CONE BUNDLES

BY

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Abstract. A theory of normal bundles for locally knotted codimension two embeddings of PL manifolds is developed. The classifying space for this theory is Cappell and Shaneson's space $BRN_2$.

Cone bundles are a generalization of blockbundles [9], allowing local knotting of the base in the total space. They are a type of "mockbundle" [8], [2], closely related to the theory of stratified polyhedra [12], and designed to provide a simple foundation for Cappell and Shaneson's theory of singularities of PL embeddings [3], [4]. A similar definition has been given by Matumoto and Matsumoto [6].

§1 contains the basic definitions. A classifying space for cone bundles is constructed in §2. §3 contains a proof that the total space of a cone bundle over a manifold is a manifold. Finally, cone bundles are related to the topology of stratified polyhedra in §4.

The geometric idea for cone bundles comes from my paper [7] on cone complexes. I thank Sylvain Cappell for encouraging me to develop this idea.

I will work in the category of polyhedra and piecewise linear maps [11]. In particular, all manifolds and homeomorphisms will be piecewise linear.

1. Thickening. Let $M$ be a compact $n$-manifold. A codimension $q$ thickening of $M$ is a compact $(n + q)$-manifold $W$ containing $M$ as a subpolyhedron, such that $W$ collapses to $M$. Furthermore, $\partial W \cap M = \partial M$, and there is a collar of $\partial W$ in $W$ which restricts to a collar of $\partial M$ in $M$. (This is called a "very proper" thickening in [4].)

The thickenings $V$ and $W$ of $M$ are equivalent if there is a homeomorphism between $V$ and $W$ which is the identity on $M$.

If $q > 2$, a codimension $q$ thickening $W$ of $M$ is an "abstract regular neighborhood" of $M$ [9, p. 14], since $M$ is locally flat in $W$. But if $q = 2$, $M$ can be locally knotted in $W$. $M$ can be locally knotted in a codimension one thickening if and only if the PL Schoenflies conjecture is false.

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Let $K$ be a (PL) cell complex. A $q$-cone bundle $\xi/K$ consists of a polyhedron $E(\xi)$ containing $|K|$ such that

(i) For each $p$-cell $\sigma_i \in K$ there is a $(p + q)$-ball $\beta_i \subset E(\xi)$, containing $\sigma_i$, such that $(\beta_i, \sigma_i)$ is homeomorphic with the cone on a sphere pair. (N.B. this sphere pair may be nonlocally flat or knotted.) $\beta_i$ is called the block over $\sigma_i$.

(ii) $E(\xi)$ is the union of the blocks $\beta_i$.

(iii) The interiors of the blocks are disjoint.

(iv) $\beta_i \cap \beta_j$ is the union of the blocks over the cells contained in $\sigma_i \cap \sigma_j$.

By the Zeeman unknotting theorem, a $q$-cone bundle is a blockbundle [9] if $q > 2$, i.e., $(\beta_i, \sigma_i)$ is homeomorphic with a standard sphere pair for each $\sigma_i \in K$. The following results show that cone bundles bear essentially the same relation to thickenings that blockbundles bear to abstract regular neighborhoods.

**Lemma 1.** Let $N$ be a manifold containing the manifold $M$ as a subpolyhedron. Suppose that $\partial N \cap M = \partial M$ and $(\partial N, \partial M)$ is collared in $(N, M)$. Then there is a cone bundle $\xi/K$ with $|K| = M$ such that $E(\xi)$ is a regular neighborhood of $M$ in $N$.

**Proof.** $\xi$ is constructed in the same manner as a normal blockbundle for a locally flat submanifold, using dual cells. (The usual construction must be slightly modified near the boundary; cf. [7, p. 284].)

**Lemma 2.** If $\xi/K$ is a cone bundle, and $|K|$ is a compact manifold, then $E(\xi)$ is a thickening of $K$.

**Proof.** It is easy to see that $E(\xi)$ collapses to $|K|$, since $E(\xi)$ can be triangulated as a stellar neighborhood of $|K|$, by induction on the dimension of $\xi$ (cf. [7, p. 274]). The fact that $E(\xi)$ is a manifold will be proved in §3.

