HAUSDORFF CONTENT AND RATIONAL APPROXIMATION IN FRACTIONAL LIPSCHITZ NORMS

BY

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ABSTRACT. For $0 < \alpha < 1$, we characterise those compact sets $X$ in the plane with the property that each function in the class $\text{lip}(\alpha, X)$ that is analytic at all interior points of $X$ is the limit in $\text{Lip}(\alpha, X)$ norm of a sequence of rational functions. The characterisation is in terms of Hausdorff content.

1. If $E$ is a closed subset of the complex plane $\mathbb{C}$, and $f$ is a bounded complex-valued function on $E$ we define the modulus of continuity $\omega_f$ by setting

$$\omega_f(r) = \sup\{|f(x) - f(y)| : x, y \in E, |x - y| < r\}$$

whenever $r > 0$. Thus $\omega_f$ is a nondecreasing function, $\omega_f(0) = 0$, and $f$ is uniformly continuous on $E$ if and only if $\omega_f$ is continuous at zero. For $0 < \alpha < 1$ we define

$$\|f\|_{\alpha,E} = \sup\{r^{-\alpha}\omega_f(r) : r > 0\},$$

$$\text{Lip}(\alpha, E) = \{f : \|f\|_{\alpha,E} < \infty\},$$

$$\text{lip}(\alpha, E) = \{f \in \text{Lip}(\alpha, E) : r^{-\alpha}\omega_f(r) \to 0 \text{ as } r \downarrow 0\}.$$

When given the norm

$$\|f\|_{\alpha,E} = \|f\|_{\alpha,E} + \|f\|_{u,E}$$

(where $\|f\|_{u,E}$ is the sup norm), $\text{Lip}(\alpha, E)$ becomes a Banach algebra, and $\text{lip}(\alpha, E)$ is a closed point-separating subalgebra [9]. This paper concerns the question of approximation in $\text{Lip}(\alpha, X)$, for compact sets $X$, by rational functions with poles off $X$.

Before stating the main result, we must define the Hausdorff contents $M^B$ and $M_*^B$. A measure function is a nonnegative increasing function defined on

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If \( h \) is a measure function and \( F \subseteq \mathbb{C} \), then the Hausdorff content \( M_h(F) \) is the infimum of all sums

\[
\sum_{S \in \mathcal{S}} h(\text{diam } S),
\]

where \( \mathcal{S} \) runs over all countable coverings of \( F \) by closed (or open) balls. In case \( h(r) = r^\beta \) for some \( \beta > 0 \), we write \( M_h = M^\beta \). The set function \( M^\beta_* \) is defined by setting

\[
M^\beta_*(F) = \sup \{ M_h(F) : h \text{ is a measure function, } h(r) < r^\beta, r^{-\beta}h(r) \rightarrow 0 \text{ as } r \downarrow 0 \}. \]

**Theorem.** Let \( X \) be a compact subset of \( \mathbb{C} \), and let \( 0 < \alpha < 1 \). In order that every function in \( \text{lip}(\alpha, X) \) which is analytic on the interior of \( X \) be the limit in \( \text{Lip}(\alpha, X) \) norm of a sequence of rational functions, it is necessary and sufficient that there exist a constant \( \mu > 0 \) such that

\[
M^{1+\alpha}(D \setminus X) > \mu M^{1+\alpha}_*(D \setminus \text{int } X)
\]

whenever \( D \) is an open disc.

It is worth noting that the condition for approximation is purely metric, in contrast to the conditions which have been obtained for uniform approximation [12].

The necessity of the condition is proved in §§2–8. We introduce capacities in §2 and show that if two spaces have the same closure then the corresponding capacities coincide. In §§3–7 we apply a generalisation of Melnikov’s Theorem [10] in order to relate the capacities corresponding to rational functions and \( \text{lip } \alpha \) analytic functions to the contents \( M^{1+\alpha}_1 \) and \( M^{1+\alpha}_* \). The proof of sufficiency in §§10–15 is modelled on the Vitushkin approximation scheme [12], [6], [8] as modified by Davie [3]. We make heavy use of the metric character of the capacities. We give some applications in §§16–23.

Throughout the paper, \( \alpha \) is fixed, \( 0 < \alpha < 1 \); \( \mathbb{Z} \) denotes the set of integers, and \( \mathbb{Z}^+ = \mathbb{Z} \cap \mathbb{R}^+ \); \( \Sigma \) is the Riemann sphere; \( \mathcal{O} \) is the space of complex-valued \( C^\infty \) functions with compact support. If \( f \) is continuous on \( \mathbb{C} \) and \( \varphi \in \mathcal{O} \), we define

\[
T_\varphi f(z) = \frac{1}{\pi} \int \frac{f(\xi) - f(z)}{z - \xi} \frac{\partial \varphi}{\partial \xi} \, dm(\xi),
\]

where \( m \) denotes Lebesgue measure on the plane. For an exposition of the properties of this “\( T_\varphi \)-operator”, see [6]. A set \( B \) of continuous functions on \( \mathbb{C} \) is said to be \( T\text{-invariant} \) if \( T_\varphi f \in B \) whenever \( f \in B \) and \( \varphi \in \mathcal{O} \). The operator \( T_\varphi \) is bounded with respect to the \( \text{Lip}(\alpha, \mathbb{C}) \) norm, for each \( \varphi \in \mathcal{O} \).
In fact
\[ \|T\varphi f\|_{a,C} < K \eta(d) \{\|\varphi\|_u + d\|\nabla \varphi\|_u \}, \]
where \( K \) is a constant depending only on \( \alpha \),
\[ d = \text{diam spt } \varphi, \quad \eta(d) = \sup \{s^{-\alpha} \omega_f(s) : 0 < s < d \}. \]

The symbol \( X \) always stands for a compact subset of \( \mathbb{C} \), \( \mathcal{R}(X) \) is the subspace of \( \text{Lip}(\alpha, \mathbb{C}) \) consisting of those functions which agree on some neighbourhood of \( X \) with a rational function, and \( \mathcal{R}_0(X) \) is the space of functions in \( \text{Lip}(\alpha, \mathbb{C}) \) which are analytic on a neighbourhood of \( X \). If \( B \) is any subspace of \( \text{Lip}(\alpha, X) \), then the closure of \( B \) with respect to the norm \( \|\cdot\|_{a,X} \) is denoted \( [B]_{a,X} \), or just \( [B]_a \). If \( B \) contains the constants, then this coincides with the closure with respect to the norm \( \|\cdot\|_{a,X} \). For any \( X \),
\[ [\mathcal{R}(X)]_{a,X} = [\mathcal{R}_0(X)]_{a,X}. \]

This assertion is the \( \alpha \) version of Runge’s Theorem, and the classical proof of Runge’s Theorem is easily modified to prove it.

As a technical convenience, we assume that the diameter of \( X \) does not exceed \( \frac{1}{4} \).

I am grateful to T. Gamelin and J. Garnett for valuable conversations. The Decay Lemma of §12 is the work of Garnett. I am grateful to the referee for suggesting the argument of §4.

