \langle t, z, k; T^n \rangle \), where \( T \) is either empty or is cyclically reduced and involves \( t \) and \( z \). Let \( x = t^{p_z} \) and \( y = t^{m_k} \). Then on the generators \( t, x, y, \text{sgp} \{ t, z, k \} \) has presentation

\[
\langle t, x, y; T^n \rangle
\]

where \( T_1 \) is obtained from \( T \) by replacing each occurrence of \( z \) by \( t^{-p} x \) and each occurrence of \( k \) by \( t^{-m} y \), and cyclically reducing. Now using Newman's Spelling Theorem for the presentation (3.14) it can easily be shown that \( \text{sgp} \{ x, y \} \cap \text{sgp} \{ t, y \} = \text{sgp} \{ y \} \). It therefore follows from (3.13) that \( \text{sgp} \{ x, y \} \cap \text{sgp} \{ t, c_0, d_0, \ldots \} = \text{sgp} \{ y \} \), as required.

There remains the situation when \( k = 1 \). To deal with this situation proceed similarly as above, but take \( \mathcal{K} \) to be empty. The equation (3.13) is readily established. Moreover \( \text{sgp} \{ t, z \} \), when presented on \( t \) and \( x \) (= \( t^{p_z} \)), is a one-relator group where the relator when cyclically reduced is either empty or is an \( n \)th power which involves \( x \). Consequently \( \text{sgp} \{ t^{m}, x \} \cap \text{sgp} \{ t \} = \text{sgp} \{ t^{m} \} \) by Newman's Spelling Theorem. It thus follows from (3.13) that \( \text{sgp} \{ t^{m}, x \} \cap \text{sgp} \{ t, c_0, d_0, \ldots \} = \text{sgp} \{ t^{m} \} \), as required.

**Case 4: \( z \) is \( t \)-free, \( z \notin A_{-1} \cup A_1 \).**
Subcase 4.1: \( p = m = 0 \). Since \( L \) has property-I it follows that \( \text{sgp} \{z,k\} \cap A_{-1} = \text{sgp} \{k\} \). Thus \( \text{sgp} \{z,k\} \cap \text{sgp}(c_i (i \in \mathbb{Z}), d_i (i \in \mathbb{Z}), \ldots) = \text{sgp} \{k\} \). Consequently \( \text{sgp}(z,k) \cap \text{sgp}(c_0, d_0, \ldots) = \text{sgp}(k) \), for it is clear that \( \text{sgp}(z, k) \cap \text{sgp}(c_0, d_0, \ldots) = \text{sgp}(z,k) \cap \text{sgp}(c_i (i \in \mathbb{Z}), d_i (i \in \mathbb{Z}), \ldots) \).

Subcase 4.2: \( p = 0, m \neq 0 \). It follows from (2.1) that if \( q \) is an integer such that \( z^q \neq 1 \) then \( z^q \notin A_{-1} \cup A_1 \). Consequently, if \( W \) is a word of the form

\[
z^{q_0}(t^{m_k})^1 z^{q_1}(t^{m_k})^2 z^q \ldots (t^{m_k})^r z^{q_r},
\]

where \( r \geq 0, q_i \neq 1 \) (i = 0, 1, \ldots, r), \( l_i \neq 0 \) (i = 1, 2, \ldots, r), then for every integer \( s \), \( t^s W \) is \( t \)-reduced. Thus \( W \notin \text{sgp} \{t, c_0, d_0, \ldots\} \) so that \( \text{sgp} \{z, t^{m_k}\} \cap \text{sgp} \{t, c_0, d_0, \ldots\} = \text{sgp} \{t^{m_k}\} \), as required.

Subcase 4.3: \( p \neq 0, m = 0 \). It can be assumed that \( k > 0 \). For \( \text{sgp} \{t^p z, k\} = t^p \text{sgp} \{z^{p-1}, k^p\} t^{-p} \), and \( \text{sgp} \{t^p z, k\} \cap \text{sgp} \{t, c_0, d_0, \ldots\} = \text{sgp} \{k\} \) if and only if \( \text{sgp} \{z^{p-1}, k^p\} \cap \text{sgp} \{t, c_0, d_0, \ldots\} = \text{sgp} \{k^p\} \).

