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HOPF INVARIANTS AND BROWDER'S WORK ON THE KERVAIRE INVARIANT PROBLEM

BY

WARREN M. KRUEGER

ABSTRACT. In this paper we calculate certain functional differentials in the Adams spectral sequence converging to Wu cobordism whose values may be thought of as Hopf invariants. These results are applied to reobtain Browder's characterization: if $q + 1 = 2^k$, there is a $2q$ dimensional manifold of Kervaire invariant one if and only if h_k^2 survives to $E_\infty(S^0)$.

Introduction. In 1969 Browder [2] showed that there is a $2q$ dimensional framed manifold with Kervaire invariant one if and only if $q + 1 = 2^k$ and h_k^2 survives to $E_\infty(S^0)$. More precisely in case $q + 1 = 2^k$, if (M^{2q}, F) is a framed manifold whose Thom invariant is $f \in \pi_{2q}(S^0)$, then Browder showed that the Kervaire invariant, $K(M^{2q}, F)$, is one if and only if f projects to h_k^2 in $E_\infty^{2,2^{k+1}}(S^0)$. On the other hand, the present author [3] introduced homotopy invariants I_1 and I_2 which are defined on certain subgroups of the homotopy groups $\pi_*(X)$ of a spectrum X and which take their values in the E_2 term of an Adams spectral sequence for X. For the mod 2 Adams spectral sequence of S^0 and an element $f \in \pi_{2q}(S^0)$, $I_2(f)$ is always defined, and if $q + 1 = 2^k$, h_k^2 survives if and only if there is $f \in \pi_{2^{k+1}-2}(S^0)$ so that $I_2(f) = h_k^2$. Thus we can restate Browder's theorem (at least in dimension $q + 1 = 2^k$) as follows.

THEOREM 1 (BROWDER). *If $(M^{2^{k+1}-2}, F)$ is a framed manifold with Thom invariant $f \in \pi_{2^{k+1}-2}(S^0)$, then*

$$I_2(f) = K(M^{2^{k+1}-2}, F)h_k^2$$

in $E_2^{2,2^{k+1}}(S^0)$.

It is the main purpose of this paper to show that the invariant I_2 not only serves to give a convenient expression for Browder's theorem but in fact serves as a vehicle for proving it.

Our efforts in this direction center on the mod 2 Adams spectral sequence which converges to Wu cobordism. If $MO\langle v_{q+1}\rangle$ denotes the Thom spectrum for Wu $(q + 1)$-cobordism theory, the Wu manifolds $(S^q \times S^q, \bar{q})$ con-

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structed by Browder [2, §5] have Thom invariant $\bar{j}\omega \in \pi_{2q}(MO\langle v_{q+1}\rangle)$. Then the main result of the paper explicitly determines $I_2(\bar{j}\omega)$ as a subset of $E_2^{2,2^{k+1}}(MO\langle v_{2^k}\rangle)$ in case $q + 1 = 2^k$.

The paper is organized along the following lines.

In §1, after giving a preamble of notation and terminology basic to our work, we state the main theorem (Theorem 3) and deduce Theorem 1 from it.

§2 is devoted to setting up the homological algebra which is preliminary to the detailed study of $I_2(\bar{j}\omega)$.

In §3 we prove Theorem 3 except for the verification of a geometric result (Lemma 8) whose details are given in §4.

1. **Statement of main results.** We begin by establishing the basic geometric and algebraic contexts for the paper. The geometric setting is the stable homotopy category constructed by Adams in [1]. A notation from that setting which shall be convenient is the following: if f: $X \to Y$ is a morphism, then its Puppe cofiber triangle is denoted by

$$\begin{tikzcd} X \arrow[rr, "f"] & & Y \arrow[dl, "P(f)"] \\ & C(f) \arrow[ul, "Q(f)"] & \end{tikzcd}$$

where the degree of $P(f) = 0$ and the degree of $Q(f) = -1$. It will also be convenient to name the objects and morphisms in the stable category which correspond to the Thom complexes arising in Browder's work [2]. Let MO and $MO\langle v_{q+1}\rangle$ denote the spectra determined by $T\gamma_n$ and $T\bar{\gamma}_n$ respectively and $\bar{p}$: $MO\langle v_{q+1}\rangle \to MO$ denote the morphism which corresponds to $T\bar{\gamma}_n \to T\gamma_n$. Also if H_q denotes the Eilenberg-Mac Lane space of type $(\mathbf{Z}_2, q)$, let $\bar{j}$: $\underline{H}_q \to MO\langle v_{q+1}\rangle$ denote the morphism which corresponds to the fiber inclusion j: $H_q \to BO\langle v_{q+1}\rangle$, where $\underline{X}$ denotes the suspension spectrum of the pointed CW complex $(X, *)$.

Let $\mathcal{Q}^*$ denote the mod 2 Steenrod algebra and $\mathcal{Q}_*$ its dual, the latter supplied with the $\mathbf{Z}_2$ base $\{\xi^R\}$ and the former with the dual base $\{\mathrm{Sq}^R\}$ of Milnor [10]. Homology and cohomology will be taken with $\mathbf{Z}_2$ coefficients and the basic algebraic context will be the category of $\mathcal{Q}_*$ comodules.

Having disposed of these preliminaries, we may now state our main results whose proofs are distributed over the next three sections.