2. We follow established custom in denoting the algebra of all continuous complex-valued functions on \( X \) by \( C(X) \) and denoting the subalgebra of functions analytic on \( \text{int}(X) \) by \( A(X) \). We further define
\[ A^a(X) = \text{Lip}(\alpha, X) \cap A(X), \quad A_a(X) = \text{lip}(\alpha, X) \cap A(X), \]
so that \( A^a \) and \( A_a \) are closed subalgebras of \( \text{Lip} \alpha \). In view of the extension theorem [11, Chapter VII], a subspace \( V \subset A^a(X) \) may be regarded as a subspace of \( \text{Lip}(\alpha, \mathbb{C}) \) (we may identify \( V \) with the set of functions in \( \text{Lip}(\alpha, \mathbb{C}) \) whose restrictions to \( X \) lie in \( V \)), so \( T \)-invariance makes sense for such subspaces. To each \( T \)-invariant subspace \( V \) of \( A^a(X) \) we associate a capacity \( \gamma(V, \circ) \), a nonnegative increasing function defined on the family \( \{D\} \) of open discs: we say a function \( f \in V \) is \( D \)-admissible if \( f \) is analytic off a compact subset of \( D \), \( f(\infty) = 0 \), and \( \|f\|_{a,C} < 1 \); we set
\[ \gamma(V, D) = \sup \{|f'(\infty)| : f \in V, f \text{ is } D \text{-admissible}\}. \]

**Lemma.** Let \( V \) and \( W \) be \( T \)-invariant subspaces of \( A^a(X) \). Suppose \( V \) and \( W \) have the same closure in \( \text{Lip}(\alpha, X) \) norm. Then \( \gamma(V, D) = \gamma(W, D) \) for every open disc \( D \).

**Proof.** It suffices to show that
\( \gamma(V, D) = \gamma([V]_a, D) \).

It is clear that
\( \gamma(V, D) < \gamma([V]_a, D) \).

To prove the opposite inequality, let \( D \) be a fixed open disc and let \( \varepsilon > 0 \) be given. Choose \( f \in [V]_a \) such that \( f \) is \( D \)-admissible and
\[ |f'(\infty)| > \gamma([V]_a, D) - \varepsilon. \]

Choose a sequence \( \{ f_n \} \) of elements of \( V \) such that \( \| f_n - f \|_{a,x} \to 0 \). For each \( n \) the extension theorem ensures the existence of a function
\[ g_n \in \text{Lip}(\alpha, C) \]
such that \( g_n = f_n - f \) on \( X \) and \( \| g_n \|_{a,C} < 4 \| f_n - f \|_{a,x} \). Let \( h_n = f + g_n \). Then \( h_n \in V \) and \( \| h_n - f \|_{a,C} \to 0 \) as \( n \to + \infty \). Choose \( \varphi \in \mathbb{D} \) such that \( \text{spt} \varphi \subset D \) and \( \varphi \equiv 1 \) on a neighbourhood of the set of singularities of \( f \). Then \( T_\varphi f = f, T_\varphi h_n \in V, \) and
\[
\| T_\varphi h_n - f \|_{a,C} = \| T_\varphi (h_n - f) \|_{a,C} < K \| h_n - f \|_{a,D} \left\{ \| \varphi \|_a + \text{diam } D \| \nabla \varphi \|_a \right\},
\]
by §1. Thus \( \| T_\varphi h_n - f \|_{a,C} \to 0 \), and hence \( (T_\varphi h_n)'(\infty) \to f'(\infty) \), so that
\[
\gamma(V, D) > \gamma([V]_a, D) - \varepsilon.
\]

Since this holds for each \( \varepsilon > 0 \), we conclude that \((*)\) holds.

We do not know whether or not the converse to this lemma is true in general.

3. In order to apply Lemma 2 to rational approximation we have to describe the capacities \( \gamma(V, \cdot) \) in the cases \( V = \mathcal{R}(X) \) and \( V = A_a(X) \). **Melnikov's Theorem** provides the key. It relates certain capacities to the Hausdorff contents \( M_h \). Before stating it we define a special class of "modulus of continuity functions".

Consider a concave increasing function \( \omega(r) \), defined for \( r > 0 \) and constant for \( r > 1 \), with \( \omega(0) = 0 \), and such that
\begin{enumerate}
    \item \( \omega(r) \) exists for \( r > 0 \);
    \item there exists a constant \( L_1 > 0 \) such that \( \omega(r) < L_1 r \omega'(r) \) for \( 0 < r < \frac{1}{2} \);
    \item there exists a constant \( L_2 > 1 \) such that \( r \omega'(r) < (L_2 - 1) \omega(r)/L_2 \) for \( 0 < r < \frac{1}{2} \).
\end{enumerate}

Such a \( \omega \) we call a **modulated function**. To each modulated function is associated a measure function \( h \), defined by \( h(r) = r \omega(r) \), and a capacity \( \tau(\omega, \cdot) \) defined on arbitrary bounded sets \( E \subset \mathbb{C} \) by
\[
\tau(\omega, E) = \sup \{ \| f'(\infty) \| : f \text{ is analytic on a neighbourhood of } \Sigma \setminus E, f(\infty) = 0, \omega_f < \omega \}.
\]
Here \( \omega_f \) refers to the modulus of continuity of \( f \) as a function on \( C \).

**Melnikov's Theorem.** Let \( \omega \) be a modulated function. Then there is a constant \( K(\omega) \) such that

\[
K^{-1} M_h(E) < \tau(\omega, E) < KM_h(E)
\]

whenever \( E \) is compact or \( E \) is open and bounded. \( K(\omega) \) may be taken to be \( K_0(L_1 + L_2) \), where \( K_0 \) is a certain universal constant.

Actually, this is a slight extension of Melnikov's result. He proved it in case \( \omega(r) = r^\beta \) for some \( \beta, 0 < \beta < 1 \), and in that case \( K(\omega) \) may be taken to be \( K_0 \beta^{-1}(1 - \beta)^{-1} \). His proof \[10\] carries over with trivial changes. We omit the details.

An example of a modulated function other than the various \( r^\beta, 0 < \beta < 1 \), is obtained by fixing \( 0 < \delta < 1 \) and setting

\[
\omega(r) = \begin{cases} 
  r^\delta (\delta^{-1} - \log 2r), & 0 < r < \frac{1}{2}, \\
  \delta^{-1/2}, & \frac{1}{2} \leq r < \infty.
\end{cases}
\]

**4. Lemma.** Let \( \omega(r) \) be a nonnegative function such that \( \omega(r) < r^\alpha \) and \( r^{-\alpha} \omega(r) \rightarrow 0 \). Let \( \varepsilon > 0 \) and \( \beta > \alpha \) be given. Then there exists a modulated function \( \omega_1(r) \) with the following properties:

1. \( (1 - \varepsilon) \omega(r) < \omega_1(r) < \omega(r) \) for \( 0 < r < \frac{1}{2} \),
2. \( \alpha \omega_1(r) < r \omega_1(r) < \beta \omega_1(r) \) for \( 0 < r < \frac{1}{2} \),
3. \( r^{-\alpha} \omega_1(r) \rightarrow 0 \) as \( r \downarrow 0 \).

**Proof.** In proving this, we may suppose that \( \beta < \alpha(1 - \varepsilon)^{-1} \). Choose a monotonically-decreasing sequence of piecewise smooth functions \( \psi_j \) such that

4. \( \beta (1 - \varepsilon) \omega(r)/\alpha \leq \psi_j(r) < r^\alpha \),
5. \( \alpha \psi_j(r) < r \psi_j(r) < \beta \psi_j(r) \),
6. \( \psi_j(r) < r^\alpha/j \) in a neighbourhood of the origin.

