ONE-PARAMETER GROUPS OF ISOMETRIES
ON HARDY SPACES OF THE TORUS:
SPECTRAL THEORY(1)

BY

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Abstract. The spectral theory of the infinitesimal generator of an arbitrary
one-parameter group of isometries on $H^p$ of the torus, $1 < p < \infty, p \neq 2,$
is considered. In particular, the spectrum of the generator is determined.

0. Introduction. The space $H^p$ $(1 < p < +\infty)$ of the torus $C^2$ is defined (see
[8]) as the subspace of complex $L^p(C^2)$ consisting of those functions whose
double Fourier coefficients vanish on the complement of $P$, where $P$ is the
"positive set" in the character group of $C^2$, $P = \{(m, n): n > 0\} \cup \{(m, 0): m > 0\}$. Throughout what follows we shall use $(z, w) (z = e^{i\theta}, w = e^{2i\psi})$ to
denote the general point of $C^2$, $Z^p$ denotes the closure in $L^p(C^2)$ of the
polynomials in $z$, and we identify $Z^p$ with $H^p(e^{i\theta}/2\pi)$.

The strongly continuous one-parameter groups $\{T_t\}$ of isometries on
$H^p(C^2), 1 < p < \infty, p \neq 2,$ were described in [3]. The present paper is
devoted to the spectral analysis of their infinitesimal generators, and thereby
completes the investigation initiated in [3].

It is known that $\{T_t\}$ leaves $Z^p$ invariant (see (1.1) below), and so the study
of $\{T_t\}$ and its infinitesimal generator $A$ naturally splits into two cases
according to whether $\{T_t|Z^p\}$ has a bounded generator ($\S 1$) or an unbounded
generator ($\S 2$). In the former case there are a real constant $\rho$ and a real-valued
measurable function $\delta(\cdot)$ on the circle $C$ such that

\[(0.1) \quad (T_t f)(z, w) = e^{i\rho t} f(z, e^{i\delta(z)}w) \quad \text{for } f \in H^p(C^2).\]

As shown in $\S 1$, in this case the spectrum of $A, \Lambda(A)$, is the closure of the set
$i\{\rho + \cup_{n=0}^{\infty} n \text{ [ess. range (}\delta\text{)]} \}$. The dependence on $\delta(\cdot)$ of the fine structure of
$\Lambda(A)$ is examined in the process.

When $\{T_t|Z^p\}$ has an unbounded generator, it is shown in [3, Theorems
(2.22) and 2.24]) (by cocycle methods reminiscent of [6, Lecture V]) that there

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(1) This research was supported by a Natural Science Foundation grant.

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are real constants $\rho$ and $\delta$ (with $\rho$ unique), a unique nontrivial one-parameter group $\{\phi_t\}$ of Möbius transformations of the disc $D$, and a unimodular measurable function $u(\cdot)$ on $C$ such that

$$
(T_t f)(z,w) = e^{i\rho t} [\phi_t'(z)]^{1/p} f(\phi_t(z), e^{i\delta t} u(\phi_t(z))u(z)w),
$$

for $f \in H^p(C^2)$.

The properties of the group $\{T_t\}$ depend on the nature of the set $S$ of common fixed points, in the extended plane, of the group $\{\phi_t\}$. The set $S$ is known to be: (i) a doubleton set of symmetric points with respect to $C$, (ii) a singleton subset of $C$, or (iii) a doubleton subset of $C$ (see [2]), and correspondingly, we shall say that $\{\phi_t\}$ is of type (i), (ii), or (iii). If $\{\phi_t\}$ is of type (ii) or (iii), we show in Theorem (2.1) that $\Lambda(A)$ is the imaginary axis. If $\{\phi_t\}$ is of type (i), we show in Theorem (2.2) that $\Lambda(A)$ is either a bilateral arithmetic progression of pure-imaginary numbers, or the whole imaginary axis.

The following notation will be used throughout. The symbols $Z$, $Z^+$, $R$, and $C$ will denote, respectively, the set of all integers, the set of nonnegative integers, the real line, and the complex plane. Normalized Lebesgue measure on $C$ (resp., on $C^2$) will be symbolized by $\mu$ (resp., $\bar{\mu}$).

The authors are indebted to Professor Robert P. Kaufman for valuable comments and suggestions.

1. **Spectral analysis when $\{T_t|\mathbb{Z}^p\}$ has a bounded generator.** For convenience we record here the following theorem of Lal and Merrill [8]:

**Theorem.** If $T$ is a linear isometry of $H^p(C^2)$ onto $H^p(C^2)$ ($1 \leq p < \infty, p \neq 2$), then there are $\alpha \in C$, a Möbius transformation of the disc $\phi$, and a measurable function $\sigma: C \to C$ such that for all $f \in H^p(C^2)$:

$$
(Tf)(z,w) = \alpha[\phi'(z)]^{1/p} f(\phi(z), \sigma(z)w).
$$

On the other hand, if $1 \leq p < \infty$, and $\alpha$, $\phi$, and $\sigma$ are as above, the right-hand side of (1.1) defines a linear isometry of $H^p(C^2)$ onto $H^p(C^2)$.

For $1 \leq p < \infty$, let $\mathcal{K}_p$ be the set of all strongly continuous one-parameter groups $\{T_t\}$ of isometries of $H^p(C^2)$ such that $\{T_t|\mathbb{Z}^p\}$ has a bounded infinitesimal generator (if $p = 2$, we further require that each $T_t$ have the form (1.1)). The following theorem describes $\mathcal{K}_p$ [3, Theorem (1.5)].

**Theorem.** Let $\{T_t\} \in \mathcal{K}_p$, $1 \leq p < \infty$. Then there are a real constant $\rho$ and a real-valued measurable function $\delta(\cdot)$ on $C$ such that

$$
(T_t f)(z,w) = e^{i\rho t} f(z, e^{i\delta t}w) \quad \text{for } t \in R, f \in H^p(C^2).
$$

Conversely, for any such $\rho$ and $\delta(\cdot)$, (1.3) defines a group $\{T_t\} \in \mathcal{K}_p$.
Remark. It is easy to see that for the group \( \{T_t\} \) given by (1.3), \( \rho \) is uniquely determined, and \( \delta(\cdot) \) is uniquely determined up to equality almost everywhere.

We also have the following description of the infinitesimal generator \([3, \text{Theorem (1.7)}]\).