THEOREM 2. *Let $a \in \pi_q(\underline{H}_q) \simeq \mathbf{Z}_2$ be the generator. Then*

$$\mathrm{Ext}_{\mathcal{Q}_*}^{1,q+1}\left(\mathbf{Z}_2, H_*(MO\langle v_{q+1}\rangle)\right)$$

is isomorphic to $\mathbf{Z}_2$ generated by $I_1(\bar{j}a)$. Further if $q + 1 = 2^k$ and i: $S^0 \to MO\langle v_{2^k}\rangle$ is the morphism of spectra induced by (pt.$\to BO_n\langle v_{q+1}\rangle$), *then*

$i_{\#}(h_k) = I_1(\bar{j}a)$, where $h_k \in \mathrm{Ext}_{\mathscr{Q}_*}^{1,2^k}(\mathbf{Z}_2, \mathbf{Z}_2)$ is the generator.

Let $\bar{q}$: $S^q \times S^q \to BO_n\langle v_{q+1}\rangle$ be the composition

$$S^q \times S^q \xrightarrow{a\times a} H_q \times H_q \xrightarrow{\mu} H_q \xrightarrow{j} BO_n\langle v_{q+1}\rangle$$

where μ is the group structure map and where we have taken the liberty of letting "a" also denote the essential map $S^q \to H_q$. Browder [2] has shown that the Thom invariant corresponding to the Wu manifold $(S^q \times S^q, \bar{q})$ is $\bar{j}\omega$, where $\omega \in \pi_{2q}(\underline{H_q}) \simeq \mathbf{Z}_2$ is the generator. Our principal result determines $I_2(\bar{j}\omega)$ for $q + 1 = 2^k$.

THEOREM 3. *If $q + 1 = 2^k$, then $I_1(\bar{j}\omega) = 0$, so that $I_2(\bar{j}\omega)$ is defined. In this case $I_2(\bar{j}\omega)$ is given by*

$$(*) \qquad I_2(\bar{j}\omega) = \{h_k \cdot I_1(\bar{j}a)\}$$

in $\mathrm{Ext}_{\mathscr{Q}_*}^{2,2^{k+1}}(\mathbf{Z}_2, H_*(MO\langle v_{2^k}\rangle))/\mathrm{im}\ d_2^{0,2^{k+1}}$ and this element is not equal to 0.

To illustrate the position of Theorem 3 relative to Browder's work on the Kervaire invariant problem, we use it to prove Theorem 1.

Let i: $\underline{S}^0 \to MO\langle v_{q+1}\rangle$ be as in Theorem 1. Browder has shown that there is a $2q$ dimensional framed manifold of Kervaire invariant one if and only if

$$\mathrm{im}\{i_*: \pi_{2q}(\underline{S}^0) \to \pi_{2q}(MO\langle v_{q+1}\rangle)\} \neq 0.$$

For $q + 1 = 2^k$, we deduce from Theorem 3 that im $i_* = 0$ if and only if h_k^2 fails to survive to E_∞.

First suppose that im $i_* = 0$. Then

$$P(i)_*: \pi_{2^{k+1}-2}(MO\langle v_{2^k}\rangle) \to \pi_{2^{k+1}-2}(C(i))$$

is monic. Now

$$i_{\#}(h_k^2) = h_k \cdot i_{\#}(h_k) = h_k \cdot I_1(\bar{j}a)$$

which survives since $\{h_k \cdot I_1(\bar{j}a)\} = I_2(\bar{j}\omega) \neq 0$. Since $P(i)_*$ is monic, $P(i)_*(g) \neq 0$ for any $g \in \pi_{2^{k+1}-2}(MO\langle v_{2^k}\rangle)$ which projects to $\{h_k \cdot I_1(\bar{j}a)\}$ in $E_\infty(MO\langle v_{2^k}\rangle)$. Thus by Lemma 3.4.1 of [8] h_k^2 is not even permanent.

Conversely suppose that h_k^2 fails to survive but that there is an element $f \in \pi_{2^{k+1}-2}(S^0)$ so that $i_*(f) \neq 0$. By Theorem 7.2 of [2], $i_*f = \bar{j}\omega$ and by naturality

$$I_2(\bar{j}\omega) = I_2(i_*f) = i_{\#}(I_2(f)) = 0,$$

since h_k^2 fails to survive. But this contradicts Theorem 3, so im i_* must be 0.

2. Homological algebra. In this section we carry out the homological algebra which is preliminary to the proof of Theorem 3. For a spectrum X, let $E(X) = \{E_r(X), d_r\}$ denote the mod 2 Adams spectral sequence whose E_2

term is $\mathrm{Ext}_{\mathcal{Q}^*}^{*,*}(\mathbf{Z}_2, H_*(X))$ and whose limit is $\pi_*(X)$. Occasionally we shall regard E as a functor on the stable category and denote induced maps on E_1 by a subscript "*" and induced maps on (sub) quotients of E_1 by a subscript "#" (which convention is consistent with the above usage of sharped morphism letters for induced homomorphisms on Ext). In [3], it was shown that

$$E_1^{s,t}(X) = \left[\bar{\mathcal{Q}}_*^s \otimes H_*(X)\right]_t$$

with differential d_1^s equal to

$$(j \otimes 1)(c \otimes 1)\psi_{\bar{\mathcal{Q}}_*^s \otimes H_*(X)},$$

where $\bar{\mathcal{Q}}_* = \mathcal{Q}_*/\mathcal{Q}_0, j\colon \mathcal{Q}_* \to \bar{\mathcal{Q}}_*$ is the quotient, $\bar{\mathcal{Q}}_*^s$ denotes the s-fold tensor product of $\bar{\mathcal{Q}}_*$, $c\colon \mathcal{Q}_* \to \mathcal{Q}_*$ is the canonical anti-automorphism and $\psi_{\bar{\mathcal{Q}}_*^s \otimes H_*(X)}$ is the tensor product $\mathcal{Q}_*$ comodule structure map for $\bar{\mathcal{Q}}_*^s \otimes H_*(X)$. In what follows, $\mathrm{Ext}_{\mathcal{Q}^*}^s(\mathbf{Z}_2, H_*(X))$ shall be regarded as a subquotient of $\bar{\mathcal{Q}}_*^s \otimes H_*(X)$.