The result is easily established if \( k = 1 \), so assume \( k \neq 1 \). Then \( zkz^{-1} \notin A_1 \) by (2.1). Moreover if \( k^* \) is the \( t \)-reduced form of \( t^{-p} k t^p \) then \( z^{-1} k^* z \notin A_{-1} \), again by (2.1). Consequently (see §2.2(A)), there are just two standard \( L \)-elements, namely \( k \) and \( z^{-1} k^* z \).

Suppose that \( t^s w \) is an element of \( \text{sgp} \{t, c_0, d_0, \ldots\} \) which is equal to a word in \( t^p z, k, z^{-1} k^* z \). Here \( s \) is an integer and \( w \) is a word in \( c_i (i \in \mathbb{Z}), d_i (i \in \mathbb{Z}), \ldots \). Let \( W \) be an element of minimal \( t \)-length from the set

\[
\{ Z: Z \text{ is a word in } t^p z, k, z^{-1} k^* z, \text{ and } Z = t^s w \}.
\]

Now by Britton's lemma the \( t \)-reduced form \( W \) of \( W \) involves \( |s| \) \( t \)-symbols and all the exponents to which \( t \) occurs in \( W \) have the same sign. It therefore follows from Proposition 1, Corollary that

\[
W = Y_1(t^p z)^e Y_2(t^p z)^e \ldots Y_r(t^p z)^e Y_{r+1},
\]

where \( r \geq 0, ep = s \), the \( Y_i \) are words in \( k, z^{-1} k^* z \).

Now if \( r = 0 \) then \( Y_1 = w \), so that \( w \) is equal to a power of \( k \). For

(3.15) \[
\text{sgp} \{k, z^{-1} k^* z\} \cap A_{-1} = \text{sgp} \{k\},
\]

since \( L \) has property-I.

In order to complete the proof that \( \text{sgp} \{t^p z, k\} \cap \text{sgp} \{t, c_0, d_0, \ldots\} = \text{sgp} \{k\} \), it suffices to establish that \( r \geq 0 \). Suppose by way of contradiction that \( r > 0 \), and assume for definiteness that \( e = 1 \). Then \( Y_1 \in A_{-1} \), so that \( Y_1 \) is equal to an element \( k^\# \) of \( \text{sgp} \{k\} \), by (3.15). Thus \( k^\# Y_2 \in A_{-1} \) (even if \( r = 1 \)). But this implies \( z Y_2 \in A_{-1} \), which contradicts the fact that \( L \) has property-NI (see Lemma 7).
Subcase 4.4: $p \neq 0, m \neq 0$. Replacing $t^m k$ by $t^{-m} k'$ if necessary, where $k'$ is the $t$-reduced form of $t^m k^{-1} t^{-m}$, it can be supposed that $m$ and $p$ have the same sign. Now the minimality of $V$ implies that in $t$-reducing $z^{-1} t^{-p} t^m k$ at most $|m|/2$ $t$-symbols from $t^m k$ are used up. Thus if $l = m - p$ then $l$ is nonzero and has the same sign as $m$. Consider the pair $t^p z, z^{-1} t^l k$. Then all four of the products $t^p z z^{-1} t^l k, t^p z k^{-1} t^{-l} z, z^{-1} t^{-p} z^{-1} t^l k, z^{-1} t^{-p} k^{-1} t^{-l} z$ are $t$-reduced, so it follows that a freely reduced word $W$ in $t^p z, z^{-1} t^l k$ is $t$-reduced. It must be ascertained whether $W$ can be equal to an element of $sgp \{t, c_0, d_0, \ldots\}$. It will be shown by induction on the length of $W$ (as a word in $t^p z, z^{-1} t^l k$) that if $W$ defines an element of $sgp \{t, c_0, d_0, \ldots\}$ then $W$ is a power of $t^p z z^{-1} t^l k$ ($= t^m k$).