Such \( \psi_j \)'s may be constructed as follows: Choose \( \delta_j > \alpha \), put

\[
\varphi_j(r) = \max\{r^\delta/r^\delta, r^\delta\}, \quad \text{and}
\]

\[
\psi_j(r) = \min\{\alpha \int_0^r \frac{\varphi_j(s)}{s} \, ds, \psi_{j-1}(r)\}.
\]

If \( \delta_j \) is sufficiently close to \( \alpha \), properties (4), (5) and (6) are satisfied, as is seen by a routine calculation.

Set \( \varphi(r) = \lim \psi_j(r) \). It follows easily that

\[
\omega_1(r) = \alpha \int_0^r \frac{\varphi(s)}{s} \, ds
\]
satisfies properties (1), (2), and (3). Verification is again routine. This completes the proof.

Fix $\beta = (1 + \alpha)/2$. For each $f \in \text{lip}(\alpha, \mathbb{C})$ with $\|f\|_{\alpha} < 1$, and each $\varepsilon > 0$, choose a modulated function $\omega_1(r)$ such that

\begin{align*}
(1 - \varepsilon)\omega_1(r) &< \omega_1(r) < r^\alpha, \\
\alpha \omega_1(r) &< r \omega_1(r) < \beta \omega_1(r), \\
r^{-\alpha} \omega_1(r) &\to 0 \text{ as } r \downarrow 0.
\end{align*}

Let $\mathcal{F}_\alpha$ denote the family of all functions $\omega_1$ obtained in this way. Clearly, we may apply Melnikov’s Theorem to all $\omega_1 \in \mathcal{F}_\alpha$ at once, using the same constant $K$.

5. Corollary. Let $X \subset \mathbb{C}$ be compact, $V = \mathcal{A}(X)$. Then for all open discs $D$

\begin{equation*}
K^{-1}_\gamma(V, D) \leq M^{1+\alpha}(D \setminus X) \leq K_\gamma(V, D),
\end{equation*}

where $K$ depends only on $\alpha$.

Proof. Choose a sequence of open sets $\{U_n\}_n \uparrow X$ such that each set $\text{bdy}(U_n)$ is a finite union of smooth curves. Then

\begin{equation*}
M^{1+\alpha}(D \setminus X) = \lim_{n \uparrow \infty} M^{1+\alpha}(D \setminus U_n).
\end{equation*}

Next, for $n = 1, 2, 3, \ldots$, we have

\begin{equation*}
A^\alpha(X_n) \subset V \subset \bigcup_{m = 1}^{\infty} A^\alpha(X_m),
\end{equation*}

where $X_n = \text{clos}(U_n)$. Hence for each open disc $D$,

\begin{equation*}
\gamma(A^\alpha(X_n), D) \leq \gamma(V, D) \leq \lim_{m \uparrow \infty} \gamma(A^\alpha(X_m), D).
\end{equation*}

Applying Melnikov’s Theorem with $\omega(r) = r^\alpha$ and $E = D \setminus X_n$ (so that $\tau(\omega, E) = \gamma(A^\alpha(X_n), D)$), we obtain

\begin{equation*}
K^{-1}_\gamma(A^\alpha(X_n), D) \leq M^{1+\alpha}(D \setminus X_n) \leq K_\gamma(A^\alpha(X_n), D),
\end{equation*}

for $n = 1, 2, 3, \ldots$, where $K$ depends only on $\alpha$. Taking limits we get the desired result.

6. In the definition of $M^{1+\alpha}$ it suffices to consider those $h$ of the form $r\omega(r)$ for $\omega \in \mathcal{F}_\alpha$.

7. Corollary. Let $W = A^\alpha(X)$. Then for all open discs $D$,

\begin{equation*}
K^{-1}_\gamma(W, D) \leq M^{1+\alpha}(D \setminus \text{int } X) \leq K_\gamma(W, D),
\end{equation*}

where $K$ depends only on $\alpha$.

Proof. Let $f \in W$ be $D$-admissible, and let $\varepsilon > 0$ be given. Then there exists $\omega \in \mathcal{F}_\alpha$ such that $(1 - \varepsilon)\omega \leq \omega$. Thus
(1 - \varepsilon) |f'(\infty)| < \tau(w, D \setminus \text{int } X).

If h(r) = r\omega(r), then Melnikov’s Theorem yields
\[ \tau(\omega, D \setminus \text{int } X) \leq K(\omega)M_{h}(D \setminus \text{int } X). \]
Thus
\[ (1 - \varepsilon)\gamma(W, D) < KM_{\ast}^{1+\alpha}(D \setminus \text{int } X), \]
where \( K = \sup\{K_{0}(L_{1} + L_{2}): \omega \in \mathcal{F}_{\alpha}\} \) depends only on \( \alpha \). This proves the first inequality.

For the second, fix \( \omega \in \mathcal{F}_{\alpha} \), and let \( h(r) = r\omega(r) \). Let \( f \in C(\Sigma) \) be analytic off \( (D \setminus \text{int } X) \), with \( \omega_{\gamma} < \omega \), \( f(\infty) = 0 \). Then \( f \in W \) and \( f \) is \( D \)-admissible. Hence \( |f'(\infty)| < \gamma(W, D) \). Thus \( \tau(\omega, D \setminus \text{int } X) < \gamma(W, D) \). By Melnikov’s Theorem
\[ K(\omega)^{-1}M_{h}(D \setminus \text{int } X) \leq \gamma(W, D). \]
Since this holds for every \( \omega \in \mathcal{F}_{\alpha} \), we conclude that
\[ K^{-1}M_{\ast}^{1+\alpha}(D \setminus \text{int } X) < \gamma(W, D), \]
with \( K \) as above.

8. Combining the results of §§1, 2, 5, and 7, we deduce the necessity of the condition of the theorem. In fact, if \([\mathbb{R}]_{\alpha} = A_{\alpha}(X)\), then
\[ M^{1+\alpha}(D \setminus X) \geq KM_{\ast}^{1+\alpha}(D \setminus \text{int } X), \]
for every open disc \( D \), where \( K > 0 \) is a constant which depends only on \( \alpha \).

9. Remark. One might wonder whether it is always possible, given a modulated function \( \omega \), to find functions \( f \in A(X) \) such that \( \omega_{\gamma} < \omega \) but \( \omega_{\gamma}^{-1}\omega(r) \to 0 \) as \( r \to 0 \). Putting it another way, if \( \omega_{1}(r)\omega_{2}(r)^{-1} \to 0 \) as \( r \to 0 \), are there any functions \( f \) in \( A(X) \) such that \( \omega_{\gamma} < \omega_{2} \) but \( \omega_{\gamma} \neq o(\omega_{1}) \)? The answer is yes. This follows from some results of Dolženko [4].

10. The first step towards proving the sufficiency of the approximation condition is a lemma which gives an estimate for the uniform norm in terms of the Lip \( \alpha \) norm.

**Lemma.** Suppose \( E \subset \mathbb{C} \) is bounded, \( f \) is analytic on \( \Sigma \setminus E \), \( f(\infty) = 0 \), and \( f \in \text{Lip}(\alpha, \mathbb{C}) \). Then
\[ \|f\|_{u,C} < 2^{1+\alpha}(\text{diam } E)^{\alpha}\|f\|_{\alpha,C}. \]

**Proof.** There is a circle \( C \) of radius \( \text{diam } E \) which encloses \( E \). Since \( f(\infty) = 0 \), then \( \int f \, d\theta = 0 \). Hence, if \( f = u + iv \), then \( \int_{C} u \, d\theta = \int_{C} v \, d\theta = 0 \). Thus \( u \) and \( v \) each have a zero on \( C \). Thus for \( x \) inside \( S \),
\[ |u(x)| < (2 \, \text{diam } E)^{\alpha}\|f\|_{\alpha}, \quad |v(x)| < (2 \, \text{diam } E)^{\alpha}\|f\|_{\alpha}, \]
hence
and the result follows by the maximum principle.

