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BOCHNER IDENTITIES FOR FOURIER TRANSFORMS[1]

BY

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ABSTRACT. Let G be a compact Lie group and R an orthogonal representation of G acting on $\mathbf{R}^n$. For any irreducible unitary representation π of G and vector v in the representation space of π define $\mathcal{S}(\pi, v)$ to be those functions in $\mathcal{S}(\mathbf{R}^n)$ which transform (under the action R) according to the vector v. The Fourier transform $\mathcal{F}$ preserves the class $\mathcal{S}(\pi, v)$. A Bochner identity asserts that for different choices of G, R, π, v the Fourier transform is the same (up to a constant multiple). It is proved here that for G, R, π, v and G', R', π', v' and a map T: $\mathcal{S}(\pi, v) \to \mathcal{S}(\pi', v')$ which has the form: restriction to a subspace followed by multiplication by a fixed function, a Bochner identity $\mathcal{F}'Tf = cT\mathcal{F}f$ for all $f \in \mathcal{S}(\pi, v)$ holds if and only if $\Delta'Tf = c_1T\Delta f$ for all $f \in \mathcal{S}(\pi, v)$. From this result all known Bochner identities follow (due to Harish-Chandra, Herz and Gelbart), as well as some new ones.

1. **Introduction.** The classical Bochner identities deal with the Fourier transform in $\mathbf{R}^n$ of the product of a radial function $f(|x|)$ times a spherical harmonic $Y_k(x)$ of degree k. Bochner [1] proves

$$(1.1) \qquad [f(|x|)Y_k(x)]^\wedge(y) = \left(\frac{i}{2\pi}\right)^k Y_k(y)\mathcal{H}_{(n/2)+k}(f)(|y|)$$

where

$$(1.2) \quad \mathcal{H}_{(n/2)+k}(f)(r) = (2\pi)^{(n/2)+k}\int_0^\infty \frac{J_{(n/2)+k-1}(rt)}{(rt)^{(n/2)+k-1}} f(t)t^{n+2k-1}\,dt.$$

Now there are two aspects to this result: the explicit formula given by (1.2) and the observation that (1.1) depends (aside from a constant) only on $(n/2) + k$. It is this second aspect only that we generalize in this paper.

Let G be a compact Lie group and R a *real* finite dimensional representation of G (*not* assumed irreducible), which we may assume to be orthogonal without loss of generality. In other words, R is a continuous homomorphism of G into $O(n)$ for some n, and we will write $g^{-1}x$ in place of $R(g)x$ for $g \in G$, $x \in \mathbf{R}^n$. If π is any irreducible unitary representation of G on the

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complex vector space V and v any nonzero vector in V, we say a function f in $\mathcal{S}(\mathbf{R}^n)$ *transforms according to* the vector v and the representation π, $f \in \mathcal{S}(\pi, v)$ if the linear span of translates $f(g^{-1}x)$ as g varies over G is equivalent as a representation of G to π with $f(x)$ corresponding to v under the equivalence. We write $\mathcal{S}_I$ for the G-invariant functions in $\mathcal{S}$ and note that every $\mathcal{S}(\pi, v)$ is an $\mathcal{S}_I$-module. It is easy to verify that $f \in \mathcal{S}(\pi, v)$ if and only if $\sum_{j=1}^m a_j f(g_j^{-1}x) \equiv 0$ whenever $\sum_{j=1}^m a_j \pi(g_j)v = 0$, and the equivalence of representations is given by the intertwining operator

$$I\left(\sum_{j=1}^m a_j f\left(g_j^{-1}x\right)\right) = \sum_{j=1}^m a_j \pi(g_j)v.$$

In the case where R is the standard representation of $SO(n)$, and π is the irreducible representation on spherical harmonics of degree k, the functions in $\mathcal{S}(\pi, v)$ are exactly the radial functions times the spherical harmonic v.

The Fourier transform

$$\mathcal{F}f(y) = \int_{\mathbf{R}^n} f(x)e^{ix\cdot y}\,dx$$

clearly preserves the class $\mathcal{S}(\pi, v)$. The question we pose is whether the Fourier transform restricted to $\mathcal{S}(\pi, v)$ is essentially the same for different choices of G, R, π and v. That is, given G', R', π' and v' as above, does there exist a map T: $\mathcal{S}(\pi, v) \to \mathcal{S}(\pi', v')$ such that $\mathcal{F}'T = cT\mathcal{F}$ for some constant c? We investigate this question in the case where G' is a subgroup of G, R' is the restriction of R to G' and a subspace $\mathbf{R}^{n'} \subseteq \mathbf{R}^n$ invariant under G', and v and v' are highest weight vectors for the representations π and π' with respect to maximal tori H and H' satisying $H' \subseteq H$ (with compatible ordering of weights). We will look for T of the form $T = M\gamma$ where γ is restriction to $\mathbf{R}^{n'}$ and M is multiplication by a function on $\mathbf{R}^{n'}$. The identity we seek to establish is

$$\mathcal{F}'M\gamma f = cM\gamma\mathcal{F}f \tag{1.3}$$

for all $f \in \mathcal{S}(\pi, v)$. To see the connection between Bochner's result (1.1) and (1.3) observe that a highest weight vector for the spherical harmonics of degree k is the function $(x_1 + ix_2)^k$. If $n \geqslant n' > 2$ with $n - n'$ even, $\mathbf{R}^{n'} \subseteq \mathbf{R}^n$ is the subspace with the last $n - n'$ components equal to zero then (1.1) implies

$$\mathcal{F}'M\gamma\left((x_1 + ix_2)^k f(|x|)\right) = i^{(n-n')/2}M\gamma\mathcal{F}\left((x_1 + ix_2)^k f(|x|)\right) \tag{1.4}$$

where M is multiplication by $(x_1 + ix_2)^{(n-n')/2}$, since both sides of the equation are equal to $(i/2\pi)^{k+(n-n')/2}\cdot(x_1 + ix_2)^{k+(n-n')/2}\mathcal{H}_{(n/2)+k}(f)(|x|)$. The implication may be reversed in this case, essentially because $\mathcal{S}(\pi, v)$ is one dimensional as an $\mathcal{S}_I$-module. In general there may be no analogue of (1.1).

We now state our main result:

THEOREM 1. *A necessary and sufficient condition for* (1.3) *to hold for all* $f \in \mathcal{S}(\pi, v)$ *is that*

$$\Delta' M\gamma f = c_1 M\gamma\Delta f \tag{1.5}$$

hold for all $f \in \mathcal{S}(\pi, v)$, *where* Δ *and* Δ' *are the Laplacians on* $\mathbf{R}^n$ *and* $\mathbf{R}^{n'}$. *In that case* $n - n'$ *is necessarily even,* M *is multiplication by a function which is homogeneous of degree* $(n - n')/2$ *and* $c = (i/2\pi)^{(n-n')/2}$, $c_1 = 1$.

The significance of this result is that it reduces a question concerning Fourier transforms, which are difficult to compute, to a question concerning Laplacians, which are easier to compute. From it we obtain new and simpler proofs of all Bochner identities known to the author, as well as new ones.

We will prove the main theorem in §2. In §3 we apply it to obtain a theorem of Harish-Chandra concerning the adjoint representation of a compact semisimple Lie group. In §§4 and 5 we give applications to the classical groups $SO(n)$, $U(n)$ and $Sp(n)$ acting on certain matrix spaces. Some of the results for $SO(n)$ were proved by Herz [7] and Gelbart [3].

For the applications it is convenient to deal with a general positive-definite bilinear form (inner product) $B(x, y) = \sum\sum b_{ij}x_iy_j$ in some coordinate system. We must then assume $R(G)$ preserves B and define the Fourier transform by

$$\mathcal{F}f = \int f(y)e^{iB(x,y)}(\det b)^{1/2}\, dy,$$

etc. The theorem is then true for $\Delta = \sum\sum(b^{-1})_{ij}(\partial/\partial x_i)(\partial/\partial x_j)$, etc.

Finally we leave open the possibility that our results may be extended to noncompact groups whose action preserves a nondegenerate quadratic form, using the results of [9]. In outline the same proof seems plausible, but there are many technical difficulties to be overcome. We have formulated Lemma 2 in §3 looking forward to such generalizations.