The result is clear if $W$ is empty. Suppose $W$ is nonempty and that $W = t^s w$ where $w \in sgp \{c_i (i \in \mathbb{Z}), d_i (i \in \mathbb{Z}), \ldots\}$. Then $t^{-s} W$ must $t$-reduce to a $t$-free word so that $W$ must have initial segment $t^p z$ or $k^{-1} t^{-l} z$. Suppose for example that $W = t^p z W'$. Then $W'$ is nonempty, for $t^p z \neq t^s w$ since $x \not\in A_{-1} \cup A_1$. Thus $W'$ can start with $t^p z, k^{-1} t^{-l} z$ or $z^{-1} t^l k$. In the former two cases however, $t^{-s} W W'$ is $t$-reduced and therefore cannot be equal to $w$. In the latter case $W = t^p z z^{-1} t^l k W''$, and $W''$ has shorter length than $W$ and defines an element of $sgp \{t, c_0, d_0, \ldots\}$. Using the inductive hypothesis it is now concluded that $W$ is a power of $t^p z z^{-1} t^l k$, as required. The situation when $W$ has $k^{-1} t^{-l} z$ as initial segment is handled similarly.

The above cases cover all possibilities and the proof of Proposition 3 is now complete.

4. Proof of the Principal Lemma. In this section a proof of the Principal Lemma will be given (see §4.2). Before doing this, however, it is necessary to solve the following problem: given a group $B$ with presentation $\langle x, y; Q^n \rangle$ ($n > 1$), for which elements $u$ do $x$ and $u$ together generate $B$? This problem is solved in §4.1.

4.1. Certain generating pairs of one-relator groups with torsion.

**Lemma 8.** Let $B = \langle x, y; Q^n \rangle$ where $Q$ is cyclically reduced and involves $x$, and $n > 1$. If $x$ is conjugate to $x y^p$ then $p = 0$. If $x$ is conjugate to $x^{-1} y^p$ then either $p = 0$ or $Q^n$ is a cyclic permutation of $(x y^p)^{\pm 1}$, where $p = -2l$.

**Proof.** The proof requires three case distinctions. Throughout the proof frequent use (without mention) will be made of Collins' lemma characterizing conjugacy in HNN groups (see [1, General Lemma 3]).

**Case 1:** $\sigma_y(Q) = 0$.

Then every relator must have zero-sum exponent on $y$. Thus if $W^{-1} x W y^{-p} x^{-\varepsilon}$ is a relator for some word $W$ then $p = 0$.

**Case 2:** $\sigma_x(Q) = 0$. 

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Then $Q$ involves both $x$ and $y$. Calculations will be done relative to the HNN presentation of $B$ with stable letter $x$ and fixed generator $y$.

Now if $x$ is conjugate to $x^e y_0^i$ then $e = 1$ and there is a freely reduced word $u$ in the generators of the associated subgroup $K_{-1}$ of $B$ such that $uxu^{-1} = xy_0^i$. Let $u^*$ be the word obtained from $u$ by replacing $y_i$ by $y_{i+1}$ for each generator $y_i$ appearing in $u$. Then $u^*$ is the $x$-reduced form of $x^{-1}ux$ and $u^* = y_0^i u$. Now if $u$ is nonempty then $u^*$ is a word in the generators of $K_{1}$ and $u^*$ involves a generator of the base of $B$ which does not occur in $y_0^i u$. Thus $u^* \neq y_0^i u$ by Newman’s Spelling Theorem. Consequently $u$ must be empty, so that $p = 0$ as required.

Case 3: $\sigma_x(Q) \neq 0$, $\sigma_y(Q) \neq 0$.

Let $\sigma_x(Q) = \mu$, $\sigma_y(Q) = \eta$, and let $B_1 = \langle c, d; Q_1^\mu \rangle$, where $Q_1$ is obtained from $Q$ by replacing each occurrence of $x$ by $cd^{-\eta}$ and each occurrence of $y$ by $d^\mu$, and cyclically reducing. Then $B$ is embedded into $B_1$ by the homomorphism defined by $x \mapsto cd^{-\eta}, y \mapsto d^\mu$.