(1.4) Theorem. For \( 1 \leq p < \infty \), let \( \{T_t\} \) in \( \mathcal{S}_p \) have the form (1.3), and let \( A \) be the infinitesimal generator of \( \{T_t\} \). Then the domain of \( A \), \( \mathcal{D}(A) \), consists of those functions \( f \in \mathcal{H}^p(C^2) \) for which there is a \( g \in \mathcal{H}^p(C^2) \) so that for almost all \( z \) the following hold:

(i) \( \delta(z) = 0 \) implies \( g(z, w) = 0 \) for almost all \( w \);

(ii) \( \delta(z) \neq 0 \) implies that there is a (necessarily unique) function \( F_z \) on \( C \) such that \( F_z(e^{it\psi}) \) is absolutely continuous for \( 0 \leq \psi < 2\pi \), \( f(z, w) = F_z(w) \) for almost all \( w \), and

\[
g(z, e^{it\psi}) = \delta(z) \frac{d}{d\psi} (F_z(e^{it\psi}))
\]

for almost all \( \psi \). If \( f \) belongs to \( \mathcal{D}(A) \) and \( g \) is as above, then

\[
Af = i\rho f + g.
\]

In order to describe the spectrum of \( A \) we shall need the following standard notion.

Definition. If \((X, \nu)\) is a measure space, \( \nu \geq 0 \), and \( h: X \to \mathbb{R} \) is a \( \nu \)-measurable function, the \( \nu \)-essential range of \( h \) (abbreviated \( \nu \)-ess. range \( (h) \)) is the set of all \( x \in \mathbb{R} \) such that for each open neighborhood \( N \) of \( x \), \( \nu(h^{-1}(N)) > 0 \).

We now turn our attention to proving the following theorem.

(1.5) Theorem. Let \( A \) be the infinitesimal generator of a group \( \{T_t\} \) in \( \mathcal{S}_p \), \( 1 \leq p < \infty \), and let \( \delta \) and \( \rho \) be as in (1.3). Then the spectrum of \( A \) is the closure of the set \( i(\rho + \bigcup_{n=0}^{\infty} \mu\text{-ess. range}(\delta)) \).

Theorem (1.5) will follow from a succession of results established below, which also provide information about the fine structure of \( \Lambda(A) \).

Since a change in \( \rho \) only translates the generator and its spectrum, we shall for convenience study the group \( \{S_t\} \) given by

\[
(S_t f)(z, w) = f(z, e^{i\delta(z)t} w).
\]

In order to avoid interruption of the flow of ideas later on, we now introduce some terminology and state some convenient facts whose simple proofs will be omitted.

Definition. Let \( \nu \) be the product measure on \( \mathbb{Z}^+ \times C \), where \( \mathbb{Z}^+ \) is endowed with discrete measure and \( C \) has Lebesgue measure \( \mu \). Let \( \Delta: \mathbb{Z}^+ \times C \to \mathbb{R} \) be
the function defined by $\Delta(n,z) = n\delta(z)$. Denote by $\delta$ the $\nu$-essential range of $\Delta$.

(1.6) For each $\lambda \in \mathbb{R}$, $\lambda \in \delta$ if and only if for each $\varepsilon > 0$, there are $n_\varepsilon \in \mathbb{Z}^+$ and $M_\varepsilon \subseteq C$ such that $\mu(M_\varepsilon) > 0$ and $|\lambda - n_\varepsilon \delta(z)| < \varepsilon$ for $z \in M_\varepsilon$. (In particular $0 \in \delta$.)

(1.7) $\delta$ is the closure of $\bigcup_{n=0}^{\infty} [\mu\text{-ess. range} (\delta)]$.

(1.8) For each $\lambda \in \mathbb{R}$, $\lambda$ is an accumulation point of $\delta$ if and only if for each $\varepsilon > 0$, there are $n_\varepsilon \in \mathbb{Z}^+$, $\eta_\varepsilon > 0$, and $M_\varepsilon \subseteq C$ such that $\mu(M_\varepsilon) > 0$ and $\eta_\varepsilon \leq |\lambda - n_\varepsilon \delta(z)| < \varepsilon$ for $z \in M_\varepsilon$.

(1.9) If for every $\gamma > 0$ the inverse-image $\delta^{-1}\{(0,\gamma)\}$ (resp., $\delta^{-1}\{(-\gamma,0)\}$) has positive measure, then $\delta$ contains every positive (resp., every negative) real number.

We shall require some further notation. For $f \in L^p(d\theta)$ and $n \in \mathbb{Z}$, let

$$a_f^{(n)}(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z, e^{i\theta}) e^{-in\theta} d\theta.$$ Clearly, $\|a_f^{(n)}\|_p \leq \|f\|_p$, where the $p$-norms are taken, respectively, in $L^p(d\theta/2\pi)$ and $L^p(d\bar{\mu})$. The closed linear span in $L^p(d\bar{\mu})$ of the functions $z^m w^n$, $m, n \in \mathbb{Z}$, $n \geq 1$, will be denoted by $W^p$. It is known [9, Lemma 5] that $H^p(C^2) = Z^p \oplus W^p$. It is easy to see that the projection on $Z^p$ along $W^p$ is given by $f \mapsto a_f^{(0)}$. Notice that if $f \in H^p(C^2)$, then for almost all $z \in C$, $a_f^{(n)}(z) = 0$ for $n = 1, 2, \ldots$. Thus for almost all $z \in C$, $f(z, \cdot) \in H^p(d\psi/2\pi)$.

Returning now to the group $\{S_t\}$, we set $E_{\delta\lambda} = \{z \in C: \delta(z) = \lambda\}$ for $\lambda \in \mathbb{R}$, $\lambda \neq 0$, and $n = 1, 2, \ldots$. Then $\{E_{\delta\lambda}\}_{n=1}$ is a sequence of disjoint measurable sets.

(1.10) Theorem. Let $\delta$ be the infinitesimal generator of the group $\{S_t\}$ acting in $H^p(C^2)$, $1 \leq p < \infty$, by

$$(S_t f)(z,w) = f(z, e^{i\delta(z)t}w).$$

Then for $\lambda \in \mathbb{R}$, $\lambda \neq 0$, we have:

(i) The closure of the range of $(\delta - i\lambda)$ is the set $\{g \in H^p(C^2):$ for $n = 1, 2, \ldots$, $a_g^{(n)}(z) = 0$ for almost all $z \in E_{\delta\lambda}\}.$

(ii) The null-space of $(\delta - i\lambda)$ is the set $\{f \in W^p: \text{for } n = 1, 2, \ldots, a_f^{(n)}(z) = 0 \text{ for almost all } z \in C \setminus E_{\delta\lambda}\}.$

(iii) The space $H^p(C^2)$ is the direct sum of the closure of the range of $(\delta - i\lambda)$ and the null space of $(\delta - i\lambda)$.

Proof. Let $\Gamma$ be the set of all $g \in H^p(C^2)$ such that for $n = 1, 2, \ldots$, $a_g^{(n)}$ vanishes almost everywhere in $E_{\delta\lambda}$. The fact that $\Gamma$ is closed follows from the observation that for each $n, g \mapsto \chi_{E_{\delta\lambda}} a_g^{(n)}$ (where $\chi$ denotes characteristic function) defines a bounded operator from $H^p(C^2)$ into $L^p(d\theta/2\pi)$. Now suppose $(\delta - i\lambda)f = g$. From (1.4)(ii) it follows that if $\delta(z) \neq 0$,
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\[ \delta(z) \frac{dF(e^{i\psi})}{d\psi} - i\lambda F(z) = g(z, e^{i\psi}), \]

for almost all \( \psi \). Taking Fourier coefficients we get

\[ i(n\delta(z) - \lambda) a_f^{(n)}(z) = a_g^{(n)}(z). \]

It is now clear that \( \Gamma \) contains the range of \((\theta - i\lambda)\), and hence its closure.