The above estimate is somewhat crude, in that it depends only on the
diameter of $E$. A more refined version is obtain in §14.

11. Now fix $X$ compact in $C$ and abbreviate $\mathcal{R} = \mathcal{R}(X)$, $A = A_\alpha(X)$, $\gamma(D) = \gamma(\mathcal{R}, D)$, $\gamma_A(D) = \gamma(A, D)$. Let $c(D)$ denote the centre of the disc $D$, and let $\tau D$ denote the disc with centre $c(D)$ and radius equal to $\tau$ times the radius of $D$. For any function $f$ which is analytic on a neighbourhood of $\infty$ we may write

$$f(z) = a_0 + \frac{a_1}{z - c(D)} + \frac{a_2}{(z - c(D))^2} + \ldots$$

for large $z$. Here $a_0 = f(\infty)$, $a_1 = f'(\infty)$, and we define $\beta(f, D) = a_2$. If $a_0 = a_1 = 0$, then $\beta(f, D)$ does not depend on $D$.

**Lemma.** Let $D$ be an open disc of radius $r$, and let $f \in \mathcal{R}$ be $D$-admissible. Then

$$|\beta (f, D)| \leq K \gamma(D),$$

where $K$ is a constant depending only on $\alpha$. For $f \in A$ the same inequality holds, but with $\gamma$ replaced by $\gamma_A$.

**Proof.** Let $f \in \mathcal{R}$ be $D$-admissible. Then $f$ is analytic off $D$, $f(\infty) = 0$, and $\|f\|_\infty < 1$. We define the function $g \in \mathcal{R}$ by setting

$$g(z) = (z - c(D))f(z) - f'(\infty).$$

Then $g(\infty) = 0$, $g'(\infty) = \beta(f, D)$, and we claim that $\|g\|_\alpha \leq K_8 r$, where $K_8$ depends only on $\alpha$.

In proving this claim we may assume $c(D) = 0$. Let $z, w \in C$, $z \neq w$. We consider four cases, which together cover all the possibilities.

**Case 1.** $z, w \in 3D$. Then

$$\frac{|zf(a) - wf(w)|}{|z - w|^\alpha} \leq \frac{|z||f(z) - f(w)| + |z - w||f(w)|}{|z - w|^\alpha} \leq 3r \|f\|_\alpha + (6r)^{1-\alpha} \|f\|_\infty \leq K_7 r \|f\|_\alpha \text{ by §10} \leq K_7 r.$$

**Case 2.** $z, w \in C \setminus 2D, |z - w| \geq r$. Then
\[ \frac{|zf(z) - wf(w)|}{|z - w|^a} < \frac{|zf(z)|}{r^a} + \frac{|wf(w)|}{r^a} < 2r^{1-a}(|f(z)| + |f(w)|) < K_1 r^{1-a} \left( \frac{r\|f\|_u}{|z|} + \frac{r\|f\|_u}{|w|} \right) < K_2 r^{1-a}\|f\|_u < K_3 r\|f\|_a < K_3 r. \]

In the third inequality we used the uniform norm decay estimate [6, p. 201], and in the fifth we again applied §10.

Case 3. \( z, w \in \mathbb{C} \setminus 2D, |z - w| < r \). Then
\[ \frac{|zf(z) - wf(w)|}{|z - w|^a} = \frac{1}{|z - w|^a} \left| \frac{1}{2\pi i} \int_{|\zeta| = r} \xi f(\zeta) \left\{ \frac{1}{\zeta - z} - \frac{1}{\zeta - w} \right\} d\zeta \right| \leq \frac{K_4 r\|f\|_u}{|z - w|^a} \int_{|\zeta| = r} \frac{|z - w|}{|\zeta - z| |\zeta - w|} |d\zeta| < K_5 r^{1+a}\|f\|_a |z - w|^{-a} < K_5 r. \]

Case 4. \( z \in 2D, w \not\in 3D \). Then
\[ \frac{|zf(z) - wf(w)|}{|z - w|^a} < \frac{|zf(z)|}{r^a} + \frac{|wf(w)|}{r^a} < 2r^{1-a}\|f\|_u + \frac{|w|}{r^a} \cdot \frac{r\|f\|_u}{|w|} < K_6 r^{1-a}\|f\|_u < K_7 r. \]

Hence the claim is true, so that \((K_{6r})^{-1}g\) is \(D\)-admissible. Thus
\[ |\beta(f, D)| = |g'(\infty)| < K_5 r\gamma(D). \]

The assertion about \(A\) is proved similarly.

12. Decay Lemma (Garnett). Let \( D \) be a disc of radius \( r \), and let \( z \in \mathbb{C} \), with \( d = \text{dist}(z, D) \geq r \). Then

\begin{align*}
(1) & \quad |f(z)| < K_\gamma(D)\|f\|_a/d \\
(2) & \quad |f'(z)| < K_\gamma(D)\|f\|_a/d^2
\end{align*}

whenever \( f \in \mathcal{R} \). There is a similar estimate for \( f \in A \), with \( \gamma \) replaced by \( \gamma_A \).

Proof. (1) \( D \setminus X \) may be covered by a finite collection \( \{S_j\} \) of open squares with sides parallel to the axes, such that
and no square is contained in the union of the rest. Arrange the squares in an order of nondecreasing side-lengths, and form \( H_1 = S_1 \), \( H_2 = S_2 \setminus S_1 \), \( H_3 = S_3 \setminus S_1 \setminus S_2 \), and so on. For each \( i \), let \( \Gamma_i = \text{bdy} \ H_i \), and choose \( \zeta_j \in \text{int} \ H_1 \). Observe that the length of \( \Gamma_i \) is at most \( 4 \text{side } S_i \). Then

\[
|f(z)| = \left| \frac{1}{2\pi i} \sum_j \int_{\Gamma_j} \frac{f(\zeta)}{\zeta - z} \, d\zeta \right| < \frac{1}{2\pi} \sum_j \left| \int_{\Gamma_j} \frac{f(\zeta) - f(\zeta_j)}{\zeta - z} \, d\zeta \right| < K_1 \sum_j \frac{(\text{side } S_j)^{1+\alpha}}{d} \|f\|_\alpha \leq \frac{K_2 M^{1+\alpha} (D \setminus X) \|f\|_\alpha}{d}
\]

\[
< \frac{K_3 \gamma(D) \|f\|_\alpha}{d}, \quad \text{by Corollary 5.}
\]

The estimate for \( f'(z) \) is obtained in a similar way.