Consider first the situation when $Q_1$ involves both $c$ and $d$. Suppose that $cd^{-\eta}$ and $(cd^{-\eta})^j d^{p\rho}$ are conjugate in $B_1$. If $e = 1$ then $p = 0$ since $\sigma_y(Q_1) = 0$. Suppose $e = -1$. Then $\eta + p\rho = -\eta$ and $cd^{-\eta}$ is conjugate to $c^{-1}d^{-\eta}$. It can be assumed without loss of generality that $\mu$ is negative. The calculations in the next three paragraphs will be done with respect to the HNN presentation of $B_1$ with stable letter $d$ and fixed generator $c$.

Now if $cd^{-\eta}$ and $c^{-1}d^{-\eta}$ are conjugate then there is a freely reduced word $u$ in the generators of the associated subgroup $K_{-1}$ of $B_1$ such that

$$d^\kappa c_0^\delta d^\theta = u^{-1} d^\rho c_0^{-\delta} d^\tau u,$$

where $\kappa, \rho > 0, \theta, \tau \geq 0, \delta = \pm 1, \kappa + \theta = \rho + \tau = -\eta$. Moreover, it is no loss of generality to assume that $\kappa \geq \rho$ (so that $\tau \geq \theta$).

Suppose first that $u$ is nonempty and let $q$ and $s$ be respectively the least and greatest integers $i$ for which $c_i$ occurs in $u$. Let $u_j (-q \leq j \leq M - s)$ be the word obtained from $u$ by replacing each generator $c_i$ appearing in $u$ by $c_{i+j}$ (so that $u_j$ is the $d$-reduced form of $d^{-j}u d^j$). Now if exactly $2r$ $d$-symbols are used up in $d$-reducing $d^r u d^{-\theta}$ then it follows from Lemma 3 that $r \leq q$ and $d^r u d^{-\theta} = d^{r-\theta} u_{-r} d^{-\theta-r}$. Thus

$$d^{-\kappa} u^{-1} d^\rho c_0^{-\delta} d^{r-\theta} u_{-r} d^{-\theta-r} = c_0^\delta.$$

In order for this equation to hold all of $d^\rho$ must be used up in $d$-reducing $d^{-\kappa} u^{-1} d^\rho$. Consequently $0 \leq \rho \leq M - s$ and $d^{-\kappa} u^{-1} d^\rho = d^{-(\kappa-\rho)} u_{-\rho}^{-1}$, again by Lemma 3. Thus

$$d^{-(\kappa-\rho)} u_{-\rho}^{-1} c_0^{-\delta} d^{r-\theta} u_{-r} d^{-\theta-r} = c_0^\delta.$$

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Now if \( r \) were less than \( \theta \) then \( \tau - r > \kappa - \rho \) so that the \( d \)-reduced form of the left-hand side of (4.1) would involve \( d \), contrary to Britton’s lemma. Thus \( r = \theta \) and \( \kappa - \rho = \tau - r \). Moreover \( d^{-(k-\rho)}u_p^{-1}c_0^{-d}d^{r-r} \) must belong to the base. Now if \( \kappa - \rho > 0 \) then \( d^{-(k-\rho)}u_p^{-1}c_0^{-d}d^{r-r} \) can belong to the base only if \( u_p^{-1} \) belongs to the associated subgroup \( K_1 \), and so \( \rho + s < M \) by (2.2). Consequently \( \kappa - \rho < M - (\rho + s) \) and \( d^{-(k-\rho)}u_p^{-1}c_0^{-d}d^{r-r} = u_\kappa^{-1}c_\kappa^{-d} \) by Lemma 3. It therefore follows from (4.1) that if \( \kappa - \rho > 0 \) then

\[
(4.2) \quad u_\kappa = c_{\kappa-\rho}u_\rho c_0^{-\delta}.
\]

This is also clearly valid if \( \kappa - \rho = 0 \). But \( u_\kappa \) is a freely reduced word in the generators of \( K_1 \) and \( u_\kappa \) involves a generator which does not occur in \( c_{\kappa-\rho}u_\rho c_0^{-\delta} \). Thus (4.2) is impossible by Newman’s Spelling Theorem.