Let $\omega \in \pi_{2q}(\underline{H}_q) \simeq \mathbf{Z}_2$ be the generator. Since ω is determined by the Hopf construction applied to

$$S^q \times S^q \xrightarrow{a \times a} H_q \times H_q \xrightarrow{\mu} H_q,$$

it is easy to see that ω_* is 0 on homology, so that $I_1(\omega)$ is defined. Its value is given in

PROPOSITION 4.

$$I_1(\omega) = \left\{ \sum_{r=1}^{q+1} (r, q+1-r)\xi_1^r \otimes \bar{g}(q+1-r) \right\}$$

in $\mathrm{Ext}_{\mathcal{Q}_*}^{1,2q+1}(\mathbf{Z}_2, H_*(\underline{H}_q))$, *where* $\bar{g}(s) \in H_{q+s}(\underline{H}_q)$ *is the element in basis dual to the base* $\{\mathrm{Sq}^R g\}$ *for* $H^{q+s}(\underline{H}_q)$ *corresponding to* $\mathrm{Sq}^s g$, g *is the generator of* $H^q(\underline{H}_q)$ *and* (r, s) *is the binomial coefficient* $(r+s)!/r!s!$.

PROOF. Let $\bar{x}(q+1) \in H_{2q+1}(C(\omega))$ be the element in the basis dual to the basis $\{\mathrm{Sq}^R x\}$ for $H^{2q+1}(C(\omega))$ corresponding to $\mathrm{Sq}^{q+1}x$, where $x \in H^q(C(\omega))$ maps to $g \in H^q(\underline{H}_q)$ by the map $P(\omega)^*$. Thus $\bar{x}(q+1)$ maps to $\iota \in H_{2q}(\underline{S}^{2q})$ via $Q(\omega)_*$. Consequently the calculation of $I_1(\omega)$ amounts to the calculation of $d_1^0(\bar{x}(q+1))$, which in turn, amounts to the calculation of $\psi_{C(\omega)}(\bar{x}(q+1))$, where $\psi_{C(\omega)}$ is the $\mathcal{Q}_*$ comodule structure map for $H_*(C(\omega))$.

To that end let $\varepsilon_q\colon \underline{H}_q \to \underline{H}_{q+1}$ be the obvious map of spectra of degree +1. Since $\varepsilon_q\omega = 0$, ε_q extends to a map $\varepsilon_q'\colon C(\omega) \to \underline{H}_{q+1}$ of degree +1 for which $\varepsilon_q'^*\colon H^j(\underline{H}_{q+1}) \to H^{j-1}(C(\omega))$ is an isomorphism for $j < 2q+1$ and a monomorphism for $j = 2q+2$. Now let $y \in H^{q+1}(\underline{H}_{q+1})$ be the generator; then $\varepsilon_q'^*(y) = x$ so that $\varepsilon_q'^*(\mathrm{Sq}^{q+1}y) = \mathrm{Sq}^{q+1}x$ and hence

$$\varepsilon_{q*}'(\bar{x}(q+1)) = \bar{y}(q+1),$$

where $\bar{y}(s)$ and $\mathrm{Sq}^s y$ are corresponding elements of the obvious dual bases. From basic properties,

$$\psi_{C(\omega)}(\bar{x}(q+1)) = 1 \otimes \bar{x}(q+1) + S,$$

while

$$\begin{aligned}(1 \otimes \varepsilon'_{q*})(\psi_{C(\omega)}(\bar{x}(q+1))) &= \psi_{\underline{H}_{q+1}}(\bar{y}(q+1)) \\ &= \sum_{r=0}^{q+1} (r, q+1-r)\xi_1^r \otimes \bar{y}(q+1-r)\end{aligned}$$

since $H_*(\underline{H}_{q+1})$ is an extended $\mathcal{Q}_*$ comodule in dimensions $\leqslant 2q+2$. However

$$(1 \otimes \varepsilon'_{q*})(S) = \sum_{r=0}^{q+1} (r, q+1-r)\xi_1^r \otimes \bar{y}(q+1-r),$$

so

$$\psi_{C(\omega)}(\bar{x}(q+1)) = \sum_{r=0}^{q+1} (r, q+1-r)\xi_1^r \otimes \bar{x}(q+1-r).$$

Thus

$$\begin{aligned}d_1^0(\bar{x}(q+1)) &= (j \otimes 1)(c \otimes 1)\psi_{C(\omega)}(\bar{x}(q+1)) \\ &= \sum_{r=1}^{q+1} (r, q+1-r)\xi_1^r \otimes \bar{x}(q+1-r).\end{aligned}$$

Since $P(\omega)_*(\bar{g}(s)) = \bar{x}(s)$ for $s \leqslant q$, we have

$$I_1(\omega) = \left\{ \sum_{r=1}^{q+1} (r, q+1-r)\xi_1^r \otimes \bar{g}(q+1-r) \right\}.$$

COROLLARY 5. *If* $q+1 = 2^k$, *then* $I_1(\bar{j}\omega)$ *is* 0.