In this work we deal with functions in $\mathcal{S}$ for convenience only. Theorem 1 extends to L^2 functions or even tempered distributions rather easily. Indeed if we define $\mathcal{S}'(\pi, v)$ to be all tempered distributions transforming according to π and v then $\mathcal{S}(\pi, v)$ is a dense subspace of $\mathcal{S}'(\pi, v)$. Thus (1.3) holds for $f \in \mathcal{S}'(\pi, v)$ once we have the existence and continuity of $M\gamma f$ for $f \in \mathcal{S}'(\pi, v)$. But this is an easy consequence of (1.5)—even though γf may be undefined. For example, if $\gamma: \mathcal{S}(\mathbf{R}^3) \to \mathcal{S}(\mathbf{R}^1)$ is defined by $\gamma f(x) = f(x, 0, 0)$ then $\gamma\delta$ is undefined, but $x\gamma\delta = (2\pi)^{-1}\delta'$, as may be seen by approximating δ by $(\pi\varepsilon)^{-3/2}e^{-|x|^2/\varepsilon}$.

2. **Proof of Theorem 1.** The necessity of (1.5) is trivial. Indeed

$$\Delta f = -(2\pi)^{-2n}\mathcal{F}^3\left(|x|^2\mathcal{F}f\right)$$

and

$$\Delta' f' = -(2\pi)^{-2n'} \mathfrak{F}'^3\left(|x'|^2 \mathfrak{F}' f'\right)$$

and $\mathcal{S}(\pi, v)$ is preserved by multiplication by $|x|^2$. Thus

$$\begin{aligned}\Delta' M\gamma f &= -(2\pi)^{-2n'} \mathfrak{F}'^3\left(|x'|^2 \mathfrak{F}' M\gamma f\right) \\ &= -(2\pi)^{-2n'} c \mathfrak{F}'^3\left(|x'|^2 M\gamma \mathfrak{F} f\right) = -(2\pi)^{-2n'} c \mathfrak{F}'^3\left(M\gamma\left(|x|^2 \mathfrak{F} f\right)\right) \\ &= -(2\pi)^{-2n'} c^4 M\gamma\left(\mathfrak{F}^3\left(|x|^2 \mathfrak{F} f\right)\right) = (2\pi)^{2(n-n')} c^4 M\gamma \Delta f\end{aligned}$$

establishing (1.5) with $c_1 = (2\pi)^{2(n-n')} c^4$.

For the converse we first deduce some consequences of (1.5). Let us assume that $\mathcal{S}'(\pi, v)$ contains a spherical harmonic f of degree k such that $M\gamma f$ is not identically zero. Then $\Delta' M\gamma f = c_1 M\gamma \Delta f = 0$ since f is harmonic. Now if $h(|x|)$ is any radial function in $\mathcal{S}(\mathbf{R}^n)$ then $hf \in \mathcal{S}(\pi, v)$ so $\Delta' h(|x'|) M\gamma f = c_1 M\gamma \Delta hf$. Both sides of this equality may be computed using the identities

$$\Delta(f_1 f_2) = (\Delta f_1) f_2 + f_1 \Delta f_2 + 2\nabla f_1 \cdot \nabla f_2$$

and

$$\Delta h(|x|) = h''(|x|) + (n-1) h'(|x|)/|x|$$

and the fact that $x \cdot \nabla f = kf$ since f is homogeneous of degree k. We find

$$\begin{aligned}& h''(|x'|) M\gamma f + \frac{n'-1}{|x'|} h'(|x'|) M\gamma f + \frac{2h'(|x'|)}{|x'|} x' \cdot \nabla' M\gamma f \\ &\quad = c_1\left(h''(|x'|) M\gamma f + \frac{n-1}{|x'|} h'|x'|\right) M\gamma f + \frac{2k}{|x'|} h'(|x'|) M\gamma f.\end{aligned}$$

Choosing $h(|x|) = |x|^{2m}$ locally this becomes

$$2x' \cdot \nabla' M\gamma f = \left[c_1(2m + n - 2) - (2m + n' - 2) + 2c_1 k\right] M\gamma f$$

and since $M\gamma f$ is not identically zero we must have $c_1 = 1$ hence $x' \cdot \nabla' M\gamma f = (k + (n - n')/2) M\gamma f$. We have proved that $M\gamma f$ is homogeneous of degree $k + (n - n')/2$ and also harmonic. But the only homogeneous harmonic functions are spherical harmonics, hence $n - n'$ must be even, and we have shown that *if $f \in \mathcal{S}'(\pi, v)$ is a spherical harmonic of degree k then $M\gamma f$ is a spherical harmonic of degree $k + (n - n')/2$.*

Now we prove (1.3). If $\mathcal{S}(\pi, v)$ contains only the zero function there is nothing to prove, so assume otherwise. Let $f \in \mathcal{S}(\pi, v)$ be not identically zero and expand the restriction of f to each sphere in a spherical harmonic series. We have $f = \sum_{k=0}^{\infty} f_k$ where

$$f_k(x) = \int_{G_0} f(g^{-1}x) d_k \chi_k(g)\, dg$$

where $G_0 = O(n)$ and χ_k and d_k are the character and dimension of the representation of G_0 on spherical harmonics of degree k. Since $f \in \mathcal{S}$ so does each f_k and the series converges uniformly. Furthermore, since the projection $f \to f_k$ commutes with all elements of G_0 (characters are central) hence all $g \in G$, we also have each $f_k \in \mathcal{S}(\pi, v)$.

Now f_k need not be a product of a radial function times a spherical harmonic. However, its restriction to each sphere is spherical harmonic which must belong to $\mathcal{S}'(\pi, v)$–thus such functions exist as assumed above. Furthermore, if we choose an orthonormal basis Y_{k_j} of spherical harmonics of degree k in $\mathcal{S}'(\pi, v)$ we may write $f_k = \sum Y_{k_j}(x) h_{k_j}(|x|)$. Thus it suffices to prove

$$\mathcal{F}' M\gamma(Yh) = cM\gamma\mathcal{F}(Yh)$$

whenever $Y \in \mathcal{S}'(\pi, v)$ is a spherical harmonic and h is radial. But since $M\gamma Y$ is a spherical harmonic of degree $k + (n - n')/2$ this follows immediately from Bochner's theorem:

$$\mathcal{F}(Yh) = (i/2\pi)^k Y \mathcal{H}_{(n/2)+k}(h)$$

and

$$\mathcal{F}' M\gamma(Yh) = (i/2\pi)^{k+(n-n')/2} M\gamma Y \mathcal{H}_{(n/2)+k}(h).$$

Thus the constant $c = (i/2\pi)^{(n-n')/2}$ which is consistent with $c_1 = 1$ and $c_1 = (2\pi)^{2(n-n')} c^4$.

REMARK. In some of the applications (1.5) will hold for all f in a subspace $\mathcal{S}_0(\pi, v)$ of $\mathcal{S}(\pi, v)$. The above proof shows that (1.3) then follows for all $f \in \mathcal{S}_0(\pi, v)$ provided $\mathcal{S}_0(\pi, v)$ is generated as an $\mathcal{S}_I$-module by the spherical harmonics it contains.

The following lemma gives conditions on M sufficient for (1.5) to hold.

LEMMA 1. *Let $n - n'$ be even and let M be multiplication by a spherical harmonic $M(x')$ homogeneous of degree $(n - n')/2$. Then* (1.5) *holds if and only if either*(2.1) $\Delta' M\gamma Y = 0$ *for every spherical harmonic Y in $\mathcal{S}'(\pi, v)$, or*

(2.2) $2\,\nabla' M \cdot \nabla' \gamma f = M\gamma\Delta'' f$ *for all $f \in \mathcal{S}(\pi, v)$, where Δ'' is the Laplacian in the variables orthogonal to $\mathbf{R}^{n'}$.*

PROOF. Since we are assuming $\Delta' M = 0$ we have

$$\Delta' M\gamma f = M\gamma\Delta' f + 2\,\nabla' M \cdot \nabla' \gamma f.$$

Also $M\gamma\Delta f = M\gamma\Delta' f + M\gamma\Delta'' f$ so (1.5) is equivalent to (2.2). Next assume (2.1), which is a special case of (2.2). By the proof of the theorem it suffices to establish (1.5) for $f = Yh(|x|)$, where Y is a spherical harmonic of degree k in $\mathcal{S}'(\pi, v)$. But

$$\Delta' M\gamma(Yh) = 2\,\nabla'(M\gamma Y) \cdot \nabla' h(|x'|) + M\gamma Y \Delta' h(|x'|)$$

by (2.1), and $M\gamma\Delta(Yh) = 2M\gamma(\nabla Y \cdot \nabla h) + M\gamma Y\gamma(\Delta h)$. An easy computation shows that both sides in (1.5) are equal to

$$M\gamma Y\big(h''(|x'|)\big) + (n - 1 + 2k)h'(|x'|)/|x'|.$$

Thus (2.1) also implies (1.5).