Now suppose that \( u \) is empty. Then \( d^{k-\rho}c_0^{-\delta}d^{r-r} \) can belong to the base only if \( u_0 \) belongs to the associated subgroup \( K_0 \), and so \( \rho + s < M \) by (2.2). Consequently \( \kappa - \rho < M - (\rho + s) \), whereas if \( \kappa = \rho \), it would assert that \( c_0^2 = 1 \), when in fact \( c_0 \) has infinite order.

It has now been established that if \( Q_1 \) involves both \( c \) and \( d \) then \( cd^{-\eta} \) and \( (cd^{-\eta})^{(d_p)} \) are conjugate only if \( \epsilon = 1 \) and \( p = 0 \).

There remains the situation when \( Q_1 \) does not involve \( d \). This can happen only if \( \eta = l \mu \) for some integer \( l \), and \( Q \) is a cyclic permutation of \( (xy^l)^n \). Let \( b = xy^l \). Then \( B = \langle b, y; b^{\mu \eta} \rangle \). Suppose \( by^{-l} \) and \( (by^{-l})^s y^p \) conjugate in \( B \). Clearly if \( \epsilon = 1 \) then \( p = 0 \). On the other hand, if \( \epsilon = -1 \) then it follows from the solution to the conjugacy problem for free products that \( b^2 = 1 \) and \( p = -2l \).

**Lemma 9.** Let \( B = \langle x, y; Q^n \rangle \) where \( Q \) is cyclically reduced and involves \( x \), and \( n > 1 \). If \( g^{-1}xg = x^\alpha y^\beta \) then there are integers \( \alpha, \beta \) such that \( g = x^\alpha y^\beta \).

**Proof.** If \( p = 0 \) then \( g \in sgp \{x\} \) since \( sgp \{x\} \) is malnormal in \( B \) (see [8, Lemma 2.1]).

Suppose \( p \neq 0 \). Then it follows from Lemma 8 that \( \epsilon = -1 \), \( B = \langle x, y; (xy^l)^4 \rangle \) and \( p = -2l \). Thus

\[
y^{-l}g^{-1}xgy^l = y^{-l}x^{-1}y^{-l} = x.
\]

Consequently \( g y^l \in sgp \{x\} \) by malnormality.

**Lemma 10.** Let \( B = \langle x, y; Q^n \rangle \) where \( n > 1 \), and where \( Q \) is a nonempty cyclically reduced word which is not a true power. If \( (x, u) \) generates \( B \) then \( u \) is expressible in the form \( x^\alpha y^\epsilon x^\beta \) for certain integers \( \alpha, \beta \), unless some cyclic permutation of \( Q^n \) has the form \( yx^l \).

**Proof.** Perhaps somewhat surprisingly, the proof is by induction on the length of \( Q \).
If $Q$ has length 1 then $B$ is the free product of its cyclic subgroups $\text{sgp} \{x\}$ and $\text{sgp} \{y\}$, and the result follows easily using the theory of free products.

Now assume that $Q$ has length greater than 1 (so that $Q$ involves $x$ and $y$), and suppose $(x, u)$ generates $B$. There are several cases to consider.

**Case 1:** $\sigma_x(Q) = 0$.

Calculations will be done relative to the HNN presentation of $B$ with stable letter $x$ and fixed generator $y$. Notice that the base of $B$ has property-I by Theorem 4, and so the results of §§3.1, 3.2 apply.

Now there are integers $\theta, \rho$ and an $x$-reduced word $w$ such that $u = x^\theta w x^\rho$, where $x^\theta w x^\rho$ is $x$-reduced and where each of the words $x^\pm 1 w, w x^\pm 1$ is $x$-reduced. Since $(x, w)$ generates $B$ it is clear that $w \neq 1$. Also, $w$ must be $x$-free, for if $w$ involved $x$ then $\text{sgp} \{x, w\}$ would intersect the base $H$ of $B$ trivially. Moreover $w \in K_{-1} \cup K_1$, for if $w \notin K_{-1} \cup K_1$ then using the fact that $K_{-1}$ and $K_1$ are malnormal in $H$ it is not difficult to show that $\text{sgp} \{x, w\} \cap H = \text{sgp} \{w\}$, which is a contradiction since $H$ is not cyclic. Conjugating the pair $(x, w)$ by a power of $x$ if necessary it can be supposed that $w \in K_{-1} \backslash K_1$. Then the standard $H$-elements $w^{(0)}, \ldots, w^{(a)}$ generate $H$ by Proposition 1(iii), so that $\lambda = M$. Thus $w = y^q$ for some integer $q$ by Lemma 3, and $|q| = 1$ by Proposition 2, Corollary.