**Lemma 3.** If $W$ is a thickening of the compact manifold $M$, there is a cone bundle $\xi/K$, $|K| = M$, such that the thickening $E(\xi)$ of $M$ is equivalent to $W$.

**Proof.** This follows from Lemma 1 and the uniqueness of regular neighborhoods [11, p. 33].

**Remarks.** (1) If $\xi/K$ and $\eta/L$ are cone bundles with $|K| = |L| = M$, and the thickenings $E(\xi)$ and $E(\eta)$ of $M$ are equivalent, one might expect (by analogy with blockbundles) that $\xi$ and $\eta$ have isomorphic "subdivisions". However, this is not true—one has to introduce the weaker relation of concordance (§2) in order to get a bijection between classes of bundles and classes of thickenings.

(2) In Matumoto and Matsumoto’s definition of “$RN_2$-bundles” [6], condition (i) in the definition of a 2-cone bundle is weakened to the condition that $(\beta_i, \sigma_i)$ is an arbitrary ball pair. Lemma 2 is not true for their bundles, since the total space need not collapse to the base space.
A 2-cone bundle $\xi/K$ has a canonical "Noguchi characteristic class" $n \in H^2(K; \gamma)$ (twisted coefficients), where $\gamma$ is the Fox-Milnor cobordism group (cf. [4]). $n$ is represented by the cocycle which assigns to each 2-cell $\sigma \in K$, the cobordism class of the knot $(\partial \beta, \partial \sigma)$, where $\beta$ is the block over $\sigma$. Thus $n$ is the primary obstruction to making $\xi$ a blockbundle. (The analogous higher obstructions are not defined a priori since $\partial \sigma_i$ is not necessarily locally flat in $\partial \beta_i$ if $\dim \sigma_i > 2$.)

2. A classifying space. The following definitions come from [9].

If $\xi/K$ is a cone bundle and $L$ is a subcomplex of $K$, the restriction $\xi|L$ is defined by putting $\beta_i(\xi|L) = \beta_i(\xi)$ for each $\sigma_i \in L$.

The cone bundles $\xi_0$, $\xi_1/K$ are isomorphic if there is a homeomorphism $h$: $E(\xi_0) \rightarrow E(\xi_1)$ such that $h$ is the identity on $|K|$ and $h(\beta_i(\xi_0)) = \beta_i(\xi_1)$ for each $\sigma_i \in K$.

The cone bundles $\xi_0$, $\xi_1/K$ are concordant if there is a cone bundle $\eta/(K \times I)$ such that $\eta((K \times \{i\})$ is isomorphic with $\xi_i$, $i = 0, 1$. Here $I = [0, 1]$ and $K \times I$ is the usual product complex. (Two blockbundles are concordant if and only if they are isomorphic [9, p. 6]. This is not true for 2-cone bundles.)

We will construct a classifying space for concordance classes of 2-cone bundles analogous to the classifying space $B\overline{P}L_q$ for $q$-blockbundles. (The same construction also works for 1-cone bundles.)

Let $\mathcal{C}(K)$ be the set of concordance classes of 2-cone bundles over $K$. $\mathcal{C}$ is a contravariant functor from the category with objects PL cell complexes and morphisms generated by isomorphisms and inclusions of subcomplexes, to the category of (based) sets. (The base point of $\mathcal{C}(K)$ is the class of the trivial bundle over $K$.)

**Theorem 1.** $\mathcal{C}$ has a unique extension to the category of CW complexes and homotopy classes of maps.

**Proof.** This is a corollary of the "mockbundle" recipe for homotopy functors [2, 1]. It is clear that cone bundles can be glued (axiom G [2, p. 15]), so we only have to verify the extension axiom (E, [2, p. 15]). That is, if $e$: $K_0 \rightarrow K$ is an elementary expansion, and $\xi_0/K_0$ is a cone bundle, we must construct a cone bundle $\xi/K$ such that $\xi|K_0 = \xi_0$. We follow [2, p. 21]. $K = K_0 \cup \{\sigma, \tau\}$, where $\sigma$ is a principal cell of $K$ and $\tau$ is a free face of $\sigma$. Let $J$ be the subcomplex of $\partial \sigma$ consisting of all the faces of $\sigma$ except $\tau$. Now $|J|$ is a ball, and $E(\xi_0|J)$ is a thickening of $|J|$ by Lemma 2, so $E(\xi_0|J)$ is a ball. Let $(B, C)$ denote the ball pair $(E(\xi_0|J), |J|)$. Identifying $(\sigma, C, \tau)$ with $(C \times I, C \times \{0\}, (\partial C \times I) \cup (C \times \{1\}))$, we define the extension $\xi/K$ as follows. The block of $\xi$ over $\sigma$ is the cone on the boundary of the ball $B \times I$, and the block over $\tau$ is the cone on the boundary of the ball...
\[ \text{where } B^* = \text{cl}[\partial(B \times I) \setminus (B \cup (B^* \times I))], \]