3. **The adjoint representation.** Let G be a semisimple compact group and let R be the adjoint representation of G on its Lie algebra $\mathfrak{G}$, which is orthogonal with respect to minus the Killing form $-B(x,y)$. Let $\mathfrak{H}$ be a maximal abelian subalgebra. Let

$$M(H) = \prod_{j=1}^{m} \alpha_j(H) \tag{3.1}$$

be the product of the positive roots $\alpha_1, \ldots, \alpha_m$ with respect to some ordering.

THEOREM 2. *Let $f \in \mathbb{S}_I$, i.e., let f be any* ad-*invariant function in* $\mathbb{S}(\mathfrak{G})$. *Then for any $H' \in \mathfrak{H}$,*

$$\begin{aligned}\int_{\mathfrak{H}} M(H)f(H)e^{iB(H,H')}\,dH \\ = (-2\pi i)^{-m} M(H')\int_{\mathfrak{G}} f(x)e^{iB(x,H')}\,dx.\end{aligned} \tag{3.2}$$

REMARKS. This is equivalent to a result of Harish-Chandra [4, Theorem 3] which states that for any $f \in \mathbb{S}(\mathfrak{G})$, if

$$T(f)(H) = M(H)\int_G f(\mathrm{ad}\, gH)\,dg \tag{3.3}$$

then

$$T(f)^\wedge = (-2\pi i)^{-m} T(\hat{f}\,). \tag{3.4}$$

Now (3.4) is the same as (3.2) if f is ad-invariant, but it also is a consequence of (3.2) if we apply (3.2) to $\int_G f(\mathrm{ad}\, gH)\,dg$.

We also note that the subgroup of G preserving $\mathfrak{H}$ is the Weyl group W, and if f is ad-invariant then $M\gamma f$ transforms according to the alternating representation of W.

PROOF. If n and n' denote the dimensions of $\mathfrak{G}$ and $\mathfrak{H}$, it is well known that $n - n' = 2m$, so (3.2) is exactly the conclusion of Theorem 1. To show that the theorem applies we will verify that M satisfies (1.5). For this verification it is not necessary to assume that G is compact, so we will formulate it in a separate lemma:

LEMMA 2. *Let G be any semisimple Lie group with Lie algebra $\mathfrak{G}$, and let $\mathfrak{H}$ be any Cartan subalgebra of $\mathfrak{G}$. Let $M(H)$ be given by* (3.1) *where the roots are with respect to $\mathfrak{G}_{\mathbf{C}}$ and $\mathfrak{H}_{\mathbf{C}}$. Then for any* ad-*invariant C^∞ function f on $\mathfrak{G}$,*

$$\Box' M\gamma f = M\gamma \Box f \tag{3.5}$$

where $\Box$ is the Casimir operator $\sum \tilde{b}_{jk}\partial^2/\partial x_j \partial x_k$ with $\tilde{b}_{jk}$ the inverse matrix of the symmetric matrix $B(x_j, x_k)$, and $\Box'$ the restriction to $\mathfrak{H}$.

PROOF. Let us first translate the hypothesis that f is ad-invariant into differential equations. If $y \in \mathfrak{G}$ then $\exp ty \in G$ so

$$f(\mathrm{ad}(\exp ty)x) = f(x)$$

hence

$$\frac{d}{dt} f(\mathrm{ad}(\exp ty)x)\Big|_{t=0} = 0$$

and

$$\frac{d^2}{dt^2} f(\mathrm{ad}(\exp ty)x)\Big|_{t=0} = 0$$

But an easy computation shows

$$\mathrm{ad}(\exp ty)x = x + t[yx] + t^2[y[yx]]/2 + o(t^2)$$

so these equations become

$$D([yx])f(x) = 0 \tag{3.6}$$

and

$$D([yx])^2 f(x) + D([y[yx]])f(x) = 0 \tag{3.7}$$

where $D(z)$ denotes the directional derivative in the z-direction, $D(z)f(x) = (df/dt(x + tz))|_{t=0}$. By polarization of (3.7) we also obtain

$$\begin{aligned} D([yx])D([zx])f(x) + D([zx])D([yx])f(x) \\ + D([y[zx]])f(x) + D([z[yx]])f(x) = 0. \end{aligned} \tag{3.8}$$

These equations provide the relationships between second and first derivatives of f needed to prove (3.5).

Let us choose a basis $H_1, \ldots, H_k, y_1, \ldots, y_{2m}$ for $\mathfrak{G}$ such that $H_1, \ldots, H_k$ forms a basis for $\mathfrak{H}$ and such that the cross-terms $B(H_i, y_j)$ all vanish. Let $\{a_{ij}\}$ be the inverse to the $k \times k$ matrix $B(H_i, H_j)$ and $\{c_{ij}\}$ be the inverse to the $2m \times 2m$ matrix $B(y_i, y_j)$. Then, as in the proof of Lemma 1, we need to show

$$\sum a_{ij} \frac{\partial^2}{\partial H_i \partial H_j} M(H) = 0 \tag{3.9}$$

and

$$2\sum a_{ij} \frac{\partial M}{\partial H_i}(H) \frac{\partial \gamma f}{\partial H_j}(H) = M(H)\gamma \sum c_{ij} \frac{\partial f}{\partial y_i \partial y_j}(H). \tag{3.10}$$

Now the proof of (3.9) is easy. If for each positive root α_r we denote by h_r the element of $\mathfrak{H}_{\mathbf{C}}$ such that $\alpha_r(H) = B(H, h_r)$, then $M(H) = \prod_{r=1}^m B(H, h_r)$ and

$$\sum a_{ij} \frac{\partial}{\partial H_i \partial H_j} M(H) = \sum_{r \neq s} \frac{M(H) B(h_r, h_s)}{B(H, h_r) B(H, h_s)}$$

by a simple computation. But this is a polynomial of degree $m - 2$ which is alternating under the Weyl group action on $\mathfrak{H}_{\mathbf{C}}$. But such a polynomial must contain M as a factor, hence it is zero.

The proof of (3.10) requires a detailed study of the root spaces of $\mathfrak{G}_{\mathbf{C}}$. Recall (see Helgason [6, pp. 140–143]) that to each positive root α_r we can associate elements Z_r and Z_{-r} of $\mathfrak{G}_{\mathbf{C}}$ so that

$$\text{(3.11)} \qquad \begin{cases} [HZ_r] = \alpha_r(H) Z_r, \\ [HZ_{-r}] = -\alpha_r(H) Z_{-r}, \\ [Z_r Z_r] = 0 \quad \text{and} \\ [Z_r Z_{-r}] = h_r. \end{cases}$$

Furthermore

$$\text{(3.12)} \qquad \begin{cases} B(Z_{\pm r}, Z_{\pm s}) = 0 \qquad \text{if } r \neq s, \\ B(Z_r, Z_r) = B(Z_{-r}, Z_{-r}) = 0, \\ B(Z_r, Z_{-r}) = 1 \quad \text{and } B(Z_r, H) = 0. \end{cases}$$

We write $Z_r = X_r + iY_r$ and $Z_{-r} = X_{-r} + iY_{-r}$ where $X_{\pm r}$ and $Y_{\pm r}$ are in $\mathfrak{G}$.

Now we distinguish three types of roots, real roots $\alpha_r = \bar{\alpha}_r$, imaginary roots $\alpha_r = -\bar{\alpha}_r$ (this is the only case that occurs if G is compact) and complex roots $\alpha_r \neq \pm \bar{\alpha}_r$. We choose a basis for a complement of $\mathfrak{H}$ in $\mathfrak{G}$ (denoted $y_1, \ldots, y_{2m}$ above) as follows:

(i) for each positive real root we may choose $Z_{\pm r}$ so that $Y_{\pm r} = 0$; we take then X_r and X_{-r} in the basis;

(ii) For each positive imaginary root we may choose $Z_{-r} = \bar{Z}_r$; we take X_r and Y_r in the basis;

(iii) for each pair of positive complex roots α_r and $\alpha_s = \pm \bar{\alpha}_r$ (one and only one of these is a positive root) we may choose $Z_s = \bar{Z}_{\pm r}$ and $Z_{-s} = \bar{Z}_{\pm r}$; we take then X_r, Y_r, X_{-r} and Y_{-r} in the basis.