PROOF. By naturality $I_1(\bar{j}\omega) = \bar{j}_\#(I_1(\omega))$. Now if $q+1 = 2^k$, $I_1(\omega) = \{\xi_1^{2^k} \otimes \bar{g}\}$ and so $\bar{j}_\# I_1(\omega) = \{\xi_1^{2^k} \otimes \bar{j}_*(\bar{g})\}$. But $\bar{j}_*(\bar{g}) = 0$.

Our next result serves to delimit the indeterminacy of $I_2(\bar{j}\omega)$.

PROPOSITION 6. *In the long exact* Ext-*sequence associated to*

$$0 \to H_*(MO\langle v_{q+1}\rangle) \to H_*\big(C(\bar{j}\omega)\big) \to H_{*-1}(\underline{S}^{2q}) \to 0,$$

the coboundary

$$\delta\colon \mathrm{Ext}_{\mathcal{Q}_*}^{1,2q+1}\big(\mathbf{Z}_2, H_*(\underline{S}^{2q})\big) \to \mathrm{Ext}_{\mathcal{Q}_*}^{2,2q+2}\big(\mathbf{Z}_2, H_*(MO\langle v_{q+1}\rangle)\big)$$

is given by

$$\delta(h_0') = h_0 \cdot I_1(\bar{j\omega}),$$

where h_0' corresponds to $h_0 \in \mathrm{Ext}^{1,1}_{\mathcal{Q}_*}(\mathbf{Z}_2, \mathbf{Z}_2)$.

PROOF. It suffices to show that $\delta(h_0') = h_0 \cdot I_1(\omega)$ in the Ext-sequence for $\omega\colon \underline{S}^{2q} \to \underline{H}_q$ and then apply naturality. To get this latter formula, we calculate δ by means of the cobar differential.

On the one hand,

$$1 \otimes Q(\omega)_*\colon \bar{\mathcal{Q}}_* \otimes H_*C(\omega) \to \bar{\mathcal{Q}}_* \otimes H_*(\underline{S}^{2q})$$

maps $[\xi_1|\bar{x}(q+1)]$ to $[\xi_1|\iota]$ which represents h_0'. On the other hand

$$\left(1 \otimes P(\omega)_*\right)\colon \mathcal{Q}_*^2 \otimes H_*(\underline{H}_q) \to \mathcal{Q}_*^2 \otimes H_*(C(\omega))$$

maps

$$[\xi_1] \cdot \left(\sum_{r=1}^{q+1} (r, q+1-r)\left[\xi_1^r|\bar{g}(q+1-r)\right]\right)$$
$$= \sum_{r=1}^{q+1} (r, q+1-r)\left[\xi_1|\xi_1^r|\bar{g}(q+1-r)\right]$$

to

$$\sum_{r=1}^{q+1} (r, q+1-r)\left[\xi_1|\xi_1^r|\bar{x}(q+1-r)\right] = d_1^1\left[\xi_1|\bar{x}(q+1)\right].$$

Consequently, $\delta(h_0') = h_0 I_1(\omega)$ according to the definition of δ and the product in $\mathrm{Ext}_{\mathcal{Q}_*}$.

We close this section with a proof of Theorem 2. To facilitate matters we set down some notation and isolate a lemma, which appears to be useful in computing certain Ext groups. First, following Liulevicius [6], [7], we identify $H_*(MO)$ with the extended comodule

$$\mathcal{Q}_* \otimes \mathbf{Z}_2[u_2, u_4, u_5, \ldots, u_\lambda, \ldots]$$

(as comodules), where degree $u_\lambda = \lambda \neq 2^s - 1$ and u_0 is the unit of the polynomial ring. If $U \in H^0(MO)$ is the Thom class, then its dual in $H_0(MO)$ is $1 \otimes u_0$.

LEMMA 7. *Suppose given a commutative ladder*

$$\begin{array}{ccccc}
C^0 & \xrightarrow{d^0} & C^1 & \xrightarrow{d^1} & C^2 \\
\downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} \\
C'^0 & \xrightarrow{d'^0} & C'^1 & \xrightarrow{d'^1} & C'^2
\end{array}$$

of K-modules (K a commutative ring with unit) in which $d^1 d^0 = 0$, $\operatorname{im} d'^0 = \ker d'^1$ *and* f_1 *and* f_2 *are isomorphisms. Then there is a short exact sequence*

$$0 \to \frac{\ker d'^0}{\ker d'^0 \cap \operatorname{im} f_0} \xrightarrow{\alpha} \frac{C'^0}{\operatorname{im} f_0} \xrightarrow{\beta} \frac{\ker d^1}{\operatorname{im} d^0} \to 0$$

where α *is induced by inclusion and* β *by* $f_1^{-1} d'^0$.

PROOF. Clearly $\ker d^1/\operatorname{im} d^0$ is isomorphic to $\operatorname{im} d'^0/d'^0(\operatorname{im} f_0)$ via f_1. But projection of $C'^0/\operatorname{im} f_0$ into this last group has kernel $(\ker d'^0 + \operatorname{im} f_0)/\operatorname{im} f_0$ which is isomorphic to $\ker d'^0/\ker d'^0 \cap \operatorname{im} f_0$.