where \( B^* = \text{cl}(\partial B \setminus \bigcup \beta_i(\xi_0)), \) union over all \( i \) such that \( \sigma_i \subset \partial C. \)

**Theorem 2.** \( \mathcal{K} \) is a representable functor.

**Proof.** Let \( G \) be the (based) \( \Delta \)-set whose \( k \)-simplexes are \( 2 \)-cone bundles over \( \Delta^k \) (the standard \( k \)-simplex) which are embedded blockwise in \( \Delta^k \times R^\infty \), and let \( \gamma \) be the canonical cone bundle on \( G \) (cf. [8, p. 131] and [2, p. 37]). \( G \) is a Kan \( \Delta \)-set by the extension axiom (see the proof of Theorem 1) and general position. It follows that if \( \mathcal{G} \) is the realization of \( G \), pulling back the class of \( \gamma \) induces a bijection

\[ \mathcal{K}(X) = [X, \mathcal{G}] \]

for all \( CW \) complexes \( X \), where \([ , ]\) denotes homotopy classes of maps (cf. [10, §6] and [9, §2]).

Theorems 1 and 2 are also true for Matumoto and Matsumoto's \( RN_2 \)-bundles, by the same proofs. As they have pointed out to me, any \( RN_2 \)-bundle is concordant to a cone bundle by the Alexander trick, so the corresponding homotopy functors are the same.

The thickenings \( W_0 \) and \( W_1 \) of the \( n \)-manifold \( M \) are *concordant* if there is a thickening \( Q \) of \( M \times I \) such that \( W_i \) is a regular neighborhood of \( M \times \{ i \} \) rel \( \partial M \times \{ i \} \) in \( \partial Q, i = 0, 1 \) (cf. [4]).

**Theorem 3.** If \( M \) is a compact \( n \)-manifold, \( \xi \mapsto E(\xi) \) induces a bijection between \( \mathcal{K}(M) \) and concordance classes of codimension 2 thickenings of \( M \).

**Proof.** Every thickening of \( M \) is in fact equivalent to \( E(\xi) \) for some \( \xi \) over \( M \), by Lemma 3. On the other hand, given \( \xi_0 \) and \( \xi_1 \), and a concordance \( Q \) between \( E(\xi_0) \) and \( E(\xi_1) \), a concordance between \( \xi_0 \) and \( \xi_1 \) can be constructed as a regular neighborhood of \( M \times I \) in \( Q \), by the relative version of Lemma 1.

It follows that the classifying space \( \mathcal{G} \) is (canonically homotopy equivalent with) Cappell and Shaneson's classifying space \( BRN_2 \) [3], [4]. In the same way, "oriented" \( 2 \)-cone bundles are classified by \( BSRN_2 \), and \( 2 \)-cone bundles which are blockbundles on the \((k-1)\)-skeleton are classified by \( BRN_{2,k} \).

**Remarks.** (1) The Noguchi obstruction \( n \) can be viewed as a natural transformation from \( \mathcal{K}(\cdot) \) to \( H^2(\cdot; \gamma) \), since \( n(\xi) \) depends only on the concordance class of \( \xi \). Furthermore, \( n(\xi) = 0 \) if and only if \( \xi \) is concordant to a cone bundle which is a blockbundle on the 2-skeleton (cf. [4, §3]).