We may compute the Killing form with respect to this basis using (3.12) and hence find that the operator $\sum c_{ij} \partial^2 / \partial y_i \partial y_j$ on the right-hand side of (3.10) has the form

$$\text{(3.13)} \qquad \begin{aligned} &\sum{}' 2 \frac{\partial}{\partial X_r} \frac{\partial}{\partial X_{-r}} + \sum{}'' 2 \frac{\partial^2}{\partial X_r^2} + 2 \frac{\partial^2}{\partial Y_r^2} \\ &\quad + \sum{}''' 4 \frac{\partial^2}{\partial X_r \partial X_{-r}} + 4 \frac{\partial^2}{\partial Y_r \partial Y_{-r}} \end{aligned}$$

where Σ', Σ'' and Σ''' denote summation over all positive real, imaginary and pairs of complex roots, respectively.

We are to apply this differential operator to an ad-invariant function f and evaluate at a point $H \in \mathfrak{H}$. If α_r is a real root we use (3.8) with $x = H$, $y = X_r$ and $z = X_{-r}$ to obtain

$$2\alpha_r(H)^2 \frac{\partial^2}{\partial X_r \partial X_{-r}} f(H) = 2\alpha_r(H) D(h_r) f(H)$$

or

$$M(H)\gamma 2 \frac{\partial^2}{\partial X_r \partial X_{-r}} f(H) = \frac{2M(H)}{\alpha_r(H)} D(h_r)\gamma f(H)$$

(here we have used (3.11) to compute brackets). If α_r is an imaginary root we use (3.7) twice with $x = H$ and $y = X_r$ and Y_r and add to obtain

$$M(H)\gamma\left(2\frac{\partial^2}{\partial X_r^2} + 2\frac{\partial^2}{\partial Y_r^2}\right) f(H) = \frac{2M(H)}{\alpha_r(H)} D(h_r)\gamma f(H)$$

(the directional derivative $D(z)$ is defined for $z \in \mathfrak{H}_{\mathbf{C}}$ by $D(z) = D(x) + iD(y)$ if $z = x + iy$, $x, y \in \mathfrak{H}$). Finally if α_r and α_s are complex roots with $\bar{\alpha}_r = \pm \alpha_s$ we use (3.8) twice with $x = H$, $y = Z_r$, $z = Z_{-r}$ and $x = H$, $y = Z_s$, $z = Z_{-s}$ to obtain

$$2D(Z_r)D(Z_{-r})f(H) = \frac{D(h_r)f(H)}{\alpha_r(H)}$$

and

$$2D(Z_s)D(Z_{-s})f(H) = \frac{D(h_s)f(H)}{\alpha_s(H)}.$$

But

$$D(Z_r)D(Z_{-r}) + D(Z_s)D(Z_{-s}) = 2\frac{\partial^2}{\partial X_r \partial X_{-r}} + 2\frac{\partial^2}{\partial Y_r \partial Y_{-r}}$$

so

$$M(H)\gamma\left(4\frac{\partial^2}{\partial X_r \partial X_{-r}} + 4\frac{\partial^2}{\partial Y_r \partial Y_{-r}}\right) f(H)$$
$$= \frac{2M(H)}{\alpha_r(H)} D(h_r)\gamma f(H) + \frac{2M(H)}{\alpha_s(H)} D(h_s)\gamma f(H).$$

Thus the right-hand side of (3.10) is equal to

$$2\sum \frac{M(H)}{\alpha_r(H)} D(h_r)\gamma f(H) \tag{3.14}$$

where the sum is taken over all positive roots. But the left-hand side of (3.10) is also equal to (3.14) since

$$\frac{\partial M}{\partial H_i} = \sum_r \frac{M(H)}{\alpha_r(H)} \frac{\partial B(H, h_r)}{\partial H_i}$$

and

$$\sum a_{ij} \frac{\partial B(H, h_r)}{\partial H_i} \frac{\partial \gamma f}{\partial H_j}(H) = D(h_r)\gamma f(H).$$

4. **Stiefel harmonics.** We consider the group $G = SO(n)$ (or $O(n)$) acting on $n \times m$ real matrices $(m \leqslant n) M_{n \times m}(\mathbf{R})$ by left multiplication. The subspace we consider is $M_{n' \times m}(\mathbf{R})$ for $n' < n$, obtained by setting the bottom $n - n'$ rows equal to zero, and the corresponding group $G' = SO(n')$ (or $O(n')$). If π_ω is the irreducible representation of G with highest weight $\omega = (\omega_1, \ldots, \omega_\mu)$ (here $\mu = [n/2]$) and v a highest weight vector, then $\mathbb{S}(\pi_\omega, v)$ can be described explicitly using results of [8].

We introduce polar coordinates in $M_{n \times m}(\mathbf{R})$ as follows: For any matrix $x \in M_{n \times m}(\mathbf{R})$ of rank m, the matrix $x^t x$ is an invertible positive definite $m \times m$ matrix. Let $p = \sqrt{x^t x}$ be the unique positive definite square-root and set $y = xp^{-1}$. Then $x = yp$ where $y^t y = I_{m \times m}$, so y belongs to the Stiefel manifold S_m^n.

Let us summarize briefly the relevant facts from [8]. Let $a_1 = (1, i, 0, \ldots, 0)$, $a_2 = (0, 0, 1, i, 0, \ldots, 0)$, etc. and $b = (0, \ldots, 0, 1)$ if n is odd. We denote the columns of x by $x_1, \ldots, x_m$. Let A denote any subset of $\{1, \ldots, m\}$ and let $|A|$ denote its cardinality. We define a polynomial $M(A)$ on $M_{n \times m}(\mathbf{R})$ as follows:

(i) if $|A| \leqslant \mu$, $M(A)$ is the determinant of the $|A| \times |A|$ matrix obtained from the $\mu \times m$ matrix $\{a_j \cdot x_k\}$ by selecting the first $|A|$ rows and those columns corresponding to $k \in A$;

(ii) if $|A| > \mu$, $M(A)$ is the determinant of the $|A| \times |A|$ matrix obtained by selecting the first μ rows and the last $|A| - \mu$ rows and those columns corresponding to $k \in A$ from the $n \times m$ matrix

$$\left\{ \begin{array}{c} a_j \cdot x_k \\ \overline{a_j} \cdot x_k \end{array} \right\}$$

if n is even, or

$$\left\{ \begin{array}{c} a_j \cdot x_k \\ \overline{a_j} \cdot x_k \\ b \cdot x_k \end{array} \right\}$$

if n is odd.

Let $\mathcal{Q}$ denote a finite sequence $A_1, A_2, \ldots, A_N$ of nonempty subsets of $\{1, \ldots, m\}$ of decreasing cardinality, $|A_j| \geq |A_{j+1}|$, and satisfying $|A_1| + |A_2| \leq n$. Then let $f(\mathcal{Q}) = \prod_{j=1}^{N} M(A_j)$. If n is even and $|A_1| = n/2$ let $f^-(\mathcal{Q})$ be the polynomial obtained from $f(\mathcal{Q})$ by replacing a_μ by $\overline{a_\mu}$. Finally to $\mathcal{Q}$ we associate the weight ω given by

(i) if $|A_1| < n/2$ then $\omega_j = |\{r\colon |A_r| \geq j\}|$,

(ii) if $|A_1| > n/2$ then $\omega_j = 0$ if $j > n - |A_1|$ and $\omega_j = |\{r\colon |A_r| \geq j\}|$ if $j \leq n - |A_1|$.

In [8] it is proved that $f(\mathcal{Q})$ is a spherical harmonic, $f(\mathcal{Q}) \in \mathcal{S}(\pi_\omega, v)$ and the restriction to S_m^n of all $f(\mathcal{Q})$ corresponding to a weight ω (or if $m \geq n/2$ and $\omega^- = (\omega_1, \ldots, \omega_{\mu-1}, -\omega_\mu)$ then the $f^-(\mathcal{Q})$ with $\mathcal{Q}$ corresponding to $(\omega_1, \ldots, \omega_\mu)$) spans the restriction to S_m^n of all functions in $\mathcal{S}(\pi_\omega, v)$. From this it is easy to deduce

Lemma 3. *Let $f_1, \ldots, f_q$ denote all polynomials $f(\mathcal{Q})$ with $\mathcal{Q}$ corresponding to a fixed weight ω (or $f^-(\mathcal{Q})$ when defined). Then an arbitrary function in $\mathcal{S}(\pi_\omega, v)$ has a representation*

$$\sum_{j=1}^{q} f_j(x)\, g_j(x^t x) \tag{4.1}$$

with $g_j \in \mathcal{S}(M_{m\times m}(\mathbf{R}))$.