To prove the corresponding estimate for \( f \in A \), first choose a modulated function \( \omega \) such that

\[
\frac{1}{2} \omega_j(r) < \|f\|_\alpha \omega_j(r), \quad 0 < r < \frac{1}{2},
\]

\[
\omega(r) < r^a, \quad 0 < r < \frac{1}{2},
\]

\[
r^{-a}\omega(r) \to 0 \quad \text{as } r \downarrow 0.
\]

Set \( h(r) = r\omega(r) \). An argument like that above shows that

\[
|f(z)| < K_4 M_h (D \setminus \text{int } X) \|f\|_\alpha / d,
\]

and so

\[
|f(z)| < \frac{K_4 M^{1+\alpha} (D \setminus \text{int } X) \|f\|_\alpha}{d} < \frac{K_5 \gamma(A) (D) \|f\|_\alpha}{d}, \quad \text{by §7.}
\]

13. Lemma. Let \( D \) be an open disc, \( s^{1+\alpha} = M^{1+\alpha} (D \setminus X) \), and let \( \{B_j\} \) be a family of discs of radius \( s \), each of which is contained in \( D \), such that no point belongs to more than \( p \) of the \( B_j \). Then there is a constant \( K \), depending only on \( \alpha \), such that

\[
\sum_j M^{1+\alpha} (B_j \setminus X) < K p M^{1+\alpha} (D \setminus X),
\]

and also
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whenever $f_j \in \mathbb{R}$ is $B_j$-admissible, $j = 1, 2, \ldots$

Proof. Fix $\varepsilon > 0$, and choose a covering $\{D_n\}$ of $D \setminus X$ by discs with radii $\{r_n\}$ such that each $r_n$ is no greater than $s$, and

$$\sum_n r_n^{1+\alpha} < M^{1+\alpha} (D \setminus X) + \varepsilon.$$ 

Then the $D_n$ cover each $B_j \setminus X$, and no $D_n$ meets more than $K_1 \rho$ of the $B_j$. Thus

$$\sum_j M^{1+\alpha} (B_j \setminus X) < K_1 \rho \sum_n r_n^{1+\alpha} < K_1 \rho \left\{ M^{1+\alpha} (D \setminus X) + \varepsilon \right\}.$$ 

This proves (1).

Now let $f_j \in \mathbb{R}$ be $B_j$-admissible, $j = 1, 2, \ldots$. Fix $x, y \in \mathbb{C}$ and consider

$$|f_j(x) - f_j(y)|/|x - y|^\alpha.$$ 

We divide the integers $j$ into classes $F_m$, corresponding to $m = 0, 1, 2, 3, \ldots$, as follows. We say $j \in F_m$ if $m$ is the greatest integer not exceeding $s^{-1} \min \{ \text{dist}(x, B_j), \text{dist}(y, B_j) \}$.

Observe that the number of elements in $F_m$ does not exceed $K_2 p m$.

For $m = 0$ or $1$ and $j \in F_m$ we use the crude estimate

$$|f_j(x) - f_j(y)|/|x - y|^\alpha < \| f_j \|_\alpha < 1.$$ 

For $m > 1$, $j \in F_m$ we consider two cases.

Case 1. $|x - y| > s$. Then

$$\frac{|f_j(x) - f_j(y)|}{|x - y|^\alpha} < \frac{|f_j(x)| + |f_j(y)|}{s^\alpha} < \frac{K_3 \gamma(B_j) \| f_j \|_\alpha}{(ms)^{s^\alpha}} \text{ by } \S 12$$

$$< \frac{K_3 \gamma(B_j)}{ms^{1+\alpha}}.$$ 

Case 2. $|x - y| < s$. Since $j \in F_m$ there is an arc $\Gamma$ joining $x$ to $y$ such that the length of $\Gamma$ does not exceed $6|x - y|$, and $\text{dist}(\Gamma, B_j) > ms$. Thus

$$\frac{|f_j(x) - f_j(y)|}{|x - y|^\alpha} = \frac{\int_{\Gamma} f'(z) \, dz}{|x - y|^\alpha}$$

$$\leq \frac{K_4 |x - y|^{1-\alpha} \gamma(B_j) \| f_j \|_\alpha}{(ms)^2} \leq \frac{K_4 \gamma(B_j)}{m^2 s^{1+\alpha}}.$$
Thus in either case
\[
\left| \frac{f_j(x) - f_j(y)}{|x - y|^\alpha} \right| \leq K_2 \frac{M^{1+\alpha}(B_j \setminus X)}{s^{1+\alpha}}.
\]

Let \( f = \Sigma j f_j \). Then, abbreviating \( M^{1+\alpha}(B_j \setminus X) = M_j \), we have
\[
\left| \frac{f(x) - f(y)}{|x - y|^\alpha} \right| \leq \sum_j \left| \frac{f_j(x) - f_j(y)}{|x - y|^\alpha} \right|
\leq K_6 p + \sum_j \frac{K_2 M_j}{s^{1+\alpha}}
\leq K_6 p + K_3 K_1 p \quad \text{by (1)}
= K_7 p.
\]

14. This lemma allows us to improve the estimate for \( \|f\|_u \) of §10.

**Corollary.** Let \( D \) be an open disc and let \( f \in \mathcal{R}(X) \) be \( D \)-admissible. Then
\[
\|f\|_u \leq K_\gamma(D)^{\alpha/(1+\alpha)}.
\]

**Proof.** In proving this we may assume that \( X \) contains a neighbourhood of \( 3D \setminus D \), and we do.

Cover the set of singularities of \( f \) by discs \( \frac{1}{2} B_j \subset D \) of side
\[
s = M^{1+\alpha}(D \setminus X)^{1/(1+\alpha)}
\]
in such a way that no point belongs to more than 100 of the \( B_j \). Choose functions \( \varphi_j \in \mathcal{D} \) such that \( 0 < \varphi_j < 1 \), spt \( \varphi_j \subset B_j \), \( \| \nabla \varphi_j \|_u < 4/s \), and \( \sum \varphi_j \equiv 1 \) on \( \bigcup \frac{1}{2} B_j \), which is a neighbourhood of the set of singularities of \( f \) (cf. [3]). Let \( f_j = T_{\varphi} f \). Then \( f = \Sigma f_j \), \( f_j \in \mathcal{R} \), \( f_j \) is analytic off \( B_j \), and \( f_j(\infty) = 0 \). Also \( \|f_j\|_a < K_1 \) by the \( T_{\varphi} \) estimate, so that \( K_1 f_j \) is \( B_j \)-admissible.

Fix \( z \in C \), and divide the indices \( j \) up into classes again: say \( j \in G_m \) if \( m \) is the greatest integer not exceeding \( s^{-1} \text{dist}(z, B_j) \). For \( m > 1 \) and \( j \in G_m \) we have
\[
|f_j(z)| \leq K_2 \gamma(B_j)/ms
\]
by the Decay Lemma, §12. Thus
\[ |f(z)| \leq \sum_j |f_j(z)| \leq K_3 \|f\|_u + \sum_{m=2}^{\infty} \sum_{j \in G_m} |f_j(z)| \]
\[ \leq K_4 \left( s^\alpha + \sum_{m=2}^{\infty} \sum_{j \in G_m} \frac{\gamma(B_j)}{ms} \right) = K_4 s^\alpha \left( 1 + \sum_{m=2}^{\infty} \sum_{j \in G_m} \frac{\gamma(B_j)}{ms^{1+\alpha}} \right) \]
\[ \leq K_6 s^\alpha \left( 1 + \left[ \sum_j \frac{\gamma(B_j)}{s^{1+\alpha}} \right]^{1/2} \right) \]
\[ \leq K_6 s^\alpha, \text{ by §13 and §15.} \]

Thus \( \|f\|_u \leq K_6 s^\alpha \leq K_7 \gamma(D)^{\alpha/(1+\alpha)}. \)

15. We are now in a position to prove the sufficiency of the condition for approximation. In fact, we will prove a slightly stronger statement.