In our application, $\operatorname{im} f_0 \supset \ker d'^0$ so that $\ker d^1/\operatorname{im} d^0 \simeq \operatorname{coker} f_0$ via β.

PROOF OF THEOREM 2. To apply Lemma 7 to this situation, for $\varepsilon = 0, 1, 2$ choose

$$C^\varepsilon = E_1^{\varepsilon, q+1}(MO\langle v_{q+1}\rangle), \qquad C'^\varepsilon = E_1^{\varepsilon, q+1}(MO),$$

f_ε to be the maps induced by $\bar{\rho}$: $MO\langle v_{q+1}\rangle \to MO$ and d^ε and d'^ε to be the appropriate differentials. Recalling that U is the generator of $H^0(MO)$, we have $\ker \bar{\rho}^* \subset \operatorname{im}(d_1^0)^*$ since $\ker \bar{\rho}^*$ is generated by

$$c\mathrm{Sq}^{q+1}U = (d_1^0)^*(\mathrm{Sq}^{q+1} \otimes U).$$

Thus

$$\operatorname{Ext}_{\mathscr{Q}_*}^{1,q+1}(\mathbf{Z}_2, H_*(MO\langle v_{q+1}\rangle)) \simeq \operatorname{coker} \bar{\rho}_* \simeq \mathbf{Z}_2.$$

A generator for $\operatorname{coker} \bar{\rho}_*$ is the class of $\xi_1^{q+1} \otimes u_0$. Now from considerations on the level of spaces (see [2, p. 172]) there is a map of spectra $\bar{\pi}$: $C(\bar{ja}) \to MO$ which extends $\bar{\rho}$ and induces isomorphisms on homology up through dimension $q + 1$. The element $w \in H_{q+1}(C(\bar{ja}))$, which corresponds to $\xi_1^{q+1} \otimes u_0$ via $\bar{\pi}_*$, maps to $\iota \in H_q(\underline{S}^q)$ via $Q(\bar{ja})_*$. Thus the element

$$\begin{aligned}(1 \otimes \bar{\rho}_*)\left(1 \otimes P(\bar{ja})_*\right)^{-1} d_1^0(w) &= (1 \otimes \bar{\pi}_*) d_1^0(w)\\ &= d_1^0(\bar{\pi}_*(w)) = d_1^0(\xi_1^{q+1} \otimes u_0)\end{aligned}$$

represents $(1 \otimes \bar{\rho}_*)_\#(I_1(\bar{ja}))$ in $\operatorname{im} d_1^0/d_0^1(\operatorname{im} \bar{\rho}_*)$ and the proof is complete.

3. **Proof of Theorem 3.** We have already shown that $I_1(\bar{j\omega}) = 0$ if $q + 1 = 2^k$ (Corollary 5) so that $I_2(\bar{j\omega})$ is defined in this case.

The next point we dispose of is the nontriviality of $I_2(\bar{j\omega})$. To that end, recall from Proposition 6 that

$$\delta\colon E_2^{1,2^{k+1}-1}(S^{2^{k+1}-2}) \to E_2^{2,2^{k+1}}(MO\langle v_{2^k}\rangle)$$

is 0 so that the indeterminacy of $(d_2)_{\bar{j\omega}}$ is simply $\operatorname{im} d_2^{0,2^{k+1}-1}$. Thus $I_2(\bar{j\omega})$ is an element of $E_2^{2,2^{k+1}}(MO\langle v_{2^k}\rangle)/\operatorname{im} d_2^{0,2^{k+1}-1}$. Now were $I_2(\bar{j\omega})$ to be 0, $\bar{j\omega}$ would have to have Adams filtration $\geqslant 3$. However Browder has shown that there is a stable secondary operation which detects $\bar{j\omega}$, so by a result in [4], the

Adams filtration of $\bar{j}\omega$ is 2. Thus $I_2(\bar{j}\omega) \neq 0$.

It remains for us to verify the formula (*) of Theorem 3. The outline for this part of the proof goes as follows. First we rewrite the map $\bar{j}\omega$ in a convenient form listing the necessary properties of this rewrite. These are summarized in Lemma 8 below. The various parts of the lemma are then applied to establish (*) and finally (in §4) Lemma 8 itself is verified.

LEMMA 8. *There is a morphism of spectra*

$$\alpha\colon \underline{H}_q \wedge MO\langle v_{q+1}\rangle \to MO\langle v_{q+1}\rangle$$

with the following properties:

(a) *the Thom invariant* $\bar{j}\omega$ *of* $(S^q \times S^q, \bar{q})$ *is equal to*

$$\alpha(a \wedge 1)(1 \wedge \bar{j}a)\colon \underline{S}^{2q} = \underline{S}^q \wedge \underline{S}^q \to MO\langle v_{q+1}\rangle,$$

(b) $\alpha(a \wedge 1)(1 \wedge i)\colon \underline{S}^q = \underline{S}^q \wedge \underline{S}^0 \to MO\langle v_{q+1}\rangle$ *is equal to* $\bar{j}a$,

(c) $[\alpha(a \wedge 1)]_*\colon H_*(\underline{S}^{2^k-1} \wedge MO\langle v_{2^k}\rangle) \to H_*(MO\langle v_{2^k}\rangle)$ *is* 0 *in dimensions* $\leqslant 2^{k+1} - 2$.