Proof. Clearly every function of the form (4.1) is in $\mathcal{S}(\pi_\omega, v)$ by the above. For the converse assume $F(x) \in \mathcal{S}(\pi_\omega, v)$. Then in polar coordinates

$$F(x) = F(yp) \quad (\text{for } x \text{ of rank } m)$$

and since $g^{-1}(yp) = (g^{-1}y)p$ with $g^{-1}y \in S_m^n$ we have

$$F(yp) = \sum_{j=1}^{q} f_j(y) G_j(p)$$

(this representation is not necessarily unique because the f_j need not be linearly independent). Now an examination of the form of the f_j shows that for any fixed $m \times m$ matrix r, $f_j(xr)$ is a linear combination of $f_1(x), \ldots, f_q(x)$. Since $y = xp^{-1}$ and p depends only on $x^t x$ we obtain the form (4.1) for x of rank m. Again the representation is not unique, but if we choose a linearly independent subset of $f_1, \ldots, f_q$ which spans the same space of polynomials (this is somewhat different from the construction in [8] where only restrictions to S_m^n are considered) then the summands $f_j(x)\, g_j(x^t x)$ are uniquely determined, and in fact must be given by a group convolution $\int_G F(g^{-1}x) h_j(g)\, dg$ for appropriate kernels h_j. From this it follows that (4.1) holds for all x and $f_j(x)\, g_j(x^t x)$ and hence $g_j(x^t x)$ are smooth and rapidly decreasing.

THEOREM 3. *Suppose $n > n' \geqslant 2m$ and $n - n' = 2d$ is even, and let $M = M(\{1, \ldots, m\})^d$. If $\omega = (\omega_1, \ldots, \omega_m, 0, \ldots, 0)$ is a dominant weight for $SO(n)$ and $\omega' = (\omega_1 + d, \ldots, \omega_m + d, 0, \ldots, 0)$ the $SO(n')$ weight (with v and v' corresponding weight vectors) then $f \to M\gamma f$ is an isomorphism of $\mathcal{S}(\pi_\omega, v)$ onto $\mathcal{S}(\pi_{\omega'}, v')$ and*

$$\mathcal{F}' M\gamma f = (2\pi i)^{-d} M\gamma \mathcal{F} f \quad \text{for all } f \in \mathcal{S}(\pi_\omega, v). \tag{4.2}$$

PROOF. The fact that $f \to M\gamma f$ maps $\mathcal{S}(\pi_\omega, v)$ onto $\mathcal{S}(\pi_{\omega'}, v')$ follows from the Lemma and the fact that $f(\mathcal{Q})$ is associated to the weight ω if and only if $M(\{1, \ldots, m\})^d \gamma f(\mathcal{Q})$ is associated to the weight ω'. To prove (4.2) we use Theorem 1 and Lemma 1. It suffices to show

$$2\nabla' M \cdot \nabla' \gamma F = M\gamma \nabla'' F \quad \text{for all } F \in \mathcal{S}(\pi_\omega, v) \tag{4.3}$$

since we know by [8] that M is harmonic. By Lemma 3 we may assume $F(x) = f(\mathcal{Q})(x) g(x^t x)$ for $\mathcal{Q}$ associated to the weight ω. But $\nabla' M \cdot \nabla' \gamma f(\mathcal{Q}) = 0$ since $M\gamma f(\mathcal{Q})$ is harmonic and $\gamma\Delta''(f(\mathcal{Q})g) = \gamma f(\mathcal{Q})\gamma\Delta'' g(x^t x)$ since $f(\mathcal{Q})$ does not depend on the bottom $2d$ rows of x (here we use $n' \geqslant 2m$). Thus (4.3) follows from

$$2\nabla' M \cdot \nabla' \gamma g(x^t x) = M\gamma\Delta'' g(x^t x). \tag{4.4}$$

Now we readily compute the right-hand side of (4.4) from the fact that

$$\frac{\partial}{\partial x_{jk}} \left(g(x^t x) \right) = \sum_{p=1}^{m} x_{jp} \frac{\partial g}{\partial u_{kp}} (u)$$

if $u = x^t x$. Since

$$\Delta'' = \sum_{j=n'+1}^{n} \sum_{k=1}^{m} \frac{\partial^2}{\partial x_{jk}^2}$$

and since γ sets equal to zero all the variables x_{jp} for $j \geqslant n' + 1$ we have

$$M\gamma\Delta'' g(x^t x) = (n - n') M \sum_{p=1}^{m} \frac{\partial^2 \gamma g}{\partial u_{pp}^2} (u).$$

The left-hand side of (4.4) is

$$2 \sum_{j=1}^{n'} \sum_{k=1}^{m} \frac{\partial M}{\partial x_{jk}} \sum_{p=1}^{m} x_{jp} \frac{\partial \gamma g}{\partial u_{kp}} (u)$$

so it suffices to show

$$\sum_{j=1}^{n'} \frac{\partial M}{\partial x_{jk}} x_{jp} = d\delta_{pk} M.$$

Since $M = M(\{1, \ldots, m\})^d$ this is equivalent to

$$\sum_{j=1}^{n'} x_{jp} \frac{\partial}{\partial x_{jk}} M(\{1, \ldots, m\}) = \delta_{pk} M(\{1, \ldots, m\}). \tag{4.5}$$

All that remains is to verify the algebraic identity (4.5). Recall that $M(\{1, \ldots, m\})$ is the determinant of the $m \times m$ matrix $\{a_j \cdot x_k\}$. Thus $\sum_{j=1}^{n'} x_{jp}(\partial/\partial x_{jk})M(\{1, \ldots, m\})$ is the determinant of the matrix obtained by substituting the pth column in place of the kth column. If $p \neq k$ we have a matrix with a repeated column hence determinant zero, while if $p = k$ we get the identical matrix hence determinant equal $M(\{1, \ldots, m\})$.

REMARK. If $n' = 2m$, $\omega = (\omega_1, \ldots, \omega_m, 0, \ldots, 0)$ and $\omega' = (\omega_1 + d, \ldots, \omega_{m-1} + d, -\omega_m - d)$ then we obtain an isomorphism of $\mathcal{S}(\pi_\omega, \tilde{v})$ and $\mathcal{S}(\pi_{\omega'}, v')$ and the analogue of (4.2) where now $\tilde{v}$ is a weight vector for the weight $(\omega_1, \ldots, \omega_{m-1}, -\omega_m, 0, \ldots, 0)$ and M is the polynomial obtained from $M(\{1, \ldots, m\})^d$ by replacing a_m by $\overline{a_m}$.

Next we consider the case $n' < 2m$. We should expect far fewer Bochner identities since for $m = 1$ there is only one ($f(x_1, x_2, x_3) \to x_1 f(x_1, 0, 0)$ for radial functions on $\mathbf{R}^3$). In fact we need to have $n = n' + 2$ and the identity holds only for half of $\mathcal{S}(\pi_\omega, v)$. The splitting of $\mathcal{S}(\pi_\omega, v)$ is best described by considering the full orthogonal group $O(n)$. Recall that to every representation π_ω of $SO(n)$ (with $\omega_\mu = 0$ if n is even) there are two distinct representations of $O(n)$ which yield π_ω when restricted to $SO(n)$. We denote them $\pi_\omega^\pm$, and they are distinguished by the condition $\pi_\omega^\pm(g) = \pm 1$ where g is the diagonal matrix with diagonal entries $(1, 1, \ldots, 1, -1)$. Furthermore it is easy to check that $f(\mathcal{A}) \in \mathcal{S}(\pi_\omega^+, v)$ if $|A_1| \leqslant [n/2]$ while $f(\mathcal{A}) \in \mathcal{S}(\pi_\omega^-, v)$ if $|A_1| > [n/2]$.

THEOREM 3'. *Let* $n = n' + 2$ *with* $n' < 2m$, *let* $\omega = (\omega_1, \ldots, \omega_{n'-m}, 0, \ldots, 0)$ *be a dominant weight for* $SO(n)$ (*note we must have* $n' - m < [n/2]$) *and* $\omega' = (\omega_1 + 1, \ldots, \omega_{n'-m} + 1, 0, \ldots, 0)$ *a dominant weight for* $SO(n')$ (*note* $n' - m = [n'/2]$ *only in the odd case* $n' = 2m - 1$), *with* v *and* v' *corresponding weight vectors for the representations* π_ω^+ *of* $O(n)$ *and* π_ω^- *of* $O(n')$. *Finally let* $M = M(\{1, \ldots, m\})$. *Then* $f \to M\gamma f$ *is an isomorphism of* $\mathcal{S}(\pi_\omega^+, v)$ *onto* $\mathcal{S}(\pi_\omega^-, v')$ *and* (4.2) *holds.*

PROOF. The proof is almost the same as before. The only new idea is that if $\mathcal{A} = (A_1, \ldots, A_N)$ is associated with the weight ω and $|A_1| < [n/2]$ then $(\{1, \ldots, m\}, A_1, \ldots, A_N)$ is associated with the weight ω' and $|\{1, \ldots, m\}| > [n'/2]$, and conversely.