**Case 2:** $\sigma_y(Q) = 0$.

Calculations will be done relative to the HNN presentation of $B$ with stable letter $y$ and fixed generator $x$. Notice that the base of $B$ has property-I by Theorem 4, and so the results of §3.2 apply.

If the number of generators of the base $H$ of $B$ is more than 2 then it follows from Proposition 2, Corollary that $u = x^\alpha y^\beta x^\gamma$ for suitable integers $\alpha, \beta$.

Suppose, on the other hand, that $H$ is generated by $x_0$ and $x_1$. Then again by Proposition 2, Corollary either $u$ or its inverse is expressible in the form $y h$, where $h \in H$ and $(x_0, h^{-1} x_1 h)$ generates $H$. Consequently by the inductive hypothesis either $h^{-1} x_1 h = x_0^\kappa x_1^\delta x_0^\mu$ for certain integers $\kappa, \delta, \mu$ with $\delta = \pm 1$, or $H = \langle x_0, x_1; (x_1 x_0^{-1})^\nu \rangle$ for some nonzero integer $\nu$. In the former situation it follows from Lemma 9 that there are integers $p, r$ such that $h = x_1^p x_0^r$. This is also true in the latter situation (see [9, p. ]). Thus in either situation $u$ is expressible in the form $x_0^\alpha y^\beta x_0^\gamma$.

**Case 3:** $\sigma_x(Q) \neq 0, \sigma_y(Q) \neq 0$.

Let $\sigma_x(Q) = \eta, \sigma_y(Q) = \mu$ and let $B_1 = \langle c, d; Q_1 \rangle$, where $Q_1$ is obtained from $Q$ by replacing each occurrence of $x$ by $d^u$ and each occurrence of $y$ by $c d^{-\eta}$ and cyclically reducing. Then $B$ is embedded into $B_1$ by the homomorphism $\Psi$ defined by $x \mapsto d^u, y \mapsto c d^{-\eta}$. Let $u' = \Psi(u)$. Then $(d, u')$ generates $B_1$. For $c d^{-\eta}$ can be obtained from $d^u$ and $u'$, so that $c$ can be obtained from $d$ and $u'$. Now if $Q_1$ involves $c$ and $d$ then it follows as in Case 1 that $u'$ or its
inverse is expressible in the form \( d^\kappa c d^{-\eta} d^\rho \). Since \( u' \in \Psi(B) \), \( \mu \) must divide \( \kappa \) and \( \rho \) so that \( u \) is expressible in the form \( x^\alpha y^\gamma x^\theta \), as required.

Suppose that \( Q_1 \) does not involve \( d \). Then \( \eta = l \mu \) for some integer \( l \), and either \( Q \) or its inverse is a cyclic permutation of \((yx^l)^\theta\) (thus \( |\eta| = 1 \)).

This completes the proof of the lemma.

4.2. Proof of the Principal Lemma. Let \( G = \langle a, t; R^n \rangle \) where \( R \) is cyclically reduced and not a true power, and \( n > 1 \). Now it follows from Lemma 4.1 of [11] that there is an automorphism \( \Psi \) of the free group \( F \) on \( a, t \) such that \( \Psi(R) \) has zero-sum exponent on \( t \), and it is easily seen that \( G = \langle a, t; \Psi(R)^n \rangle \).

Moreover, it follows from Corollary N4 of [3] that the cyclically reduced form of \( \Psi(R) \) is a nontrivial power of \( a \) (in which case it is \( a^{\pm 1} \)) if and only if \( R \) is a primitive. In order to determine the Nielsen equivalence classes of \( G \) it can be assumed without loss of generality that \( \Psi \) is the identity, so that \( \Psi(R) = R \).