According to the definition of I_2, the formula (*) will follow if we show that

$$h_k \cdot I_1(\bar{j}a) \in \left(P(\bar{j}\omega)\right)_{\#}^{-1} d_2^0 \left(Q(\bar{j}\omega)\right)_{\#}^{-1}(\iota).$$

What we shall arrange is the location of an element

$$z \in \left(Q(\bar{j}\omega)\right)_{\#}^{-1}(\iota) \subset E_2^{0,2^{k+1}-1}\left(C(\bar{j}\omega)\right)$$

so that

$$\left(P(\bar{j}\omega)\right)_{\#}\left(h_k \cdot I_1(\bar{j}a)\right) = d_2^{0,2^{k+1}-1}(z).$$

We record some data derived from Lemma 8 to be used in finding the element z. From part (a) there is a map $A\colon C(1 \wedge \bar{j}a) \to C(\bar{j}\omega)$ so that

$$\begin{tikzcd}
\underline{S}^{2^k-1} \wedge MO\langle v_{2^k}\rangle \arrow[r, "P(1 \wedge \bar{j}a)"] \arrow[d, "\alpha(a \wedge 1)"'] & C(1 \wedge \bar{j}a) \arrow[d, "A"] \\
MO\langle v_{2^k}\rangle \arrow[r, "P(\bar{j}\omega)"] & C(\bar{j}\omega)
\end{tikzcd}$$

commutes. We note that $P(1 \wedge \bar{j}a)_*$ and $P(\bar{j}\omega)_*$ are isomorphisms on homology in dimensions $\leqslant 2^{k+1} - 2$; consequently A_*, as well as $[\alpha(a \wedge 1)]_*$, is 0 in dimensions $\leqslant 2^{k+1} - 2$. Now let $w \in H_{2^k}(C(\bar{j}a))$ be the element defined in the proof of Theorem 2, $\sigma w \in H_{2^{k+1}-1}(C(1 \wedge \bar{j}a))$ its $(2^k - 1)$-fold suspension. Set $z = A_*(\sigma w)$. It is easy to see that not only is $Q(\bar{j}\omega)_*(z) = \iota \in$

$H_{2^{k+1}-2}(\underline{S}^{2^{k+1}-2})$ but also that $z \in E_2^{0,2^{k+1}-1}(C(\bar{j\omega})) = \ker d_1'^{0,2^{k+1}-1}$. Equally easy to see is that

$$P\big(1 \wedge \bar{j}a\big)_*\big(\xi_1^{2^k} \otimes \sigma(u)\big) = d_1^0(\sigma w)$$

in $E_1^{1,2^{k+1}}(C(1 \wedge \bar{j}a))$, where $u \in H_0(MO\langle v_{2^k}\rangle)$ corresponds to $1 \otimes u_0$ via $\bar{\rho}_*$ and σ, as above, denotes $(2^k - 1)$-fold suspension. Furthermore

$$\big[\alpha(a \wedge 1)\big]_*\big(\xi_1^{2^k} \otimes \sigma u\big) = 0 \quad \text{and} \quad A_*\big(d_1^0\sigma w\big) = 0.$$

Consequently the functional differentials $(d_1^1)_{\alpha(a\wedge 1)}$ and $(d_1^1)_A$ are defined on $\xi_1^{2^k} \otimes \sigma u$ and $d_1^0(\sigma w)$ respectively and, by naturality, their values are related by

$$(\dagger) \qquad P\big(\bar{j\omega}\big)_\#\big(d_1^1\big)_{\alpha(a\wedge 1)}\big(\xi_1^{2^k} \otimes \sigma u\big) = \big(d_1^1\big)_A\big(d_1^0(\sigma w)\big)$$

in the quotient of $E_1^{2,2^{k+1}}(C(\bar{j\omega}))$ by the total indeterminacy. The key point here is this. Since $\alpha(a \wedge 1)$ and A induce the zero homomorphism on homology in dimensions $\leqslant 2^{k+1} - 2$, they likewise induce the zero homomorphism on $E_1^{2,2^{k+1}}$. Consequently $(d_1^1)_{\alpha(a\wedge 1)}$ and $(d_1^1)_A$ take their values in $E_1^{2,2^{k+1}}(\underline{MO}\langle v_{2^k}\rangle)/\operatorname{im} d_1^1$ and $E_1^{2,2^{k+1}}(C(\bar{j\omega}))/\operatorname{im} d_1^1$ respectively, and so in $(\dagger)$ $P(\bar{j\omega})_\#$ is, after restriction, the homomorphism on $E_2 = \operatorname{Ext}$ induced by $P(\bar{j\omega})$.

On the one hand, now, it is a routine calculation, using part (b) of Lemma 8, to show that

$$\big(d_1^1\big)_{\alpha(a\wedge 1)}\big(\xi_1^{2^k} \otimes \sigma u\big) = \big\{\xi_1^{2^k} \otimes \xi_1^{2^k} \otimes \sigma u\big\}$$

which of course is $h_k \cdot I_1(\bar{j}a)$. On the other hand, by the second Peterson-Stein formula for Adams differentials [5],

$$\big(d_1^1\big)_A\big(d_1^0(\sigma w)\big) = d_2^0\big(A_*(\sigma w)\big) = d_2^0(z)$$

in $E_1^{2,2^{k+1}}(C(\bar{j\omega}))/\operatorname{im} d_1^1$. Thus $I_2(\bar{j\omega}) = \{h_k \cdot I_1(\bar{j}a)\}$ in

$$\operatorname{Ext}_{\mathcal{Q}_*}^{2,2^{k+1}}\big(\mathbf{Z}_2, H_*(MO\langle v_{2^k}\rangle)\big)/\operatorname{im} d_2^{0,2^{k+1}-1}$$

and the proof of Theorem 3 is complete subject to the verification of Lemma 8.