5. **Unitary and symplectic spaces.** Here we discuss results analogous to those of the previous section for the unitary groups $U(n)$ and the symplectic groups $Sp(n)$. We will need to use results analogous to those of [8] for these groups, which will be stated without proof. The omitted proofs are long, but are quite similar to those in [8].

We start with the unitary case first. We consider complex $n \times m$ matrices $z \in M_{n \times m}(\mathbf{C})$ $(m \leqslant n)$ with the real quadratic form tr z^*z. The unitary group

$U(n)$ acts on $M_{n\times m}(\mathbf{C})$ by left multiplication. We let $U_m^n = \{z: z^*z = I\}$. As a homogeneous space it may be identified with $U(n)/U(n-m)$. We have a polar coordinate decomposition for every $z \in M_{n\times m}(\mathbf{C})$ of rank m as $z = u|z|$ where $|z|$ is the unique $m \times m$ positive definite square root of z^*z and $u = z|z|^{-1} \in U_m^n$.

Now if $(\omega_1, \ldots, \omega_n)$ is a dominant weight for $U(n)$, π_ω the corresponding representation and v a corresponding weight vector, we wish to define a collection of spherical harmonics in $\mathcal{S}(\pi_\omega, v)$ analogous to what we had for $SO(n)$. We do this by considering pairs $\mathcal{A}$, $\mathcal{B}$ with $\mathcal{A} = (A_1, \ldots, A_r)$ and $\mathcal{B} = \{B_1, \ldots, B_s\}$ where each A_j and B_j is a subset of $\{1, \ldots, m\}$ subject to the conditions of decreasing cardinality, $|A_j| \geqslant |A_{j+1}|$ and $|B_j| \geqslant |B_{j+1}|$, and that $|A_1| + |B_1| \leqslant n$. For each such pair (we allow $\mathcal{A}$ or $\mathcal{B}$ to be empty, with the obvious modifications) we define a polynomial on $M_{n\times m}(\mathbf{C})$

$$f(\mathcal{A}, \mathcal{B}) = \prod_{j=1}^{r} M(A_j) \prod_{k=1}^{s} N(B_k)$$

where $M(A)$ denotes the determinant of the $|A| \times |A|$ submatrix of z obtained by selecting the first $|A|$ rows and the columns corresponding to elements of A, while $N(B)$ denotes the determinant of the $|B| \times |B|$ submatrix of $\bar{z}$ obtained by selecting the *last* $|B|$ rows and the columns corresponding to elements of B.

THEOREM A. *$f(\mathcal{A}, \mathcal{B})$ is a spherical harmonic (in fact it is annihilated by every $U(n)$-invariant constant coefficient differential operator without constant term) and $f(\mathcal{A}, \mathcal{B}) \in \mathcal{S}(\pi_\omega, v)$ where $\omega = (\omega_1, \ldots, \omega_n)$ is given by*

$$\omega_j = |\{k: |A_k| \geqslant j\}| \quad \text{if } j \leqslant |A_1|,$$
$$\omega_j = 0 \quad \text{if } j > |A_1| \text{ and } j < n - |B_1|,$$
$$\omega_j = -|\{k: |B_k| \geqslant n + 1 - j\}| \quad \text{if } j \geqslant n - |B_1|.$$

Furthermore, when restricted to U_m^n, the functions $f(\mathcal{A}, \mathcal{B})$ span the restriction of $\mathcal{S}(\pi_\omega, v)$ to U_m^n.

REMARKS. Let us say that a pair $\mathcal{A}$, $\mathcal{B}$ is admissible if

(i) $|\{r \in A_1: r \leqslant m - j\}| + |\{r \in B_1: r \leqslant m - j\}| \leqslant n - j$ for $j = 0, 1, \ldots, m-1$;

(ii) if we write $A_j = \{i_1, \ldots, i_p\}$ and $A_{j+1} = \{i'_1, \ldots, i'_q\}$ in increasing order ($i_1 < i_2 < \cdots$ and $i'_1 < i'_2 < \ldots$) then $i'_k \geqslant i_k$ for $k \leqslant q$ (note $q \leqslant p$) and

(iii) the analogous condition for B_j and B_{j+1}.

(If either $\mathcal{A}$ or $\mathcal{B}$ are empty the conditions for them are dropped.) The restrictions to U_m^n of the functions $f(\mathcal{A}, \mathcal{B})$ for admissible pairs are linearly independent and span the restriction of $\mathcal{S}(\pi_\omega, v)$ to U_m^n (together with their translates they span $L^2(U_m^n)$).

Also we may obtain spherical harmonics in $\mathcal{S}(\pi_\omega, v)$ which are invariant under right multiplication by matrices in $U(m)$ giving explicitly each representation which occurs (with multiplicity one) in the regular representation of the symmetric space $U(n)/U(n-m) \times U(m)$. These have the form (take $m \leqslant n-m$) $\prod_{k=1}^m F_k^{r_k}$ for any m-tuple of nonnegative integers $(r_1, \dots, r_m)$ where

$$F_k = \sum_{|A|=k} M(A)N(A)$$
$$= \frac{1}{k!} \sum_{i_1=1}^m \cdots \sum_{i_k=1}^m M(\{i_1, \dots, i_k\})N(\{i_1, \dots, i_k\}).$$

The associated weight ω is given by

$$\omega_j = \sum_{k=j}^m r_k \quad \text{if } j \leqslant m,$$
$$\omega_j = 0 \quad \text{if } m < j \leqslant n-m,$$
$$\omega_j = -\sum_{k=n-j+1}^m r_k \quad \text{if } n-m < j \leqslant n.$$

These results will not be used in the sequel.

LEMMA 4. *An arbitrary function in $\mathcal{S}(\pi_\omega, v)$ has a representation*

$$\sum_{j=1}^q f_j(z) g_j(z^* z)$$

with $g_j \in \mathcal{S}(M_{m\times m}(\mathbf{C}))$ where $f_1, \dots, f_q$ are all polynomials $f(\mathcal{A}, \mathcal{B})$ for pairs $\mathcal{A}$, $\mathcal{B}$ corresponding to the weight ω (if $n > 2m$ then $\mathcal{S}(\pi_\omega, v)$ is empty unless $\omega_{m+1} = \cdots = \omega_{n-m} = 0$).

PROOF. Identical to Lemma 3.

We consider now Bochner identities where $G = U(n)$, $G' = U(n')$ for $n \geqslant n' \geqslant 2m$ and γ sets equal to zero the rows $m+1, \dots, m+n-n'$ of $z \in M_{n\times m}(\mathbf{C})$ (in a natural way γf can be regarded as a function on $M_{n'\times m}(\mathbf{C})$). Note that the difference in dimension is always even, $2m(n-n')$.

THEOREM 4. *Let $n \geqslant n' \geqslant 2m$, let*

$$\omega = (\omega_1, \dots, \omega_m, 0, \dots, 0, \omega_{n-m+1}, \dots, \omega_n)$$

be a dominant weight for U_n, and let s and t be integers (possibly negative) satisfying $s + t = n - n'$, $\omega_m + s \geqslant 0$ and $\omega_{n-m+1} - t \leqslant 0$. We set $M = M(\{1, \dots, m\})^s N(\{1, \dots, m\})^t$. Then $f \to M\gamma f$ is an isomorphism of $\mathcal{S}(\pi_\omega, v)$ onto $\mathcal{S}(\pi_{\omega'}, v')$ where

$$\omega' = (\omega_1 + s, \ldots, \omega_m + s, 0, \ldots, 0, \omega_{n-m+1} - t, \ldots, \omega_n - t)$$

is a dominant $U(n')$ *weight, and furthermore*

$$\mathfrak{F}' M\gamma f = (2\pi i)^{-m(n-n')} M\gamma \mathfrak{F} f$$

for all $f \in \mathbb{S}(\pi_\omega, v)$.

PROOF. There are some technical problems if either s or t is negative, for then M has singularities. These problems may be circumvented, however, by repeated application of the result in the case of nonnegative s and t (if say $s < 0$ just compare f and $M\gamma f$ with $M(\{1, \ldots, m\})^{|s|} f$ defined on $M_{(n+|s|)\times m}(\mathbf{C})$). Thus we may assume M is a polynomial.