To prove the reverse inclusion, we first observe that if \( n \) is any positive integer and \( h \in L^p(d\theta/2\pi) \) vanishes almost everywhere on \( E_{n,\lambda} \), then \( h(z)w^n \) is in the closure of the range of \((\theta - i\lambda)\). In fact, for each positive integer \( k \), let \( \chi_k \) be the characteristic function of the set \( \{ z \in \mathbb{C} : |n\delta(z) - \lambda| > 1/k \} \). Now define

\[ f_k(z, w) = \left[ \frac{\chi_k(z)}{i(n\delta(z) - \lambda)} \right] h(z)w^n. \]

It is easy to check that \((\theta - i\lambda)f_k = \chi_k(z)h(z)w^n\). It follows from the dominated convergence theorem that \( h(z)w^n \) is in the closure of the range of \((\theta - i\lambda)\).

Let now \( g \in \Gamma \). It is clear from the preceding observation that for each \( n = 1, 2, \ldots, a_f^{(n)}(z)w^n \) is in the closure of the range of \((\theta - i\lambda)\). Also, it is obvious from (1.4) that \( \theta a_f^{(0)} = 0 \) and \( a_f^{(0)} \) is in the range of \((\theta - i\lambda)\). Let \( g_n(z, w) \) be the \( n \)th Cesàro mean of the series \( \sum_{k=0}^{\infty} a_g^{(k)}(z)w^k \). By the foregoing, \( g_n \) belongs to the closure of the range of \((\theta - i\lambda)\). From standard facts about the convergence of Cesàro means \( \{ g_n \} \) converges to \( g \) in \( H^p(C^2) \), and the proof of (1.10)(i) is complete.

Now we turn to the proof of (1.10)(ii). For any \( f \in H^p(C^2) \), consider the series

\[ \sum_{n=1}^{\infty} \chi E_{n,\lambda}(z) a_f^{(n)}(z)w^n. \]

Since the \( E_{n,\lambda} \)'s are disjoint, for each \( z \) this series has at most one term not identically zero in \( w \). We have

\[ \left| \sum_{n=1}^{k} \chi E_{n,\lambda}(z) a_f^{(n)}(z)w^n \right| \leq \left[ \frac{1}{2\pi} \int_0^{2\pi} |f(z, e^{i\psi})|^p d\psi \right]^{1/p}. \]

Let \( h(z, w) \) denote the pointwise sum of the series in (1.11). It is clear from (1.12) that this series converges to \( h \) in \( L^p(C^2) \), and therefore \( h \in W^p \), with \( \|h\|_p \leq \|f\|_p \). Let us define \( Q : H^p(C^2) \to H^p(C^2) \) by setting \( Qf = h \). From the foregoing \( Q \) is a well-defined idempotent linear map with \( \|Q\| \leq 1 \).

It is easy to see that the null space of \( Q \) consists of all \( f \in H^p(C^2) \) such that
for \( n = 1, 2, \ldots \), \( a_f^{(n)}(z) = 0 \) for almost all \( z \) in \( E_{n,\lambda} \). Thus by (1.10)(i) we have

\[
(1.13) \text{The null space of } Q \text{ is the closure of the range of } (\mathcal{A} - i\lambda).
\]

Next we note that if \( f \in H^p(C^2) \), then for almost all \( z \), \( a_f^{(n)}(z) = \chi_{E_{n,\lambda}}(z)a_f^{(n)}(z) \) for \( n \geq 1 \), \( a_f^{(0)}(z) = 0 \), and \( a_f^{(n)}(z) = a_f^{(n)}(z) = 0 \) for \( n < 0 \). From this we see that the range of \( Q \) consists of all \( f \in W^p \) such that for \( n = 1, 2, \ldots \), \( a_f^{(n)}(z) = 0 \) for almost all \( z \) in \( C \setminus E_{n,\lambda} \). From the definition of \( Q \) and Theorem (1.4) we find that the range of \( Q \) is contained in the null space of \( (\mathcal{A} - i\lambda) \). Conversely, suppose \( \mathcal{A}f = i\lambda f \). It is easy to see from Theorem (1.4) that the range of \( \mathcal{A} \) is contained in \( W^p \), and so \( f \in W^p \). It follows readily from Theorem (1.4) that for almost all \( z \), \( (n\delta(z) - \lambda)a_f^{(n)}(z) = 0 \) for every integer \( n \). Thus \( f \) is in the range of \( Q \), as characterized above. In summary we have:

\[
(1.14) \text{The null space of } (\mathcal{A} - i\lambda) \text{ is the range of } Q, \text{ and the range of } Q \text{ consists of all } f \in W^p \text{ such that for } n = 1, 2, \ldots, \ a_f^{(n)}(z) = 0 \text{ for almost all } z \text{ in } C \setminus E_{n,\lambda}.
\]

Thus (1.10)(ii) is established, and, in view of (1.13), so is (1.10)(iii).

Now we take up the companion theorem (for the case \( \lambda = 0 \)) to Theorem (1.10).

\[
(1.15) \text{Theorem. Let } N(\delta) = \{z \in C: \delta(z) = 0\} \text{ and let } K(z, w) = \chi_{N(\delta)}(z).
\]

Then with the same notation as in (1.10), we have:

(i) The closure of the range of \( \mathcal{A} \) is \( (1 - K)W^p \).

(ii) The null-space of \( \mathcal{A} \) is \( Z^p + KW^p \).

(iii) The space \( H^p(C^2) \) is the direct sum of the closure of the range of \( \mathcal{A} \) and the null-space of \( \mathcal{A} \).

\text{Proof. Let } g \text{ be in the range of } \mathcal{A}. \text{ Then by Theorem (1.4), } (1 - K)g = g \text{ and as noted earlier } g \in W^p. \text{ Obviously, } (1 - K)W^p \text{ is closed and so by the foregoing contains the closure of the range of } \mathcal{A}. \text{ To see the reverse inclusion, for } k = 1, 2, \ldots, \text{ let } \chi_k \text{ be the characteristic function of the set } \{z \in C: |\delta(z)| \geq k^{-1}\}. \text{ Let } m, n \text{ be integers with } n > 0 \text{ and define}

\[
f_k(z, w) = \left[ \frac{\chi_k(z)}{i\delta(z)} \right]z^mw^n.
\]

In a fashion similar to the proof of (1.10)(i) it follows that \( (1 - K)z^mw^n \) is in the closure of the range of \( \mathcal{A} \), and this completes the proof of (1.15)(i).