(2) In [4], Cappell and Shaneson completely determine the homotopy type of \( BSRN_2 \). An interesting problem is to give a geometric description of the resulting \( H \)-space structure on \( BSRN_2 \).
3. Collared complexes. A collared complex $C$ on a polyhedron $X = |C|$ is a locally finite covering of $X$ by compact subpolyhedra, together with a subpolyhedron $\delta \alpha$ of each element $\alpha$ of $C$ such that

(i) for each $\alpha \in C$, $\delta \alpha$ is a union of elements of $C$,
(ii) if $\alpha$ and $\beta$ are distinct elements of $C$, $\alpha^o \cap \beta^o$ is empty, where $\alpha^o = \alpha \setminus \delta \alpha$,
(iii) $\delta \alpha$ is collared in $\alpha$ for each $\alpha \in C$. (i) and (ii) imply that $\alpha \cap \beta$ is a union of elements of $C$.)

Collared complexes are Alan's "general complexes" [1]. Examples of collared complexes are cell complexes, manifold complexes [5], and cone complexes [7].

The usefulness of collared complexes comes from the following proposition, derived from the proof of a lemma of Cohen and Sullivan [5, p. 142]. (See also [2, p. 21] and [7, p. 278].)

If $C$ is a collared complex and $\alpha \in C$, let $L(\alpha)$ be the geometric realization of the nerve of the finite partially ordered set $\{ \beta \in C, \alpha < \beta \}$, where $\alpha < \beta$ means $\alpha \subseteq \delta \beta$.

**Proposition 1.** If $C$ is a collared complex on $X$, $\alpha \in C$, and $x \in \alpha^o$, then

$$\text{lk}(x; X, \alpha) \cong (L(\alpha) \ast \text{lk}(x; \alpha), \text{lk}(x; \alpha)),$$

where $\text{lk}$ denotes the link, and $\ast$ denotes the join.

**Proof.** Use induction on the "depth" of $\alpha$, i.e. the length of a maximal chain $\alpha < \alpha_1 < \cdots < \alpha_n$ in $C$.

With this proposition, we can prove Lemma 2, by induction on the dimension of the base. Let $\xi/K$ be a cone bundle, with $|K|$ a manifold (with boundary). If $\sigma_i \in K$, and $\beta_i$ is the block of $\xi$ over $\sigma_i$, let $\delta \beta_i = E(\xi \setminus \delta \sigma_i)$. By induction hypothesis, $\delta \beta_i$ is a codimension 0 submanifold of $\delta \beta_i$, so $\delta \beta_i$ is collared in $\beta_i$. Thus the set of blocks of $\xi$ forms a collared complex $C$ on $E(\xi)$. The map $\beta_i \mapsto \sigma_i$ is an incidence preserving bijection between $C$ and $K$. Therefore $L(\beta_i) = L(\sigma_i)$ for all $\sigma_i \in K$. Thus the proposition implies $E(\xi)$ is a manifold (with boundary), since each block $\beta_i$ is a manifold and $|K|$ is a manifold.

4. Geometry of codimension 2 thickenings. Let $W$ be a codimension 2 thickening of the compact $n$-manifold $M$. If $x \in M$, the intrinsic dimension $d(x; W, M)$ is the smallest integer $k$ such that $x$ is in the $k$-skeleton of every (PL) cell complex on $W$ which has $M$ as a subcomplex. The $k$th intrinsic stratum $S_k$ of $M$ in $W$ is

$$\{ x \in M \setminus \partial M, d(x; W, M) = k \} \cup \{ x \in \partial M, d(x; \partial W, \partial M) = k - 1 \}.$$

(Cf. [12, p. 13]. Recall that $(\partial W, \partial M)$ is collared in $(W, M)$.) $S_k$ is a $k$-dimensional submanifold of $M$, and $\text{cl}(S_k) = \bigcup_{j < k} S_j$. $S_n$ is the set of
locally flat points of $M$ in $W$, and $S_k$ can be thought of as the points at which the “degree of local knottedness” of $M$ in $W$ is $n - k$. Let $\mathcal{S} = \{S_k\}$ denote this intrinsic stratification of $M$ in $W$.