Repeating the arguments in the proof of Theorem 3, we see that the crucial identity we must prove is

$$2\nabla' M \cdot \nabla' \gamma g(z^* z) = M\gamma \Delta'' g(z^* z)$$

where

$$\Delta'' = 4 \sum_{k=1}^{m} \sum_{j=m+1}^{m+n-n'} \frac{\partial}{\partial \bar{z}_{jk}} \frac{\partial}{\partial z_{jk}}$$

and

$$2\nabla' M \cdot \nabla' \gamma g(z^* z) = 4 \sum_{k=1}^{m} \sum_{j\in D} \frac{\partial M(z)}{\partial z_{jk}} \frac{\partial \gamma g(z^* z)}{\partial \bar{z}_{jk}} + \frac{\partial M(z)}{\partial \bar{z}_{jk}} \frac{\partial \gamma g(z^* z)}{\partial z_{jk}}$$

(here $D = \{1, \ldots, m, m+n-n'+1, \ldots, n\}$). To prove this we use (here $w = z^* z$)

$$\frac{\partial}{\partial z_{jk}} g(z^* z) = \sum_{i=1}^{m} \bar{z}_{ji} \left(\frac{\partial g}{\partial w_{ik}}(z^* z) + \frac{\partial g}{\partial \bar{w}_{ki}}(z^* z) \right)$$

and

$$\frac{\partial}{\partial \bar{z}_{jk}} g(z^* z) = \sum_{i=1}^{m} z_{ji} \left(\frac{\partial g}{\partial \bar{w}_{ik}}(z^* z) + \frac{\partial g}{\partial w_{ki}}(z^* z) \right).$$

Since γ sets equal to zero the variables z_{jk} for $m < j \leqslant m+n-n'$ we find

$$M\gamma \Delta'' g(z^* z) = 4(n-n') M \cdot \sum_{k=1}^{m} \left(\frac{\partial \gamma g}{\partial w_{kk}}(z^* z) + \frac{\partial \gamma g}{\partial \bar{w}_{kk}}(z^* z) \right).$$

On the other hand,

$$2\,\nabla' M\cdot\nabla'\gamma g(z^*z)=4\sum_{k=1}^{m}\sum_{j\in D}\sum_{i=1}^{m} z_{ji}\frac{\partial M}{\partial z_{jk}}\left(\frac{\partial\gamma g}{\partial w_{ki}}(z^*z)+\frac{\partial\gamma g}{\partial\bar{w}_{ik}}(z^*z)\right)$$
$$+\bar{z}_{ji}\frac{\partial M}{\partial\bar{z}_{jk}}\left(\frac{\partial\gamma g}{\partial w_{ik}}(z^*z)+\frac{\partial\gamma g}{\partial\bar{w}_{ki}}(z^*z)\right)$$
$$=4(s+t)M\cdot\sum_{k=1}^{m}\left(\frac{\partial\gamma g}{\partial w_{kk}}(z^*z)+\frac{\partial\gamma g}{\partial\bar{w}_{kk}}(z^*z)\right)$$

since we have

$$\sum_{j\in D} z_{ji}\frac{\partial M}{\partial z_{jk}}=s\delta_{ik}M$$

and

$$\sum_{j\in D}\bar{z}_{ji}\frac{\partial M}{\partial\bar{z}_{jk}}=t\delta_{ik}M.$$

Next we consider the case $2m > n' \geqslant m$. Here we have Bochner identities where the mapping has the form $f\to M_1\gamma M_2^{-1}f$ (in fact we may have $\gamma f\equiv 0$ for the class of functions considered). We must assume that the $U(n)$ weight ω satisfies either $\omega_j = s \geqslant 0$ for $n'-m<j\leqslant m$ and $\omega_{n-m+1}\leqslant 0$, or $\omega_j = -t\leqslant 0$ for $n-m<j\leqslant n-n'+m$ and $\omega_m\geqslant 0$. We will discuss the first case only, leaving the essentially symmetric second case to the reader.

THEOREM 4′. *Let $n\geqslant n'\geqslant m$ but $n'<2m$, and let s and t be nonnegative integers such that $t-s=n-n'$. Let ω be a dominant weight for $U(n)$ such that $\omega_j=s$ for $n'-m<j\leqslant m$ and $\omega_{n-m+1}\leqslant 0$ and let $\omega'=(\omega'_1,\dots,\omega'_{n'})$ be the dominant $U(n')$ weight given by*

$$\omega'_j=\begin{cases}\omega_j-s & \text{if } 1\leqslant j\leqslant n'-m,\\ -t & \text{if } n'-m<j\leqslant m+n'-n,\\ \omega_{j+n-n'}-t & \text{if } m+n'-n<j\leqslant n',\end{cases}$$

in case $n<2m$, and

$$\omega'_j=\begin{cases}\omega_j-s & \text{if } 1\leqslant j\leqslant n'-m,\\ \omega_{j+n-n'}-t & \text{if } n'-m<j\leqslant n',\end{cases}$$

in case $n\geqslant 2m$. Let γ be the restriction to the set of $z\in M_{n\times m}(\mathbf{C})$ with rows $n'-m+1$ through $n-m$ equal to zero (we identify γf naturally as a function on $M_{n'\times m}(\mathbf{C})$). Let $M_2=M(\{1,\dots,m\})^s$ on $M_{n\times m}(\mathbf{C})$ and let $M_1=N(\{1,\dots,m\})^t$ on $M_{n'\times m}(\mathbf{C})$. Then $f\to M_1\gamma M_2^{-1}f$ is an isomorphism of $\mathcal{S}(\pi_\omega,v)$ onto $\mathcal{S}(\pi_{\omega'},v')$ and

$$\mathfrak{F}' M_1 \gamma M_2^{-1} f = (2\pi i)^{-m(n-n')} M_1 \gamma M_2^{-1} \mathfrak{F} f$$

for all $f \in \mathcal{S}(\pi_\omega, v)$.

PROOF. Let $n'' = n + s = n' + t$ and let ω'' be the dominant $U(n'')$ weight

$$\omega_j'' = \begin{cases} \omega_j' & \text{if } 1 \leqslant j \leqslant n' - m, \\ 0 & \text{if } n' - m < j \leqslant m + s, \\ \omega_{j-s} & \text{if } m + s < j \leqslant n + s. \end{cases}$$

For functions F defined on $M_{n'' \times m}(\mathbf{C})$ let $\gamma_1 F$ (and $\gamma_2 F$ respectively) be the restriction to those $z \in M_{n'' \times m}(\mathbf{C})$ with rows $n' - m + 1$ through $n'' - n + n' - m$ ($n'' - m$ respectively) equal to zero. Note that $\gamma_1 F$ ($\gamma_2 F$ respectively) may be naturally identified with a function on $M_{n' \times m}(\mathbf{C})$ ($M_{n \times m}(\mathbf{C})$ respectively), and with these identifications $\gamma\gamma_2 = \gamma_1$.

Now by essentially the same proof as before we have that $f \to M_1 \gamma_1 F$ ($M_2 \gamma_2 F$ respectively) is an isomorphism of $\mathcal{S}(\pi_{\omega''}, v'')$ onto $\mathcal{S}(\pi_{\omega'}, v')$ ($\mathcal{S}(\pi_\omega, v)$ respectively) and that

$$\mathfrak{F}' M_1 \gamma_1 F = (2\pi i)^{-m(n''-n')} M_1 \gamma_1 \mathfrak{F}'' F$$

and

$$\mathfrak{F} M_2 \gamma_2 F = (2\pi i)^{-m(n''-n)} M_2 \gamma_2 \mathfrak{F}'' F.$$

Now any $f \in \mathcal{S}(\pi_\omega, v)$ has the form $f = M_2 \gamma_2 F$ for some $F \in \mathcal{S}(\pi_{\omega''}, v'')$, and conversely, so $M_1 \gamma M_2^{-1} f = M_1 \gamma_1 F \in \mathcal{S}(\pi_{\omega'}, v')$ and the mapping is onto. Furthermore we have

$$\begin{aligned} \mathfrak{F}' M_1 \gamma M_2^{-1} f &= \mathfrak{F}' M_1 \gamma_1 F \\ &= (2\pi i)^{-m(n''-n')} M_1 \gamma_1 \mathfrak{F}'' F = (2\pi i)^{-m(n''-n')} M_1 \gamma M_2^{-1} (M_2 \gamma_2 \mathfrak{F}'' F) \\ &= (2\pi i)^{-m(n-n')} M_1 \gamma M_2^{-1} \mathfrak{F} M_2 \gamma_2 F = (2\pi i)^{-m(n-n')} M_1 \gamma M_2^{-1} \mathfrak{F} f. \end{aligned}$$