Suppose there exist constants \( \mu > 0, \tau > 1 \) such that for each point \( x \in \text{bdy } X \) and each disc \( D \) centered at \( x, \)
\[ M^{1+\alpha}(\tau D \setminus X) > \mu M^{1+\alpha}_{*}(D \setminus \text{int } X). \]
Then \( [\mathcal{R}]_a = A_a(X). \)

Throughout the proof \( K_1, K_2, K_3, \ldots \) stand for constants which may depend on \( \alpha, \mu, \tau \) and \( \|f\|_a, \) but not on any other variables.

Suppose \( \mu \) and \( \tau \) exist as in the statement. Then for each open disc \( D \) of radius \( r \) centered at a point of \( C \setminus \text{int } X \) we have
\[ M^{1+\alpha}(\tau D \setminus X) > 4^{-1}\mu M^{1+\alpha}_{*}(D \setminus \text{int } X), \]
hence \( \gamma(D) > K_7 \gamma_a(D) \) for each such disc \( D. \)

Fix \( f \in A. \) We shall prove that \( f \) may be approximated in \( \text{Lip}(\alpha, X) \) norm by elements of \( \mathcal{R}. \) First, we extend \( f \) to \( C \) so that the extension (also denoted by \( f \)) lies in \( \text{lip}(\alpha, C) \) and is analytic off some disc. Fix \( \delta > 0. \) Let \( (D_n)_{n=1}^{\infty} \) be a covering of \( \text{C \setminus int } X \) by open discs of radius \( \delta \) centered at points of \( \text{C \setminus int } X \) and such that no disc \( D_n \) meets more than 100 others. Let \( \{\varphi_n\}_{n=1}^{\infty} \subset \mathcal{D} \) be a sequence of functions such that \( 0 < \varphi_n < 1, \) \( \text{spt } \varphi_n \subset 2D_n, \) \( \|\nabla \varphi_n\|_u < 4\delta^{-1}, \) and \( \sum_{n=1}^{\infty} \varphi_n \equiv 1 \) on \( \bigcup_{n=1}^{\infty} D_n. \) Let \( f_n = T_{\varphi_n} f. \) Then \( f_n \in A, f_n \equiv 0 \) except for a finite number of indices \( n, \) and \( f = \sum_{n=1}^{\infty} f_n. \) Let \( \eta(r) = r^{-\alpha}\omega_j(r), \) so that \( \eta(r) \to 0 \) as \( r \downarrow 0. \) For each \( n, f_n \) is holomorphic off \( 2D_n, f_n(\infty) = 0, \) and \( \|f_n\|_a < K_2 \eta(\delta). \)

Now fix \( n \) and, following Davie [3], let
\[ r = \frac{1}{2\tau} \cdot \min\{\delta, M^{1+\alpha}(3D \setminus X)^{1/(1+\alpha)}\}. \]
Cover the (closed) set of singularities of \( f_n \) (a subset of \( 2D_n \setminus \text{int } X \)) by centered discs \( B_j \subset 2D_n \) of radius \( r, \) in such a way that no point belongs to
more than 25 of the $B_j$. Select a collection $\{\psi_j\} \subset \mathcal{D}$ of functions such that $0 < \psi_j \leq 1$, spt $\psi_j \subset 2B_j$, $\|\nabla \psi_j\|_u \leq \frac{4}{r}$, and $\Sigma \psi_j \equiv 1$ on $\bigcup B_j$, which is a neighbourhood of the set of singularities of $f_n$. Let $f^*_n = T_{\psi_j} f_n$. Then $f^*_n \in A, f^*_n$ is analytic off $2B_j$, $f^*_n(\infty) = 0$, $\|f^*_n\|_u \leq K_4 \eta(\delta)$, and $f_n = \Sigma j^*_n$. From the definition of $\gamma_A$ we deduce that

$$|f^*_n(\infty)| \leq K_4 \eta(\delta) \gamma_A(2B_j) < K_5 \eta(\delta) \gamma(2\tau B_j),$$

by hypothesis. Thus there exist functions $g^*_j \in \mathcal{R}$ such that $g^*_j$ is analytic off $2\tau B_j$, $g^*_j(\infty) = 0$, $\|g^*_j\|_u \leq K_5 \eta(\delta)$, and $g^*_j(\infty) = f^*_j(\infty)$. Let $g_n = \Sigma g^*_j$. Then $g_n \in \mathcal{R}$, $g_n$ is analytic off $3D_n$, $g_n(\infty) = 0$, and $g'_n(\infty) = f'_n(\infty)$. Also, by Lemma 13 (2), $\|g_n\|_u < K_6 \eta(\delta)$.

We have

$$\beta(f_n - g_n, D_n) = \sum_j \beta(f^*_n - g^*_j, D_n) = \sum_j \beta(f^*_n - g^*_j, g_j),$$

since $f^*_n - g^*_j$ vanishes to second order at $\infty$. Hence by Lemma 11 and Lemma 1 (1),

$$|\beta(f_n - g_n, D_n)| \leq \sum_j K_7 \gamma(B_j) \eta(\delta) \leq K_8 \gamma(2D_n)^{(2+a)/(1+a)} \eta(\delta).$$

We may choose a function $h_n \in \mathcal{R}$, analytic off $2D_n$ and vanishing at $\infty$, with $\|h_n\|_u \leq 2$ and $h'_n(\infty) = \gamma(2D_n)$. Forming

$$k_n = g_n + \beta(f_n - g_n, D_n)(h_n/\gamma)^2 \in \mathcal{R},$$

(where we have abbreviated $\gamma = \gamma(2D_n)$), we deduce that

$$\|k_n\|_u \leq \|g_n\|_u + |\beta(f_n - g_n, D_n)|\gamma^{-2}\|h_n\|_u$$

$$\leq K_9 \eta(\delta) + K_8 \gamma^{-a/(1+a)} \eta(\delta) \|h_n\|_u < K_9 \eta(\delta)$$

by Corollary 14. Also $k_n$ is analytic off $2D_n$, $k_n(\infty) = 0$, $k'_n(\infty) = g'_n(\infty) = f'_n(\infty)$, and $\beta(k_n, D_n) = \beta(g_n, D_n) + \beta(f_n - g_n, D_n) = \beta(f_n, D_n)$.

Let $q_n = f_n - k_n$. Then $f = \Sigma k_n + \Sigma q_n$. The first sum belongs to $\mathcal{R}$. We will show that the second sum tends to zero in $\text{Lip}(\alpha, \mathbb{C})$ norm as $\delta \downarrow 0$, so that $f \in [\mathcal{R}]_{\alpha, \mathbb{C}}$.

Clearly $\|q_n\|_u \leq K_{10} \eta(\delta)$, so that by Lemma 10, $\|q_n\|_u \leq K_{11} \delta^\alpha \eta(\delta)$. Fix two distinct points $x, y \in \mathbb{C}$. In order to estimate

$$|x - y|^{-\alpha} \left\{ \sum q_n(x) - \sum q_n(y) \right\}$$

we divide the indices $n$ into classes $F_m$, in the same way as in the proof of Lemma 13, with $s = 2\delta$. Thus $n \in F_m$ if $ns$ is the greatest integral multiple of $s$ not exceeding

$$\min\{\text{dist}(x, 2D_n), \text{dist}(y, 2D_n)\}.$$
The function $q_n$ has a triple zero at $\infty$, so that $\delta^{-3}(z-c_n)^3 q_n(z)$, the function, is analytic on $\Sigma \setminus 2D_n$ (here $c_n = c(D_n)$). For $z \in \text{bdy}(2D_n)$,

$$|\delta^{-3}(z-c_n)^3 q_n(z)| \leq 8\|q_n\|_u \leq K_{12}\delta^\alpha\eta(\delta),$$

hence by the maximum principle,

(*) $$|q_n(z)| \leq K_{13}\delta^{3+\alpha}\eta(\delta)d^{-3}$$

whenever $d = \text{dist}(z, 2D_n) > s$.