We now prove the less obvious inclusion in (1.15)(ii). Suppose \( \mathcal{A}f = 0 \). Write \( f = u + v, u \in Z^p, v \in W^p \). By Theorem (1.4), \( \mathcal{A}u = 0 \), and hence for almost all \( z \in C \) if \( \delta(z) \neq 0 \), \( v(z, w) \) has the same value for almost all \( w \). This constant value must be \( a_v^{(0)}(z) = 0 \) since \( v \in W^p \). Hence \( v = Kv \) and \( f \in Z^p + KW^p \).
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**Definition.** For \( \lambda \in \mathbb{R} \), we denote by \( Q_\lambda \) the projection of \( H^p(C^2) \) on the null space of \( (\alpha - i\lambda) \) along the closure of the range of \( \alpha - i\lambda \).

The stage is now set for considering the fine structure of the spectrum of \( \alpha \).

**Theorem (1.16).** The point spectrum of \( (-i\alpha) \) is

\[
\{0\} \cup \left\{ \lambda \in \mathbb{R} \mid \left( \bigcup_{n=1}^{\infty} E_{n,\lambda} \right) > 0 \right\}.
\]

**Proof.** Suppose \( \lambda \in \mathbb{R}, \lambda \neq 0 \), and for some positive integer \( j \) we have \( \mu(E_{j,\lambda}) > 0 \). Then by (1.10)(ii), the function \( \chi_{E_{j,\lambda}}(z)w^j \) is in the null-space of \( (\alpha - i\lambda) \).

**Remark.** Theorem (1.16) shows that the point spectrum of \( (-i\alpha) \) is a countable union of arithmetic progressions. Specifically, the point spectrum of \( (-i\alpha) \) is the union of \( \{0\} \) and

\[
\bigcup \{ a\mathbb{Z}^+ : a \in \mathbb{R}, \mu(\delta^{-1}(a)) > 0 \}.
\]

The set \( S \) discussed in (1.6)–(1.9) now enters the stage.

**Theorem (1.17).** Let \( \lambda \in \mathbb{R} \). The range of \( (\alpha - i\lambda) \) is closed if and only if \( \lambda \) is not an accumulation point of \( S \). In this case, there is a bounded linear operator \( J_\lambda : H^p(C^2) \to H^p(C^2) \) such that \( J_\lambda(H^p(C^2)) \subseteq \mathfrak{D}(\alpha) \) and \( (\alpha - i\lambda)J_\lambda f = f - Q_\lambda f \) for all \( f \in H^p(C^2) \).

**Proof.** We first prove the existence of \( J_\lambda \) for \( \lambda \in \mathbb{R}, \lambda \neq 0 \), where \( \lambda \) is not an accumulation point of \( S \). Let \( B_\lambda^{(+)} \) (resp., \( B_\lambda^{(-)} \)) be the set of all \( z \in \mathbb{C} \setminus \mathbb{N}(\delta) \) such that \( \text{sgn}\delta(z) = \text{sgn}\lambda \) (resp., \( \text{sgn}\delta(z) = -\text{sgn}\lambda \)), where “sgn” stands for “sign of”. For \( n = 1, 2, \ldots, \) define \( G_n \) on \( C \) by \( G_n(z) = 0 \) if \( z \in E_{n,\lambda} \cup \mathbb{N}(\delta) \), and \( G_n(z) = [\text{sgn}\delta(z)](n\delta(z) - \lambda)^{-1} \) if \( z \notin E_{n,\lambda} \cup \mathbb{N}(\delta) \). Also, let \( G_0(z) = -[\text{sgn}\delta(z)]/\lambda \) for \( z \in C \setminus \mathbb{N}(\delta), \) and \( G_0(z) = 0 \) for \( z \in \mathbb{N}(\delta). \) Since \( \lambda \) is not an accumulation point of \( S \), it follows from (1.8) and (1.9), respectively, that there are \( \epsilon_\lambda > 0 \) and \( \gamma_\lambda > 0 \) such that

\[
\|G_n\|_\infty \leq \epsilon_\lambda^{-1} \quad \text{for } n = 1, 2, \ldots,
\]

and

\[
|\delta(z)| \geq \gamma_\lambda \quad \text{for almost all } z \in B_\lambda^{(+)}.
\]

For \( z \in B_\lambda^{(-)} \) (resp., \( B_\lambda^{(+)} \)) the sequence (resp., a tail of the sequence) \( \{G_n(z)\}_{n=0}^{\infty} \) is convex. Thus by [5, Theorem (4.5)] for each \( z \in C \setminus \mathbb{N}(\delta) \) the series

\[
G_0(z) + 2 \sum_{n=1}^{\infty} G_n(z) \text{Re}(w^n)
\]

is convergent.
converges for all \( w \in C \setminus \{1\} \). Since the series in (1.20) converges trivially to 0 for \( z \in N(\delta) \), we define a function \( L(z, w) \) on \( C \times (C \setminus \{1\}) \) by (1.20). For \( z \in B_A^{(-)} \) we also have from [5, Theorem (4.5)] that

\[
L(z, w) \geq 0 \quad \text{and} \quad (2\pi)^{-1} \int_0^{2\pi} L(z, e^{i\psi}) e^{-in\psi} d\psi = G_n(z)
\]

(1.21)

\[\text{for } n = 0, 1, 2, \ldots.\]

Let \( n_{\lambda} \) be the largest integer less than or equal to \( |\lambda|/\gamma_{\lambda}^{-1} \). It follows from (1.19) that for almost all \( z \in B_A^{(+)} \) \( |G_n(z)| \leq (n\gamma_{\lambda} - |\lambda|)^{-1} \) for \( n > n_{\lambda} \). Thus by (1.18) we have for almost all \( z \in B_A^{(+)} \),

\[
\sum_{n=1}^{\infty} |G_n(z)|^2 \leq n_{\lambda} \varepsilon_{\lambda}^{-2} + \sum_{n=n_{\lambda}+1}^{\infty} (n\gamma_{\lambda} - |\lambda|)^{-2}.
\]

For such \( z \) in \( B_A^{(+)} \) we see from (1.20) that

\[
(2\pi)^{-1} \int_0^{2\pi} |L(z, e^{i\psi})|^2 \, d\psi = |G_0(z)|^2 + 2 \sum_{n=1}^{\infty} |G_n(z)|^2
\]

(1.22)

\[\leq |\lambda|^{-2} + 2n_{\lambda} \varepsilon_{\lambda}^{-2} + 2 \sum_{n=n_{\lambda}+1}^{\infty} (n\gamma_{\lambda} - |\lambda|)^{-2}.
\]

It is obvious from (1.21) that for \( z \in B_A^{(-)} \), \( (2\pi)^{-1} \int_0^{2\pi} |L(z, e^{i\psi})| \, d\psi = |\lambda|^{-1} \). Let us denote the square root of the majorant in (1.22) by \( A/\lambda \). Then it is immediate from the foregoing that:

(1.23) For almost all \( z \in C \), \( (2\pi)^{-1} \int_0^{2\pi} |L(z, e^{i\psi})| \, d\psi \leq M_\lambda \). Now define a function \( k_\lambda(\cdot, \cdot) \) by setting \( k_\lambda(z, w) = -i[\text{sgn} \delta(z)]L(z, w) \) if \( z \in C \setminus N(\delta) \), \( k_\lambda(z, w) = 0 \) if \( z \in N(\delta) \). For each \( f \in H^p(C^2) \) we see from (1.23) that the integral \( \int_0^{2\pi} k_\lambda(z, we^{-iu}) f(z, e^{iu}) \, du \) exists for almost all \( (z, w) \in C^2 \) and defines a \( \bar{\mu} \)-integrable function of \((z, w)\). We set

\[
(J_\lambda f)(z, w) = (2\pi)^{-1} \int_0^{2\pi} k_\lambda(z, we^{-iu}) f(z, e^{iu}) \, du
\]

(1.24)

\[+i\lambda^{-1} X_{N(\delta)}(z) f(z, w), \quad \text{for } f \in H^p(C^2).
\]

We first observe from (1.23) and (1.24) that for almost all \( z \in C \setminus N(\delta) \):

\[
\int_0^{2\pi} |(J_\lambda f)(z, e^{i\psi})|^p \, d\psi \leq M_\lambda^p \int_0^{2\pi} |f(z, e^{i\psi})|^p \, d\psi.
\]

(1.25)

Since \( M_\lambda \geq |\lambda|^{-1} \), it is clear that (1.25) holds for \( z \in N(\delta) \). It follows that \( J_\lambda \) is a well-defined linear transformation from \( H^p(C^2) \) into \( L^p(\bar{\mu}) \) with \( \|J_\lambda\| \leq M_\lambda \). If we denote \( J_\lambda f \) by \( g \), then it is immediate from (1.24) that for almost all \( z \in C \), \( a_g^{(n)}(z) = 0 \) for \( n < 0 \), \( a_g^{(0)}(z) = i\lambda^{-1} a_f^{(0)}(z) \). This shows that \( g \in H^p(C^2) \). Moreover, for almost all \( z \in C \setminus N(\delta) \)
(1.26) \[ a_f^{(n)}(z) = a_k^{(n)}(z) \alpha_f^{(n)}(x) \] for \( n = 0, \pm 1, \pm 2, \ldots \).

Let \( z_0 \) be an arbitrary point in \( C \setminus N(\delta) \) such that (1.26) holds. Since the \( E_{n,\lambda} \)'s are disjoint, it is clear that for all sufficiently large \( n \) (say \( n \geq n_0 > 0 \)), \( z_0 \not\in E_{n,\lambda} \), and so for \( n \geq n_0 \) (1.26) becomes

(1.27) \[ a_f^{(n)}(z_0) = -i(n\delta(z_0) - \lambda)^{-1} \alpha_f^{(n)}(z_0). \]

Consider now the function \( h(w) \) (defined for \( |w| < 1 \)) by \( \sum_{n=n_0}^{\infty} a_\delta^{(n)}(z_0) w^n \). This function is in \( H^p \) of the disc, and we now show that \( dh/dw \) is also in \( H^p \) of the disc. Clearly by (1.27)

\[ h'(w) = -i \sum_{n=n_0}^{\infty} n(n\delta(z_0) - \lambda)^{-1} \alpha_f^{(n)}(z_0) w^{n-1}. \]

So

(1.28) \[ i\delta(z_0)h'(w) = \sum_{n=n_0}^{\infty} a_\delta^{(n)}(z_0) w^{n-1} + \lambda \sum_{n=n_0}^{\infty} (n\delta(z_0) - \lambda)^{-1} \alpha_f^{(n)}(z_0) w^{n-1}. \]

The first series on the right of (1.28) is obviously in \( H^p \). It follows from Hardy's inequality \([5, p. 48]\) that \( \sum_{n=n_0}^{\infty} |n\delta(z_0) - \lambda|^{-1} |a_\delta^{(n)}(z_0)| < \infty \). So the second series on the right of (1.28) is in \( H^\infty \). By \([5, \text{Theorem (3.11)}]\), it now follows that \( g(z_0, e^{i\psi}) \) is equal for almost all \( \psi \) to an absolutely continuous function of \( \psi \). From this it is readily verified (by taking Fourier coefficients with respect to \( \psi \) for each fixed \( z \) in a set of full measure) that by virtue of Theorem (1.4) \( g \in \Theta(\delta) \) and \( (\Theta - i\lambda)g = f - Q_\lambda f \) in \( H^p(C^2) \).

Now we prove the existence of \( f_0 \) with the stated properties under the assumption that 0 is not an accumulation point of \( \delta \). For arbitrary \( f \) in \( H^p(C^2) \), write \( f = a_f^{(0)} + v \), where \( v \in W^p \). Define \( V \) on \( C^2 \) by \( V(z, e^{i\psi}) = \int_0^\psi v(z, e^{i\mu}) d\mu \). It is easy to see that \( V \in L^p(\mu) \), and \( \|V\|_p < 2\pi\|v\|_p < 4\pi\|f\|_p \), and also that \( (V(z, w) - a_f^{(0)}(z)) \) is in \( W^p \). Define \( J_0 f \) on \( C^2 \) as follows: \( (J_0 f)(z, w) = 0 \) if \( z \in N(\delta) \), and

\[ (J_0 f)(z, w) = [\delta(z)]^{-1} (V(z, w) - a_f^{(0)}(z)) \] if \( z \in C \setminus N(\delta) \).

Since 0 is not an accumulation point of \( \delta \), we get from (1.9) that there is \( \eta > 0 \) such that \( |\delta(z)| > \eta \) for almost all \( z \) in \( C \setminus N(\delta) \). It follows that \( J_0 f \in W^p \), and \( \|J_0 f\|_p < (8\pi/\eta)\|f\|_p \). It is straightforward to verify that \( \Theta(J_0 f) = f - Q_\lambda f \). We remark in passing that the foregoing shows that \( \|J_0 f\| \) has a majorant which depends only on \( \lambda \) (and \( \delta \)), but not on \( p \).

Now we show that if \( \lambda \in R \) is not an accumulation point of \( \delta \), then \( (\Theta - i\lambda) \) has closed range. In fact, suppose that \( \{g_n\}_{n=1}^{\infty} \) is a sequence from the range
of \((\varnothing - i\lambda)\), and \(\|g_n - g\|_p \to 0\). Then \((\varnothing - i\lambda)J_\lambda g_n = g_n\). Since \(J_\lambda g_n \to J_\lambda g\), and \((\varnothing - i\lambda)\) is a closed operator, \((\varnothing - i\lambda)J_\lambda g = g\).