By Lemma 3, we can assume that $W = E(\xi)$ for some 2-cone bundle $\xi/K$, $|K| = M$. Now for each block $\beta$ of $\xi$, choose a cellular subdivision of the “rim” $\beta^* = \text{cl}(\partial \beta \setminus \delta \beta)$. These cells, together with the blocks themselves, form a cell complex $\mathcal{C}$ on $W$. Choose a cone structure for each cell of $\mathcal{C}$ so that $\sigma_j$ is a subcone of $\beta_j$ for each cell $\sigma_j \in K$. (N.B. $K$ is not a subcomplex of $\mathcal{C}$.) Then the dual cone complex $\mathcal{C}^*$ on $W$ [7] will have the complex $K^*$ on $M$ as a subcomplex. (Note that the cones of $\mathcal{C}^*$ are cells, but if $\alpha \in \mathcal{C}^*$ and $\alpha \cap \partial W \neq \emptyset$, the apex of $\alpha$ lies in $\partial W$.) It follows that $\text{cl}(S_k)$ is a subcomplex of $K^*$ for all $k$, i.e. the cells of $K$ are transverse to the intrinsic stratification of $M$ in $W$. (See [7, p. 287] for a discussion of transversality to a stratification.)

If $K'$ is a subdivision of $K$, the cone bundle $\xi'/K'$ is a subdivision of $\xi/K$ if for each $\sigma \in K$, $\beta_{ij}(\xi) = \bigcup \beta_{ij}(\xi')$, where the union is taken over all blocks $\beta_{ij}(\xi')$ over cells $\tau_j \subset K'$ such that $\tau_j \subset \sigma_i$.

It is not hard to see that a subdivision $\xi'$ of $\xi$ will exist over the subdivision $K'$ of $K$ if and only if $(K')^*$ extends to a cell complex on $W = E(\xi)$ (for some cone structuring of $K'$). This is equivalent to the condition that $K'$ be transverse to the intrinsic stratification $\mathcal{S}$. Therefore, $\xi$ can be restricted to precisely those subpolyhedra of $M$ which are transverse to $\mathcal{S}$.

Thus the fact that concordance classes of cone bundles can be “pulled back” is a consequence of the geometric fact that any subpolyhedron $X$ of the manifold $M$ can be moved transverse to $\mathcal{S}$. (In fact, Stone’s transversality theorem [12] can be easily proved from the mockbundle viewpoint—cf. [7, p. 287].)

The following result is important in [4].

**Proposition 2.** Let $W$ be a codimension 2 thickening of $M$, and let $N$ be a locally flat codimension $q$ submanifold of $M$, with $\partial M \cap N = \partial N$. Suppose that $N$ is transverse to the intrinsic stratification $\mathcal{S}$ of $M$ in $W$. Then there is a cone bundle $\xi$ over $M$ with $E(\xi) = W$, and a normal blockbundle $\nu$ of $N$ in $M$ such that $E(\nu)$ is transverse to $\mathcal{S}$, and $E(\xi|E(\nu))$ is a codimension $q$ thickening of $E(\xi|N)$ equivalent to $E(q^*\nu)$, where $q: E(\xi|N) \to N$ is a homotopy inverse of the inclusion.

**Proof.** $N$ is transverse to $\mathcal{S}$ implies there is a cone bundle $\eta/L$, $|L| = M$, with $E(\eta) = W$ and $N$ a subcomplex of $L$. Let $K$ be the canonical “full” subdivision of $L$ constructed in [7, p. 276], and let $\xi$ be a subdivision of $\eta$ over $K$. (It is easy to construct $\xi$ explicitly.) Then the union of the cells in $K$ which meet $N$ is a regular neighborhood of $N$, and so this neighborhood equals $E(\nu)$ for some blockbundle $\nu$ over $N$. $E(\nu)$ is transverse to $\mathcal{S}$ since it is a
subcomplex of $K$. $E(\xi|E(\nu))$ is a manifold by Lemma 2, and it collapses to $E(\xi|N)$ since $E(\nu)$ collapses to $N$. Thus $E(\xi|E(\nu))$ is a thickening of $E(\xi|N)$. $E(\xi|N)$ is locally flat in $E(\xi|E(\nu))$ by Proposition 1, since the given collared complexes on $E(\xi|E(\nu))$ and $E(\nu)$ are abstractly isomorphic. Thus $E(\xi|E(\nu)) \supset E(\xi|N)$ is equivalent to $E(q^*\nu) \supset E(\xi|N)$ by the uniqueness of regular neighborhoods.

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