4. **Proof of Lemma 8.** We begin with the construction of α. For $n \gg 2q$, let

$$\alpha_n\colon H_q \times BO_n\langle v_{q+1}\rangle \to BO_n\langle v_{q+1}\rangle$$

be the principal fiber space structure map. Since $\operatorname{proj}_2(1 \times \pi) = \pi\alpha_n$, the bundles $\alpha_n^*(\bar{\gamma}_n)$ and $\alpha_n^*(\bar{\gamma}_n \oplus \theta^1)$ are isomorphic to $(H_q \times \mathbf{R}^0) \times \bar{\gamma}_n$ and $(H_q \times \mathbf{R}^0) \times (\bar{\gamma}_n \oplus \theta^1)$ respectively. Thus, by passing to Thom complexes, we have maps $\bar{\alpha}_n\colon (H_q)_+ \wedge MO(n) \to MO(n)$ and commutative diagrams

$$\begin{tikzcd}
(H_q)_+ \wedge MO(n) \wedge S^1 \arrow[r, "\bar{\alpha}_n \wedge S^1"] \arrow[d] & MO(n) \wedge S^1 \arrow[d] \\
(H_q)_+ \wedge MO(n+1) \arrow[r, "\bar{\alpha}_{n+1}"] & MO(n+1)
\end{tikzcd}$$

These diagrams, via the handicrafted smash products $X \wedge_{\phi,\mathbf{Z}} Y$ [1, p. 49], give a map of spectra

$$\bar{\alpha}: (\underline{H}_q)_+ \wedge MO\langle v_{q+1}\rangle \to MO\langle v_{q+1}\rangle.$$

Additionally, there is an isomorphism of spectra $\underline{X} \vee \underline{S} \simeq \underline{X}_+$, where $\underline{X}$ and $\underline{X}_+$ are the suspension spectra of the pointed spaces (X, x_0), $x_0 \in X$ and $(X_+, +)$, $+ \notin X$. Thus $\bar{\alpha}$ may be regarded as being defined on $(\underline{H}_q \vee \underline{S}) \wedge MO\langle v_{q+1}\rangle$. The map α, now, is the composite of $\bar{\alpha}$ with the inclusion

$$\underline{H}_q \wedge MO\langle v_{q+1}\rangle \to (\underline{H}_q \vee S^0) \wedge MO\langle v_{q+1}\rangle.$$

Property (a) now follows routinely from the definition of the Thom invariant and the fact that $\bar{q}$: $S^q \times S^q \to BO_n\langle v_{q+1}\rangle$ is equal to $\alpha_n(a \wedge 1) \cdot (1 \wedge \bar{j}a)$. Property (b) follows from commutative diagrams on the level of Thom complexes arising from commutative diagrams

$$\begin{tikzcd}
S^q \times \mathrm{pt} \arrow[r] \arrow[d] & H_q \arrow[d, "j"] \\
S^q \times BO_n\langle v_{q+1}\rangle \arrow[r] & BO_n\langle v_{q+1}\rangle
\end{tikzcd}$$

Turning now to part (c), we note that α_* factors through $\bar{\alpha}_{n_*}$ which, by the Thom isomorphism is

$$\alpha_{n_*}: H_*(H_q \times BO_n\langle v_{q+1}\rangle) \to H_*(BO_n\langle v_{q+1}\rangle).$$

Thus it suffices to prove

PROPOSITION 9. *Let* $\bar{g} \in H_{2^k-1}(H_{2^k-1})$ *be the generator. Then*

$$\alpha_{n_*}(\bar{g} \otimes \_): H_m(BO_n\langle v_{2^k}\rangle) \to H_{m+2^k-1}(BO_n\langle v_{2^k}\rangle)$$

is 0 *for* $m \leqslant 2^k - 1$.

PROOF. It suffices to show that, on cohomology, $\alpha_n^*(c)$ does not contain a term of the form $g \otimes c'$ with $c' \in H^{l-2^k+1}(BO\langle v_{2^k}\rangle)$ when

$$c \in H^l(BO_n\langle v_{2^k}\rangle) \quad \text{and} \quad l \leqslant 2^{k+1} - 2.$$

To show this we appeal to Theorem 7.4 in the work of J.-P. Meyer on functional operations and principal coactions [9]. Since $\{\mathrm{Sq}^R g\}$ is a transgressive $\mathbf{Z}_2$-base for $H_*(H_{2^k-1})$ in dimensions $\leqslant 2^{k+1} - 2$ and $\tau(\mathrm{Sq}^R g) = \mathrm{Sq}^R v_{2^k}$, it suffices to show

THEOREM 10. *There are no relations of the form*

$$C \cdot v_{2^k} + \sum_{R \neq (0,0,\ldots)} C_R \cdot \mathrm{Sq}^R v_{2^k} = 0$$

in $H^l(BO(n))$ *for* $l \leqslant 2^{k+1} - 1$. *Here* C *and* C_R *are elements of* $H^*BO(n)$ *subject only to the restrictions that* $C \neq 0$, $\dim \mathrm{Sq}^R > 0$ *and*

$$\dim C_R + \dim \mathrm{Sq}^R = l = \dim C + 2^k.$$

Before giving the proof, we need some tèchnical data concerning the representation of $H^*(BO(n))$ as the full symmetric algebra of $\mathbf{Z}_2[t_1, \ldots, t_n]$.