Now suppose \(\lambda \in \mathbb{R}\) is an accumulation point of \(\mathcal{E}\). If \(\lambda \neq 0\) (resp., if \(\lambda = 0\)) then by (1.8) we have for each \(\varepsilon\) such that \(0 < \varepsilon < |\lambda|\) (resp., \(0 < \varepsilon\)) that there are \(n_\varepsilon \in \mathbb{Z}^+, \eta_\varepsilon > 0\), and a set \(Y_\varepsilon \subseteq C\) such that \(\mu(Y_\varepsilon) > 0\) and \(\eta_\varepsilon < |\lambda - n_\varepsilon \delta(z)| < \varepsilon\) for \(z \in Y_\varepsilon\). Clearly \(n_\varepsilon > 0\). Define \(f\) in \(W^p\) by \(f(z, w) = \chi_\varepsilon(z)w^{n_\varepsilon}\). It is easy to see that \((\varnothing - i\lambda)f = i(n_\varepsilon \delta(z) - \lambda)f\) and so \(|(\varnothing - i\lambda)f|_p \leq \varepsilon|f|_p\). Let \(g_\varepsilon = f/|f|_p\), whence

\[
(1.29) \quad |(\varnothing - i\lambda)g_\varepsilon|_p \leq \varepsilon.
\]

Let \(\mathcal{E}\) be the operator from the graph of \((\varnothing - i\lambda)\) onto the range of \((\varnothing - i\lambda)\) defined by \(\mathcal{E}(f, (\varnothing - i\lambda)f) = (\varnothing - i\lambda)f\). If the range of \((\varnothing - i\lambda)\) were closed, then by the open mapping theorem we would have that the \(\mathcal{E}\)-image of the closed ball \(\mathcal{B}\) in the graph of \((\varnothing - i\lambda)\) centered at 0 and of radius \(2(1 + \|Q_\lambda\|)^{-1}\) would contain a closed ball around 0 in the range of \((\varnothing - i\lambda)\).

By Theorem (1.10)(i) and Theorem (1.15)(i), \(g_\varepsilon\) is in the range of \((\varnothing - i\lambda)\). With the aid of (1.29) we get that for sufficiently small \(\varepsilon\), there is \(G_\varepsilon \in \mathcal{D}(\varnothing)\) such that \(|G_\varepsilon|_p \leq [2(1 + \|Q_\lambda\|)]^{-1}\) and \(G_\varepsilon - g_\varepsilon\) is in the null-space of \((\varnothing - i\lambda)\). Since \(g_\varepsilon\) is in the range of \((\varnothing - i\lambda)\), \(G_\varepsilon - Q_\lambda G_\varepsilon = g_\varepsilon\). This gives the absurd conclusion \(1 = \|G_\varepsilon\|_p \leq (1 + \|Q_\lambda\|)\|G_\varepsilon\|_p \leq 1/2\).

We now establish Theorem (1.5) by showing that \(\Lambda(\varnothing) = \varnothing\).

**Proof of Theorem (1.5).** By Theorem (1.15)(ii) (resp., (1.6)), \(0 \in \Lambda(\varnothing)\) (resp., \(0 \in \varnothing\)). Let \(\lambda \in \mathbb{R}, \lambda \neq 0\). If \(\lambda \in \varnothing\), then \(\lambda\) is an accumulation point of \(\mathcal{E}\), or else by (1.6) and (1.8) \(\mu(E_n, \lambda) > 0\) for some positive integer \(n\). In the first instance the range of \((\varnothing - i\lambda)\) is not closed by Theorem (1.17), while in the second instance \((\varnothing - i\lambda)\) is not one-to-one by Theorem (1.16). So \(i\varnothing \subseteq \Lambda(\varnothing)\). Conversely, if \(i\lambda \in \Lambda(\varnothing)\), then the range of \((\varnothing - i\lambda)\) is not closed, or else by Theorem (1.10)(iii) \((\varnothing - i\lambda)\) is not one-to-one. In either event \(\lambda \in \varnothing\).

This completes the proof.

**Remarks.** Let \(\lambda \in \mathbb{R}\). (i) If \(\lambda \notin \varnothing\), then the operator \(J_\lambda\) of Theorem (1.17) is \((\varnothing - i\lambda)^{-1}\), the resolvent operator of \(\varnothing\) at \(i\lambda\). (ii) If \(\lambda\) is not an accumulation point of \(\varnothing\), then it is easy to see with the aid of Theorem (1.10)(ii), Theorem (1.15)(ii), and the definition of \(J_\lambda\) in the proof of Theorem (1.17) that the null space of \((\varnothing - i\lambda)\) coincides with the null space of \(J_\lambda\). Hence \(J_\lambda = J_\lambda(I - Q_\lambda)\).

In particular, the range of \(J_\lambda\) is the range of its restriction to the range of \((\varnothing - i\lambda)\). Thus if \(\lambda\) is not an accumulation point of \(\varnothing\) and \((\varnothing - i\lambda)\) is not one-to-one, then the range of \(J_\lambda\) is not \(\mathcal{D}(\varnothing)\). For example, if \(\delta = \chi_Y\), where \(0 < \mu(Y) < 1\), then \(\varnothing = \mathbb{Z}^+\), and \((\varnothing - i)\) is not one-to-one.

We close this section with a discussion of the problem of extending \(\{S_t\}, t \in \mathbb{R}\), to a semigroup of contraction operators indexed by the upper half-plane in \(\mathbb{C}\). Specifically, let \(\mathcal{H} = \{\zeta \in \mathbb{C}: \text{Im} \zeta \geq 0\}\). By the term "con-
traction operator" we shall mean an operator of norm at most one which maps $H^p(C^2)$ into itself. Suppose the mapping $t \in \mathbb{R} \mapsto S_t$ can be extended to a mapping $\xi \in \mathcal{H} \mapsto S_\xi$ so that $\{S_\xi\}, \xi \in \mathcal{H}$, is a semigroup of contraction operators which is continuous on $\mathcal{H}$ in the strong operator topology and holomorphic on the interior of $\mathcal{H}$. Let $\xi$ be the infinitesimal generator of $\{S_\alpha\}, \alpha \geq 0$. By [7, Theorem 17.9.2], $\xi = i\varphi$. By [4, VIII. 1.11], $\text{Re}[\Lambda(\xi)] \leq 0$. So $\text{Im}[\Lambda(\xi)] \geq 0$, and, by Theorem (1.5), $\delta \geq 0$. It follows that $\delta(z) \geq 0$ for almost all $z$ in $\mathbb{C}$. The next result shows that conversely if $\delta \geq 0$ a.e., then $\{S_\xi\}, t \in \mathbb{R}$, can be extended to a semigroup $\{S_t\}, \xi \in \mathcal{H}$, as above. Without loss of generality we take $\delta \geq 0$ everywhere on $\mathbb{C}$.