First note that if \( R \) is empty then trivially \( G \) has one Nielsen equivalence class.

Next suppose that \( R = a^{\pm 1} \). Then it follows from the Grushko-Neumann Theorem that every generating pair of \( G \) is Nielsen equivalent to a pair of the form \((a^\alpha, t)\) where \( \alpha \) is coprime to \( n \). Consequently \( G \) has one Nielsen equivalence class if \( n = 2 \). On the other hand, suppose \( n > 2 \). Then two pairs \((a^{\alpha_1}, t), (a^{\alpha_2}, t)\), where \( \alpha_1 \) and \( \alpha_2 \) are coprime to \( n \), are Nielsen equivalent if and only if \( a^{\alpha_1} = a^{\pm \alpha_2} \). For it follows from Theorem 3.9 of [3] that \((a^{\alpha_1}, t)\) and \((a^{\alpha_2}, t)\) are Nielsen equivalent only if \( a^{\pm \alpha_1 t^{-1}} a^{\alpha_1 t} \) is conjugate to \((a^{-\alpha_1 t^{-1}} a^{\alpha_1 t})^{\pm 1} \).

Such a conjugacy can only take place if \( a^{\alpha_1} = a^{\pm \alpha_2} \) by Theorem 4.2 of [3]. This establishes that \( G \) has \( 2^{\frac{n}{2}}(n) \) Nielsen equivalence classes. The fact that it has one \( T \)-system follows from the observation that the mapping \( a \mapsto a^\alpha, t \mapsto t \), where \( \alpha \) is coprime to \( n \), defines an automorphism of \( G \).

Now suppose that \( R \) involves \( a \) and \( t \). By assumption \( \sigma(R) = 0 \). Calculations will mainly be done with reference to the HNN presentation of \( G \) with stable letter \( t \) and fixed generator \( a \).

It follows from Theorem 6 of [9] that every generating pair of \( G \) is Nielsen equivalent to a pair of the form \((th, k)\) where \( h \) belongs to the base \( H \) of \( G \) and \( k \) is a nontrivial element of the associated subgroup \( K_{-1} \). Conjugating the pair \((th, k)\) by an element of \( K_{-1} \) if necessary it can be supposed that \( k \) is a nonempty cyclically reduced word in the generators of \( K_{-1} \). Moreover, conjugating the pair \((th, k)\) by a power of \( t \) if necessary, it can be assume that \( hkh^{-1} \in K_1 \) (see §2.2 (A)).

Let \( k^{(0)}, \ldots, k^{(q)} \) be the standard \( H \)-elements associated with \((th, k)\). Now \( H \) has property I by Theorem 4, and so it follows from Proposition I(iii) that if \( th \) and \( k \) generate \( G \) then \( k^{(0)}, \ldots, k^{(q)} \) generate \( H \). But \( H \) cannot be generated by less than \( M + 1 \) elements so that \( \lambda = M \). Thus \( t^{-M} k t^M \in H \) (see §2.2 (A)) and so \( k = a_0^q \) for some integer \( q \), by Lemma 3. Moreover, it
follows from Proposition 2, Corollary that \(|q| = 1\). Finally, the fact that 
\((t^h, a_0^0)\) generates \(G\) implies that there are integers \(\alpha, \beta\) such that 
\(t^h = a_0^\alpha \, t^\beta\). This follows from Lemma 10 by reverting back to the one-relator presentation 
of \(G\).

It has now been established that every generating pair of \(G\) is Nielsen 
equivalent to a pair of the form \((a^\alpha \, t^\beta, a^\delta)\). Since such a pair \((a^\alpha \, t^\beta, a^\delta)\) is 
obviously Nielsen equivalent to \((t, a)\) it follows that \(G\) has one Nielsen 
equivalence class, as required.

This completes the proof of the Principal Lemma.

**REFERENCES**

10. ——— *Certain subgroups of certain one-relator groups*, Math. Z. 146 (1976), 1–6.