If $k(z)$ is a bounded function, is analytic off a disc $D$ of radius $r$, and vanishes at $\infty$, and $0 < R = \text{dist}(z, D)$, then the uniform norm derivative decay estimate [12, p. 201] states that

$$|k'(z)| \leq 4r\|k\|_{u, \text{bdy}D}/R^2.$$  

If $d = \text{dist}(z, D_n) > 4s$, take $D = \frac{1}{2}dD_n$, so that $\|q_n\|_{u, \text{bdy}D} < K_{14}\delta^{3+\alpha}\eta(\delta)d^{-3}$ by (*), and conclude that

(**) $$|q_n'(z)| \leq K_{15}\delta^{3+\alpha}\eta(\delta)d^{-4}.$$  

If $n$ belongs to one of the first six $F_m$ we use the crude estimate

$$|q_n(x) - q_n(y)|/|x-y|^\alpha \leq \|q_n\|_a \leq K_6\theta(\delta).$$

If $6 < m \in \mathbb{Z}$ and $n \in F_m$, we consider two cases.

Case 1. $|x-y| < s$. We have

$$ms \leq \min\{\text{dist}(x, 2D_n), \text{dist}(y, 2D_n)\},$$

so there is a curve $\Gamma$ joining $x$ to $y$, the length of which does not exceed $\pi|x-y|$, with the property that $\text{dist}(\Gamma, 2D_n) > ms$. Thus by (**)  

$$\frac{|q_n(x) - q_n(y)|}{|x-y|^\alpha} = \frac{1}{|x-y|^\alpha}\left|\int_{\Gamma} h_n'(z) \, dz\right| < \pi K_{15}|x-y|^{1-\alpha}\delta^{3+\alpha}\eta(\delta)(ms)^{-4} \leq K_{16}\theta(\delta)m^{-4}.$$  

Case 2. $|x-y| > s$. Then by (*),

$$\frac{|q_n(x) - q_n(y)|}{|x-y|^\alpha} < \frac{|q_n(x)| + |q_n(y)|}{s^\alpha} < 2K_{13}\delta^{3+\alpha}\theta(\delta)(ms)^{-3} = K_{17}\theta(\delta)m^{-3}.$$  

Thus in either case
\[
\frac{|\sum q_n(x) - \sum q_n(y)|}{|x - y|^\alpha} < \sum_n \frac{|q_n(x) - q_n(y)|}{|x - y|^\alpha} \\
< \sum_{m=0}^5 \sum_{n \in F_m} K_6 \eta(\delta) + \sum_{m=6}^\infty \sum_{n \in F_m} K_{18} \eta(\delta) m^{-3}
\]
\[
< \left( \sum_{m=0}^5 K_{11} K_6 (m + 1) + \sum_{m=6}^\infty K_{18} K_{11} (m + 1) m^{-3} \right) \eta(\delta)
\]
\[
= K_{19} \eta(\delta).
\]
Since \( \eta(\delta) \to 0 \) as \( \delta \downarrow 0 \), this proves that \( \|\Sigma q_n\|_\alpha \to 0 \) as \( \delta \downarrow 0 \), so we are done.

16. As a special case we obtain a characterisation of those compact sets \( X \) on which all \( f \in \text{lip}(\alpha, X) \) may be approximated in \( \text{Lip}(\alpha, X) \) norm by rational functions.

**Corollary.** A necessary and sufficient condition that
\[
[\mathcal{R}_1]_\alpha = \text{lip}(\alpha, X)
\]
is that there exist \( \mu > 0 \) such that \( M^{1+\alpha}(D \setminus X) > \mu r^{1+\alpha} \) for every open disc \( D \) of radius \( r \) (\( 0 < \alpha < 1 \)).

This follows from our theorem because \( M^{1+\alpha}_*(D) = (2r)^{1+\alpha} \).

17. **Corollary.** If \( X \) has zero area and \( 0 < \alpha < 1 \), then
\[
[\mathcal{R}_1]_\alpha = \text{lip}(\alpha, X).
\]

**Proof.** Let \( D \) be any disc of radius \( r \). Then, denoting Lebesgue measure on the plane by \( m \), we have \( m(D \setminus X) = m(D) = \pi r^2 \). Let \( \{B_j\} \) be a covering of \( D \setminus X \) by discs with radii \( \{r_j\} \), \( r_j < r \). Then
\[
\sum r_j^{1+\alpha} > \frac{\sum r_j^2}{r^{1-\alpha}} > \frac{m(D \setminus X)}{\pi r^{1-\alpha}} = r^{1+\alpha},
\]
hence \( M^{1+\alpha}(D \setminus X) > r^{1+\alpha} \). Thus the condition of Corollary 16 is satisfied, with \( \mu = 1 \).

J. Garnett has shown the author how to give a direct constructive proof of this fact. There is also an entirely different proof, based on duality.

18. **Corollary.** If \( 0 < \alpha < 1 \) and \( M^{1+\alpha}_*(\text{bdy} X) = 0 \), then
\[
[\mathcal{R}_1]_\alpha = \text{lip}(\alpha, X) \cap A(X).
\]

**Proof.** If \( E_1 \) and \( E_2 \) are two subsets of \( C \), then
\[
M^{1+\alpha}_*(E_1 \cup E_2) \leq M^{1+\alpha}_*(E_1) + M^{1+\alpha}_*(E_2).
\]
This is an immediate consequence of the definition of \( M^\beta_* \) and the subadditivity of \( M^\alpha_\alpha \). It follows that
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\[ M_{*}^{1+a}(D \setminus \text{int} \, X) < M_{*}^{1+a}(\text{bdy} \, X) + M_{*}^{1+a}(D \setminus X) \]

\[ < M_{*}^{1+a}(\text{bdy} \, X) + M^{1+a}(D \setminus X) \]

hence if \( M_{*}^{1+a}(\text{bdy} \, X) = 0 \), then the condition of our theorem is satisfied, with \( \mu = 1 \).

The condition \( M_{*}^{1+a}(E) = 0 \) is equivalent to \( \mathcal{H}^{1+a}(E) < \infty \), where \( \mathcal{H}^{1+a} \) is \((1 + a)\)-dimensional Hausdorff measure \([5, (2.10)]\).

19. Before giving some examples, we need a definition. Let \( B(x, r) \) denote the disc \( \{z \in \mathbb{C} : |z - x| < r\} \). If \( E \subset \mathbb{C} \) and \( \beta > 0 \), then the \( \beta \)-dimensional upper density of \( E \) at the point \( x \in \mathbb{C} \) is defined as

\[ \limsup_{r \to 0} \frac{M^\beta(E \cap B(x, r))}{r^\beta} \]

the lower density is the corresponding lim inf, and in case these two coincide, we refer to the density.

20. Example. We construct a compact set \( X \subset \mathbb{C} \) such that \( X \) is the closure of its interior, and \([\Re]_a = A(X)\), but \([\Re]_a \neq A_a(X)\).