(1.30) Theorem. Suppose $\delta(z) \geq 0$ for all $z \in \mathbb{C}$. Then the one-parameter group $\{S_t\}$ in Theorem (1.10) has a unique extension to a semigroup $\{S_\xi\}, \xi \in \mathcal{H}$, of bounded operators on $H^p(C^2)$ such that $\{S_\xi\}$ is continuous on $\mathcal{H}$ in the strong operator topology and holomorphic on the interior of $\mathcal{H}$. The unique semigroup $\{S_\xi\}, \xi \in \mathcal{H}$, consists of contraction operators.

Proof. Uniqueness is immediate from the Schwarz reflection principle.

Suppose $\text{Im} \xi > 0$ and $f \in H^p(C^2)$. For almost all $z \in \mathbb{C}, f(z, \cdot) \in H^p(d\psi/2\pi)$. For such $z$, let $\tilde{f}(z, w)$ denote (for $|w| < 1$) the Poisson integral of $f(z, \cdot)$. We define $h_\xi$ almost everywhere on $C^2$ as follows: $h_\xi(z, w) = \tilde{f}(z, e^{i\xi(z)}w)$ if $\delta(z) > 0$, and $h_\xi(z, w) = f(z, w)$ if $\delta(z) = 0$. It is easy to see that $h_\xi \in H^p(C^2)$, and $\|h_\xi\|_p \leq \|f\|_p$. Let $S_\xi f = h_\xi$. Then $S_\xi$ is a contraction operator. For $\xi \in \mathcal{H}$, it is easy to see that if $f \in H^p(C^2)$, then for almost all $z \in \mathbb{C}, (S_\xi f)(z, w)$ has (as a function of $w$) the sequence of Fourier coefficients $\{e^{in\xi(z)}a_{\xi n}(z)\}$. It follows readily that $S_{\xi + \eta} = S_\xi S_\eta$ for $\xi, \eta \in \mathcal{H}$. To complete the proof it suffices to show that if $f_0 \in H^p(C^2)$, $q$ is the index conjugate to $p$, and $g \in L^q(\mu)$, then, as a function of $\xi$,

$$F(\xi) = \int_{C^2} g(z, w)[(S_\xi f_0)(z, w)] d\mu(z, w)$$

is continuous on $\mathcal{H}$ and analytic on the interior of $\mathcal{H}$. Since $S_\xi u = u$ for $u \in Z^p, \xi \in \mathcal{H}$, we can assume without loss of generality that $f_0 \in W^p$. So there is a sequence $\{v_k\}$ from the linear span of the functions $z^m w^n (m, n \in \mathbb{Z}, n > 0)$ such that $v_k \to f_0$ in $L^p(\mu)$. Since $\|S_\xi\| \leq 1$ for $\xi \in \mathcal{H}$, the functions (of $\xi$)

$$\int_{C^2} g(z, w)[(S_\xi v_k)(z, w)] d\mu(z, w)$$

tend to $F$ uniformly on $\mathcal{H}$ as $k \to \infty$. Accordingly we can further assume without loss of generality that $f_0(z, w) = z^m w^n$ for some $m \in \mathbb{Z}, n \in \mathbb{Z}, n > 0$. Hence $(S_\xi f_0)(z, w) = e^{in\xi(z)}z^m w^n$ for $\xi \in \mathcal{H}$. The continuity of $F$ on $\mathcal{H}$ now follows from the dominated convergence theorem.
Let $\xi_0$ be an interior point of $\mathcal{S}$, $y_0 = \text{Im} \xi_0$. Let $\{\xi_k\}$ be a sequence of points distinct from $\xi_0$ such that $|\xi_k - \xi_0| < y_0/2$ for all $k$ and $\xi_k \to \xi_0$. For all $k$

$$[F(\xi_k) - F(\xi_0)](\xi_k - \xi_0)^{-1}$$

(1.31)

$$= \int_{\mathbb{C}^2} g(z, w) z^m w^n (e^{in\delta(z)\xi_k} - e^{in\delta(z)\xi_0})(\xi_k - \xi_0)^{-1} d\mu(z, w).$$

But for each $k$ and all $z \in \mathbb{C}$

$$|e^{in\delta(z)\xi_k} - e^{in\delta(z)\xi_0}|/|\xi_k - \xi_0| \leq n\delta(z)e^{-n\delta(z)y_0/2}.$$

Since the majorant in this last inequality is a bounded function of $z$, it follows from (1.31) by dominated convergence that

$$F(\xi_0) = \inf \int_{\mathbb{C}^2} g(z, w) z^m w^n d\mu(z, w).$$

2. Spectral analysis when $\{T_t\} |Z^p|$ has an unbounded generator. For $1 < p < \infty$, let $\Omega_p$ be the set of all strongly continuous one-parameter groups $\{T_t\}$ of isometries of $H^p(C^2)$ such that $\{T_t\} |Z^p|$ is not continuous in the uniform operator topology (if $p = 2$, we further require that each $T_t$ have the form (1.1)). It is known [3, Theorem (2.22) and (2.24)] that a group $\{T_t\} \in \Omega_p, 1 < p < \infty$, can be written in the form (0.2). The unique group of Möbius transformations of the disc $\{\phi_t\}$ occurring in (0.2) is called the conformal group of $\{T_t\}$. As described in the Introduction, the conformal group of $\{T_t\}$ is assigned a type number depending on the nature of its set of common fixed points in the extended plane. $\{T_t\}$ is said to be of type (i), (ii), or (iii) according as its conformal group is.

(2.1) Theorem. Let $\{T_t\} \in \Omega_p, 1 < p < \infty$. If $\{T_t\}$ is of type (ii) or (iii), then the spectrum of its infinitesimal generator is $i\mathbb{R}$.

Proof. In the case at hand, $\{T_t\}$ can be expressed in the form (0.2) with the real constant $\delta$ equal to zero [3, Theorem (2.24)]. There is clearly no loss of generality if we establish the desired conclusion under the additional hypothesis that the real constant $\rho$ in (0.2) is zero. Let $U$ be the isometry of $H^p(C^2)$ onto itself defined by $(Uf)(z, w) = f(z, u(z)w)$, where $u$ is the unimodular measurable function on $C$ in (0.2). Since it suffices to establish the desired conclusion for the group $\{UT_tU^{-1}\}$, we assume further without loss of generality that the function $u$ in (0.2) is identically one.