Next we discuss actions of the symplectic groups $Sp(n)$. We consider "symplectic $n \times m$ matrices" $\zeta \in M_{n \times m}(\mathbf{Q})$ for $m \leqslant n$ defined by

$$M_{n \times m}(\mathbf{Q}) = \left\{ \zeta = \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} : z, w \in M_{n \times m}(\mathbf{C}) \right\}.$$

This is a $4nm$ real dimensional space with quadratic form

$$\tfrac{1}{2} \operatorname{tr} \zeta^* \zeta = \operatorname{tr}(z^* z + w^* w).$$

We let

$$Q_m^n = \{ \zeta \in M_{n \times m}(\mathbf{Q}) : \zeta^* \zeta = I \}.$$

Then $Q_n^n = Sp(n)$ and $Sp(n)$ acting on $M_{n \times m}(\mathbf{Q})$ by left multiplication preserves Q_m^n. As a homogeneous space Q_m^n may be identified with

$Sp(n)/Sp(n-m)$. Regarding $M_{n\times m}(\mathbf{Q})$ as a subspace of $M_{2n\times 2m}(\mathbf{C})$ we have a polar coordinate decomposition $\zeta = u|\zeta|$, and it is easy to check that $u \in Q_m^n$ and $|\zeta| \in M_{m\times m}(\mathbf{Q})$.

Now if $\omega = (\omega_1, \ldots, \omega_n)$ is a dominant weight for $Sp(n)$, π_ω the corresponding representation and v a corresponding weight vector, we wish to describe a sufficiently large collection of spherical harmonics in $\mathcal{S}(\pi_\omega, v)$. Let A denote any subset of $\{1, \ldots, 2m\}$ of cardinality $\leqslant n$, and let $M(A)$ denote the determinant of the $|A| \times |A|$ submatrix of ζ obtained by selecting the first $|A|$ rows and the columns corresponding to elements of A. If $\mathcal{Q} = (A_1, \ldots, A_N)$ is a sequence of such subsets let

$$f(\mathcal{Q}) = \prod_{j=1}^{N} M(A_j).$$

THEOREM B. *$f(\mathcal{Q})$ is a spherical harmonic (in fact it is annihilated by every $Sp(n)$-invariant constant coefficient differential operator without constant term) and $f(\mathcal{Q}) \in \mathcal{S}(\pi_\omega, v)$ where $\omega = (\omega_1, \ldots, \omega_n)$ is given by $\omega_j = |\{k: |A_k| \geqslant j\}|$. Furthermore, when restricted to Q_m^n, the functions $f(\mathcal{Q})$ span the restriction of $\mathcal{S}(\pi_\omega, v)$ to Q_m^n.*

REMARKS. We may again describe a linearly independent (on Q_m^n) set of functions which span the restriction of $\mathcal{S}(\pi_\omega, v)$ to Q_m^n by considering only admissible sequences $\mathcal{Q}$. To give the definition of admissibility it is convenient to associate with each set A the pair of subsets B and C of $\{1, 2, \ldots, m\}$ given by

$$B = A \cap \{1, \ldots, m\}$$

and

$$C = \{k: m + k \in A \cap \{m+1, \ldots, 2m\}\}.$$

We say $\mathcal{Q}$ is admissible if

(i) $$|\{r \in B_j: r \leqslant k\}| + |\{r \in C_j: r \leqslant k\}| \geqslant |\{r \in B_{j+1}: r \leqslant k\}| + |\{r \in C_{j+1}: r \leqslant k\}| \quad \text{for } k = 1, \ldots, m;$$

(ii) $$|\{r \in B_j: r \leqslant k\}| + |\{r \in C_j: r \leqslant k-1\}| \geqslant |\{r \in B_{j+1}: r \leqslant k\}| + |\{r \in C_{j+1}: r \leqslant k-1\}| \quad \text{for } k = 1, \ldots, m;$$

and

(iii) $$|\{r \in B_1: r \leqslant k\}| + |\{r \in C_1: r \leqslant k\}| \leqslant n + k - m \quad \text{for } k = 1, \ldots, m.$$

The proof is based on the branching theorem for $Sp(n)$ given in [5]. Also we may obtain spherical harmonics in $\mathbb{S}(\pi_\omega, v)$ which are invariant under right multiplication by matrices in $Sp(m)$ giving explicitly each representation which occurs (with multiplicity one) in the regular representation of the symmetric space $Sp(n)/Sp(n-m) \times Sp(m)$. These have the form (when $m \leq n-m$) $\prod_{k=1}^m F_k^{r_k}$ for any m-tuple of nonnegative integers $(r_1, \ldots, r_m)$ where

$$F_k = \sum_{\substack{B=C \\ |B|=K}} M(A) = \frac{1}{k!} \sum_{i_1=1}^m \cdots \sum_{i_k=1}^m M(\{i_1, \ldots, i_k, m+i_1, \ldots, m+i_k\}).$$

The corresponding weight ω is given by

$$\omega_{2j-1} = \omega_{2j} = \sum_{k=j}^m r_k \quad \text{if } j \leq m,$$

$$\omega_j = 0 \quad \text{if } j > 2m.$$

LEMMA 5. *An arbitrary function in $\mathbb{S}(\pi_\omega, v)$ has a representation*

$$\sum_{j=1}^q f_j(\zeta) g_j(\zeta^* \zeta)$$

with $g_j \in \mathbb{S}(M_{m\times m}(\mathbf{Q}))$ where $f_1, \ldots, f_q$ are all polynomials $f(\mathcal{Q})$ for $\mathcal{Q}$ corresponding to the weight ω (if $n > 2m$ then $\mathbb{S}(\pi_\omega, v)$ is empty unless $\omega_j = 0$ for $j > 2m$).

We consider now Bochner identities where $G = Sp(n)$, $G' = Sp(n')$ and γ restricts to those $\zeta \in M_{n\times m}(\mathbf{Q})$ with the last $n - n'$ rows of z and w equal to zero (naturally identified with $M_{n'\times m}(\mathbf{Q})$). We only have results in the case $n > n' \geq 2m$. Note that the difference in dimensions is always even, $4m(n-n')$.

THEOREM 5. *Let $n > n' \geq 2m$ and let $\omega = (\omega_1, \ldots, \omega_{2m}, 0, \ldots, 0)$ be a dominant weight for $Sp(n)$. Set $d = n - n'$ and $\omega' = (\omega_1 + d, \ldots, \omega_{2m} + d, 0, \ldots, 0)$ the dominant $Sp(n')$ weight. Also set $M = M(\{1, \ldots, 2m\})^d$. Then $f \to M\gamma f$ is an isomorphism of $\mathbb{S}(\pi_\omega, v)$ onto $\mathbb{S}(\pi_{\omega'}, v')$ and $\mathfrak{F}' M\gamma f = (2\pi i)^{-2md} M\gamma \mathfrak{F} f$ for all $f \in \mathbb{S}(\pi_\omega, v)$.*

PROOF. Almost identical to the proof of Theorem 4.

REFERENCES

1. S. Bochner, *Theta relations with spherical harmonics*, Proc. Nat. Acad. Sci. U.S.A. **37** (1951), 804–808. MR **14**, 43.

2. H. Boerner, *Representations of groups with special consideration for the needs of modern physics*, North-Holland, Amsterdam; Interscience, New York, 1963. MR **26** #6272.

3. S. Gelbart, *A theory of Stiefel harmonics*, Trans. Amer. Math. Soc. **192** (1974), 29–50.

4. Harish-Chandra, *Differential operators on a semisimple Lie algebra*, Amer. J. Math. **79** (1957), 87–120. MR **18**, 809.

5. G. C. Hegerfeldt, *Branching theorem for the symplectic groups*, J. Mathematical Phys. **8** (1967), 1195–1196. MR **37** #339.

6. S. Helgason, *Differential geometry and symmetric spaces*, Pure and Appl. Math., vol. 12, Academic Press, New York, 1962. MR **26** #2986.

7. C. Herz, *Bessel functions of matrix argument*, Ann. of Math. (2) **61** (1955), 474–523. MR **16**, 1107.

8. R. Strichartz, *The explicit Fourier decomposition of* $L^2(SO(n)/SO(n-m))$, Canad. J. Math. **27** (1975), 294–310.

9. ______, *Fourier transforms and non-compact rotation groups*, Indiana Univ. Math. J. **24** (1974), 499–526.

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