Fix \( \beta, \alpha < \beta < 1 \). We begin with a closed square \( P \), and inside \( P \) an arc \( \Gamma \) having positive \((1 + \beta)\)-dimensional lower density at each of its points \([7]\).

We then remove from \( P \) a sequence of thin wavy open strips \( S_1, S_2, S_3, \ldots \), so that the \( S_j \) "accumulate" only on \( \Gamma \) and accumulate at every point of \( \Gamma \), and so that \( \bigcup_j S_j \) has zero \((1 + \alpha)\)-dimensional density at each point of \( \Gamma \). Then we set \( X = P \setminus \bigcup_j S_j \). For any small disc \( D \) of radius \( r \) about any point of \( \Gamma \), \( M_{*}^{1+a}(D \setminus \text{int} \, X) \) will be bounded below by some constant times \( r^{1+a} \), whereas \( M^{1+a}(D \setminus X) \) will be \( o(r^{1+a}) \). So the condition of the theorem cannot hold for any \( \mu > 0 \). Thus \([\Re]_a \neq A_a(X)\). Since the diameters of the components of \( \mathbb{C} \setminus X \) are bounded away from zero, it follows that \([\Re]_a = A(X)\) (cf. \([6, p. 219 (8.3)]\)).

\[ \bigcup_j S_j \]

\( \Gamma \)

**Figure 1**
21. Example. We construct a set $X$ with empty interior such that the analytic polynomials $\mathcal{P}$ are uniformly dense in $C(X)$, but $[\mathcal{R}]_\alpha \neq \text{lip}(\alpha, X)$.

Choose a sequence of positive numbers $l_n$ such that $\sum l_n^\alpha < 1$. Then $\sum l_n < 1$ and we may form a Cantor set $C$ of positive length on $[0, 1]$ by deleting successively (open) intervals of length $l_n$. Let $\lambda$ denote Lebesgue measure on the line.

**Lemma.** $[0, 1] \setminus C$ has zero $\alpha$-dimensional density at $\lambda$ almost all points of $C$.

**Proof.** Let $(a_n, b_n)$ be the interval of length $l_n$ in $[0, 1] \setminus C$. Then by Fubini’s Theorem,

$$
\int_0^1 \sum_{n=1}^{\infty} \frac{l_n^\alpha}{|z - a_n|^\alpha} \, d\lambda(z) = \sum_{n=1}^{\infty} l_n^\alpha \int_0^1 \frac{d\lambda(z)}{|z - a_n|^\alpha}
$$

$$
< 2^\alpha (1 - \alpha)^{-1} \sum_{n=1}^{\infty} l_n^\alpha < \infty,
$$

so that

$$
\sum_{n=1}^{\infty} \frac{l_n^\alpha}{|z - a_n|^\alpha} < \infty
$$

for $\lambda$ almost all $z \in [0, 1]$. Similarly,

$$
\sum_{n=1}^{\infty} \frac{l_n^\alpha}{|z - b_n|^\alpha} < \infty
$$

for $\lambda$ almost all $z \in [0, 1]$. For $z \in C$ the upper $\alpha$ density of $[0, 1] \setminus C$ at $z$ is

$$
\limsup_{r \downarrow 0} \frac{M^\alpha([z - r, z + r] \setminus C)}{r^\alpha} < \limsup_{r \downarrow 0} \frac{\sum l_n^\alpha}{r^\alpha}
$$

(where the sum is taken over those $n$ for which $[a_n, b_n]$ meets $[z - r, z + r]$).

$$
< \limsup_{r \downarrow 0} \sum' \left\{ \frac{l_n^\alpha}{|z - a_n|^\alpha} + \frac{l_n^\alpha}{|z - b_n|^\alpha} \right\}
$$

$$
< \limsup_{r \downarrow 0} \sum_{n'} \left\{ \frac{l_n^\alpha}{|z - a_n|^\alpha} + \frac{l_n^\alpha}{|z - b_n|^\alpha} \right\}
$$

(\text{where } N'_r \text{ is the first index in } \Sigma')

$$
= 0
$$

for $\lambda$ almost all $z \in C$. This proves the lemma.

Now set $X = C \times [0, 1]$. Then $[\mathcal{P}]_\alpha = C(X)$ by Mergelyan’s Theorem [6], since $X$ does not separate the plane. But clearly $C \setminus X$ has zero $(1 + \alpha)$-density at $\mathcal{L}^2$ almost all points of $X$, so $[\mathcal{R}]_\alpha \neq \text{lip}(\alpha, X)$ by Corollary 16.
22. Example. The term Swiss Cheese is traditionally applied to any compact set $X$ obtained by removing from the closed unit disc an infinite sequence $\{D_n\}$ of disjoint open discs, with radii $\{r_n\}$ and centres $\{a_n\}$, such that $\Sigma r_n < 1$ and $\bigcup_n D_n$ is dense in the unit disc. For any such $X$, $[\mathcal{R}]_u \neq C(X)$ [1], [6], and hence a fortiori $[\mathcal{R}]_a \neq \text{lip}(\alpha, X)$, for $0 < \alpha < 1$.

Fix $0 < \alpha < 1$. A larger class of cheeses is obtained by relaxing the condition on the radii of the excised discs to $\Sigma r_n^{1+\alpha} < \infty$. We call such a cheese an \text{"a-cheese"}. If $X$ is an $\alpha$-cheese, then $[\mathcal{R}]_\alpha = \text{lip}(\alpha, X)$. To see this, note that by Fubini's Theorem,

$$
\int_X \sum_{n=1}^{\infty} \frac{r_n^{1+\alpha}}{|z - a_n|^{1+\alpha}} \, dm(z) = \sum_{n=1}^{\infty} \frac{r_n^{1+\alpha}}{|z - a_n|^{1+\alpha}} \int \frac{dm(z)}{2\pi(1 - \alpha)^{-1}} < \infty.
$$

Hence

$$
\sum_{n=1}^{\infty} \frac{r_n^{1+\alpha}}{|z - a_n|^{1+\alpha}} < \infty \text{ a.e. } dm.
$$

For $m$ almost all such $z$, it follows that

$$
M^{1+\alpha}(B(z, r) \setminus X)/r^{1+\alpha} \to 0
$$
as $r \downarrow 0$. Precisely speaking, the limit is zero for any $z$ for which the series converges, unless $z$ happens to belong to $\text{bdy } D_n$ for some $n$. This is seen by essentially the same argument as that of the last section.

Thus the necessary condition for rational approximation is violated, and so $[\mathcal{R}]_\alpha \neq \text{lip}(\alpha, X)$.

23. We close with some remarks about polynomial approximation. Let $\mathcal{P}$ denote the space of analytic polynomials. It is not hard to see that $[\mathcal{R}]_{\alpha,X} = [\mathcal{P}]_{\alpha,X}$ if and only if $C \setminus X$ is connected. Thus $[\mathcal{P}]_{\alpha,X} = A_\alpha(X)$ if and only if $C \setminus X$ is connected and there exists a constant $\mu > 0$ such that

$$
M^{1+\alpha}(D \setminus X) > \mu M^{1+\alpha}_*(D \setminus \text{int } X)
$$

whenever $D$ is an open disc. Also $[\mathcal{P}]_{\alpha,X} = \text{lip}(\alpha, X)$ if and only if $C \setminus X$ is connected and there exists a constant $\mu > 0$ such that

$$
M^{1+\alpha}(D \setminus X) > \mu r^{1+\alpha}
$$

whenever $D$ is an open disc and the radius of $D$ is $r$.

References


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