If $\{T_t\}$ is of type (ii) (resp., type (iii)), we represent its conformal group $\{\phi_t\}$ in the form $[1, (1.8)]$ (resp., $[1, (1.9)]$). This representation furnishes a certain nonzero real constant $c$, and a unimodular common fixed point $\alpha$ of the group.
\{\phi_i\}. Let \(\lambda\) be any real number. For each \(\eta > -p^{-1}\) (resp., for each \(\eta \in \mathbb{C}\) with \(\text{Re}(\eta) > -p^{-1}\)) define \(f_\eta\) on \(\mathbb{C} \setminus \{a\}\) by setting

\[
f_\eta(z) = (z - a)^\eta \exp(\lambda(z - a)) \quad \text{(resp., } f_\eta(z) = (z - a)^\eta).\]

Define \(F_\eta(z,w) = f_\eta(z)w\) for \(z \in \mathbb{C} \setminus \{a\}, w \in \mathbb{C}\). It is straightforward to verify that \(F_\eta \in W^p\). Denote the infinitesimal generator of \(\{T_t\}\) by \(\mathcal{A}\). By virtue of the description of \(\mathcal{A}\) in [3, (3.16) and Theorem (3.17)], the proof of [1, Theorem (3.1)] applies with obvious minor modifications to show that

\[
\|\mathcal{A}^n\| \to 0 \quad \text{as } t, \text{ approaches } -p_x \text{ from the right (resp., as } t \text{ approaches } -p_x + iXc).\]

Remark. In the type (ii) part of the proof of Theorem (2.1) above, the function \(f_\eta \in L^p(\mu)\) (and hence \(F_\eta \in W^p\)), but \(f_\eta\) need not be in \(H^p(d\theta/2\pi)\). This state of affairs accounts for the fact that the infinitesimal generator of a type (ii) group on \(H^p(d\theta/2\pi)\) has for its spectrum a proper subset of \(\mathbb{R}\) [1, Theorem (3.1)(ii)] in contrast to the outcome of Theorem (2.1) for the torus.

Now let \(\{T_t\} \in \Omega\) be of type (i) with the representation (0.2). Let \(c\) be the angular velocity of \(\{\phi_i\}\) (see [2, p. 337]; in particular \(c \in \mathbb{R}, c \neq 0\)). As shown in [3, (2.26) and Theorem (3.1)] the cohomology class of the cocycle of \(\{T_t\}\) can be identified with the element \(\delta + c\mathbb{Z}\) of the group \(\mathbb{R}/c\mathbb{Z}\). The order of this cohomology class will be called the cohomological order of \(\{T_t\}\). Thus the cohomological order of \(\{T_t\}\) is finite (resp., infinite) if and only if \(\delta/c\) is rational (resp., irrational).

(2.2) Theorem. Let \(\{T_t\} \in \Omega_p (1 < p < \infty)\) be of type (i), with infinitesimal generator \(A\). Let \(\{T_t\}\) have the form (0.2), and let \(c\) be the angular velocity of its conformal group. If the cohomological order of \(\{T_t\}\) is a positive integer \(k\), then \(A\) has pure point spectrum, and \(\Lambda(A) = i(p + cp^{-1} + ck^{-1}\mathbb{Z})\). If the cohomological order of \(\{T_t\}\) is infinite, then \(\Lambda(A) = i\mathbb{R}\).

Proof. For \(f \in L^p(\mu)\), let \(\{f_{m,n}\}\) be the double sequence of Fourier coefficients of \(f\). Let \(\delta_0\) be the unique element of \(\delta + c\mathbb{Z}\) such that \(0 < \delta_0 \leq |c|\). By [3, Theorem (4.4)] \(A\) is similar under an isometry to \(\mathcal{A} + i(p + cp^{-1})\), where \(\mathcal{A}\), the infinitesimal generator of the group \(\{S_t\}\) given by

\[
(S_t f)(z,w) = f(e^{\delta_0 t}z, e^{\delta_0 t}w), \quad \text{for } f \in H^p(\mathbb{C}^2),
\]

has the following description: \(\mathcal{A}f = g\) means \(\{g_{m,n}\} = \{i(m\delta_0 + n\delta_0)f_{m,n}\}\). In particular, for each \((m, n) \in P\) (recall the definition of \(P\) in the Introduction), \(z^{m}w^{n}\) is an eigenvector of \(\mathcal{A}\) with associated eigenvalue \(i(m\delta_0 + n\delta_0)\).

If the cohomological order of \(\{T_t\}\) is a positive integer \(k\), then \(\delta_0|c|^{-1} = jk^{-1}\), where \(j\) is a positive integer relatively prime to \(k\). Because \(j\) and \(k\) are relatively prime, any integer \(n\) can be written in the form \(n = n_1j + n_2k\),
where \( n_1 > 0 \). It follows that \( \{ mc + n\delta_0 : (m, n) \in P \} = \{ mc + n\delta_0 : m \in \mathbb{Z}, n \in \mathbb{Z} \} \). Also, since \( j \) and \( k \) are relatively prime, we have \( \{ mc + n\delta_0 : m \in \mathbb{Z}, n \in \mathbb{Z} \} = ck^{-1}\mathbb{Z} \). If \( \lambda \in \mathbb{R} \setminus (ck^{-1}\mathbb{Z}) \), let \( \lambda_1 = \lambda c^{-1}, \delta_1 = \delta_0 c^{-1} \), and define a Borel measure \( \beta \) on \( C^2 \) as follows: for each continuous complex-valued function \( f \) on \( C^2 \)

\[
\beta(f) = -c^{-1}(e^{i2\pi k\lambda_1} - 1)^{-1} \int_0^{2\pi k} f(e^{i\theta}, e^{i\delta_1 \theta}) e^{i\lambda_1 \theta} d\theta.
\]

Denoting the double sequence of Fourier coefficients of the measure \( \beta \) by \( \{ \hat{\beta}_{m,n} \} \) we have for all \( m \in \mathbb{Z}, n \in \mathbb{Z}, \)

\[
\hat{\beta}_{m,n} = i(\lambda - mc - n\delta_0)^{-1}.
\]

For \( g \in H^p(C^2) \), let \( F \) be the convolution on \( C^2, g \ast \beta \). Then for \( m \in \mathbb{Z}, n \in \mathbb{Z}, \)

\[
\hat{F}_{m,n} = i(\lambda - mc - n\delta_0)^{-1} \hat{\delta}_{m,n}.
\]

Clearly, \( F \in H^p(C^2) \) and \( (\beta - i\lambda)F \)

\[
= g.
\]

It is obvious from double Fourier coefficients that \( (\beta - i\lambda) \) is one-to-one, and thus we have established that \( \beta \) has pure point spectrum equal to \( ick^{-1}\mathbb{Z} \).

If the cohomological order of \( \{ T_j \} \) is infinite, then \( \delta_0 c^{-1} \) is irrational, and therefore \( i(\lambda \in (mc + n\delta_0) : (m, n) \in P \} \) is dense in \( i\mathbb{R} \). So \( \Lambda(\beta) = i\mathbb{R} \).

Remark. By direct calculation using (2.3), it is easy to see that for each \( s \in \mathbb{Z} \) the Fourier coefficient of \( \beta \) corresponding to the integer pair \(((-sgn c) s_j, s k)\) is \( i/\lambda \). So by Riemann-Lebesgue \( \beta \) is not absolutely continuous with respect to \( \mu \).

References