Let $\pi(m) = \{\phi | \phi$ is a partition of $m\}$. If $\phi = (c_1, \ldots, c_r) \in \pi(m)$, we define the length of ϕ, $l(\phi)$, to be r and the degree of ϕ, $n(\phi)$, to be $c_1 + \cdots + c_r$. Corresponding to the partition $\phi = (c_1, \ldots, c_r)$ let $s(\phi)$ be the "smallest" symmetric polynomial of $\mathbf{Z}_2[t_1, \ldots, t_n]$ containing the monomial $t_1^{c_1} \cdots t_r^{c_r}$. As is well known the set $\{s(\phi)\}$ forms a $\mathbf{Z}_2$-base for $H^*(BO(n))$. In each dimension m we order the base $\{s(\phi) | \phi \in \pi(m)\}$ by the order induced from the following order of $\pi(m)$. Let $\geqslant$ be any total order of $\pi(m)$ which is consistent with

(i) $\omega' \geqslant \omega$ if ω' refines ω [11, p. 183],

(ii) if ω and ω' are incompatible with respect to refinement, then $\omega' > \omega$ if $l(\omega') > l(\omega)$.

Relative to the refinement notion, we note the following two points. First, if 2ω denotes the partition obtained from ω by termwise multiplication by 2, then $2\omega'$ refines 2ω if and only if ω' refines ω. Second, if $s(\omega)$ occurs in $s(\phi) \cdot s(1^{2^k})$ then $\phi \cdot 1^{2^k}$ refines ω. (Of course n, the number of indeterminates, $\geqslant l(\phi) + 2^k$.)

Now we express v_{2^k}, the 2^k Wu class of $BO(n)$, in terms of the base $\{s(\phi)\}$.

PROPOSITION 11. *Let*

$$\mathbb{S} = \left\{\rho \in \pi(2^k) | \rho = \left(1^{a_1}, (2^2 - 1)^{a_2}, \ldots\right)\right\}.$$

Then $v_{2^k} = \sum_{\rho \in \mathbb{S}} s(\rho)$.

PROOF. Since $c\mathrm{Sq}^{2^k} = \sum_{\rho \in \mathbb{S}} \mathrm{Sq}^{R(\rho)}$ [10], $v_{2^k} U = \sum \mathrm{Sq}^{R(\rho)} U$, where $U \in H^n(MO(n))$ is the Thom class and $R(\rho) = (a_1, a_2, \ldots)$. Thus the proof of Thom's theorem (II. 10) [13, p. 42] implies $v_{2^k} = \sum_{\rho \in \mathbb{S}} s(\rho)$.

COROLLARY 12. *If* $s(\phi)$ *is the largest basis element occurring in* $C \in H^p(BO(n))$ *with* $p \leq 2^k - 1$, *then* $s(\phi \cdot 1^{2^k})$ *is the largest basis element occurring in* $C \cdot v_{2^k}$.

We need one final observation concerning how certain basis elements multiply.

LEMMA 13. *If* ϕ, ψ *and* ρ' *are partitions subject to the following conditions*:
(i) $n(\phi) \leq 2^k - 1$ *and* $n(\psi) < n(\phi)$;
(ii) $n(\rho') > 2^k$ *and* $l(\rho') \leq 2^k$;
(iii) $n(\psi) + n(\rho') = n(\phi) + 2^k$.
Then $s(\psi)s(\rho')$ *does not contain a basis element* $s(\sigma)$ *for which* σ *refines* $\phi \cdot 1^{2^k}$.

The proof of Lemma 13 is a bookkeeping exercise involving the length and degree conditions and is left to the reader.

PROOF OF THEOREM 10. Set

$$\exp = Cv_{2^k} + \sum_{R \neq (0,0,\dots)} C_R \mathrm{Sq}^R v_{2^k} \in H^l(BO(n)),$$

$l \leq 2^{k+1} - 1$. Corresponding to this element exp, we construct a group G and a homomorphism Φ: $H^l(BO(n)) \to G$ so that $\Phi(\exp) \neq 0$.

Let $s(\phi)$ be the largest term occurring in C with $\phi = (c_1, \dots, c_r)$. Following Stong [12], let

$$\mathbf{R}P_{2(\phi \cdot 1^{2^k})} = \left(\mathop{\times}_{i=1}^{r} \mathbf{R}P_{2c_i} \right) \times \mathop{\times}_{j=1}^{2^k} (\mathbf{R}P_2)_j$$

and let Φ be the projection of Stong's map Ω into $H^*(\mathbf{R}P_{2(\phi \cdot 1^{2^k})})$. Relative to $\{s(\omega)\}$, exp is equal to $s(\phi \cdot 1^{2^k})$ plus a sum of terms $s(\sigma)$ for which σ does not refine $\phi \cdot 1^{2^k}$. But Stong has shown that for such σ, $\Phi(s(\sigma)) = 0$, while $\Phi s(\phi \cdot 1^{2^k})$ equals the top dimensional cohomology class of $\mathbf{R}P_{2(\phi \cdot 1^{2^k})}$. Thus $\Phi(\exp) \neq 0$.

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Department of Mathematics, New Mexico State University, Las Cruces, New Mexico 88003