TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 228, 1977

MODULE STRUCTURE OF CERTAIN INDUCED REPRESENTATIONS OF COMPACT LIE GROUPS

BY
E. JAMES FUNDERBURK

ABSTRACT. Let G be a compact connected Lie group and assume a choice of maximal torus and positive roots has been made. Given a dominant weight λ, the Borel-Weil Theorem shows how to construct a holomorphic line bundle on whose sections G acts so that the holomorphic sections provide a realization of the irreducible representation of G with highest weight λ. This paper studies the G-module structure of the space Γ of square integrable sections of the Borel-Weil line bundle. It is found that $\Gamma = \lim_{n\to\infty}\Gamma(n)$, where $\Gamma(n) \subset \Gamma(n+1) \subset \Gamma$ and $\Gamma(n)$ is isomorphic, as G-module, to

$$V(\lambda \dotplus n\lambda) \otimes V(n\lambda^*),$$

where $V(\mu)$ denotes the irreducible representation of highest weight μ, '$\dotplus$' is the Cartan semigroup operation, and '$*$' is the contragredient operation. Similar formulas hold for powers of the Borel-Weil line bundle.

1. Introduction. In the formulation of quantum mechanics proposed by J.-M. Souriau (see [14]) one is led, given a Lie group G, to the study of certain homogeneous Hermitian line bundles with connection whose bases have a symplectic structure determined as the curvature of the given connection. It is a natural question to ask the G-module structure of the sections of such homogeneous line bundles. For the case when G is a compact connected Lie group, this paper presents an answer to this question. In the solution presented one also obtains the G-module structure of square-integrable functions on both the total space and base of the associated principal bundle.

The answer obtained may be expressed as follows. Let T be a maximal torus in the compact connected Lie group G and Λ^+ the set of dominant weights for G with respect to a fixed choice of positive roots. Set theoretically we view Λ^+ as a certain set of linear functions on the Lie algebra of G, closed under addition and containing the zero linear functional. The Cartan theory of highest weights presents an identification of Λ^+ with $\hat{G}$, the set of, necessarily finite-dimensional and unitary, irreducible representations of G.

Let $\mathfrak{S}$ denote the set of all functions from $\hat{G}$ to the nonnegative integers, $\mathbf{Z}^+$, and let $\mathfrak{R}$ denote the set of unitary equivalence classes of completely

Received by the editors November 6, 1975.

AMS (MOS) subject classifications (1970). Primary 17B10, 43A85; Secondary 22E45, 17B20.

Key words and phrases. Compact connected Lie group, highest weight, lowest weight, opposition involution, Cartan semigroup, tensor product, cyclic representation, Borel-Weil realization.

continuous unitary representations of G. Then, there is a bijection ch: $\mathfrak{R} \to \mathbb{S}$, such that $\mathrm{ch}(\rho)(\lambda)$ is the multiplicity of λ as a subrepresentation of ρ, for ρ in $\mathfrak{R}$, λ in G. Thus the function $\mathrm{ch}(\rho)$ determines the G-module structure of ρ. We refer to $\mathrm{ch}(\rho)$ as the formal character of ρ.

The elements of $\mathfrak{R}$ whose formal characters we will determine may be described as follows. Fix λ in $\hat{G}$ and denote by λ^* in $\hat{G}$ the representation contragredient to λ. Let K denote the isotropy group of λ^* in G with respect to the contragredient adjoint action of G. Let R and L denote the right and left regular representations of G in $L^2(G)$. Let Γ and Γ_k (for k in $\mathbf{Z}$) denote the elements of $\mathfrak{R}$ which are subrepresentations of L determined by the following requirements on f in $L^2(G)$:

$f \in \Gamma$ iff $R(x)f = f$, $x \in K_0$,

$f \in \Gamma_k$ iff $R(x)f = \chi(x)^k f$, $x \in K$.

Here χ: $K \to S^1$ is the homomorphism determined by λ^* ($\chi(\exp X) = e^{\lambda^*(x)}$, for X in the Lie algebra of K), and K_0 is the kernel of χ. It is clear that $\Gamma_k \subset \Gamma$ for all k, and one may show $\Gamma = \bigoplus_{k \in \mathbf{Z}} \Gamma_k$, a direct sum in $\mathfrak{R}$. Γ_1 may be interpreted as the L^2 sections of a homogeneous Hermitian line bundle with connection $E \to M$ determined by λ^*, Γ_0 as $L^2(M)$, and Γ as $L^2(P)$, $S^1 \to P \to M$ being the associated principal bundle to $E \to M$.

We express $\mathrm{ch}(\Gamma_k)$ as the limit of a certain bounded increasing sequence in $\mathbb{S}$ ($f < g$ in $\mathbb{S}$ iff $f(\lambda) < g(\lambda)$ for all λ in $\hat{G}$; a subset S of $\mathbb{S}$ is bounded iff $\{f(\lambda) | f \in \mathbb{S}, \lambda \in \hat{G}\}$ is a bounded set in $\mathbf{Z}^+$); obviously such a sequence has a unique point-wise defined limit in $\mathbb{S}$. It is shown that

$$\{\mathrm{ch}(\mu + n\nu \otimes n\nu^*)\}_{n=0}^{\infty} \quad \text{and} \quad \{\mathrm{ch}(n\nu \otimes \mu + n\nu^*)\}_{n=0}^{\infty}$$

are bounded increasing sequences in $\mathbb{S}$, for μ, ν in Λ^+. Then for k in $\mathbf{Z}^+$, our main result states

$$\mathrm{ch}(\Gamma_k) = \lim_{n\to\infty} \mathrm{ch}(k\lambda + n\lambda \otimes n\lambda^*), \qquad \mathrm{ch}(\Gamma_{-k}) = \lim_{n\to\infty} \mathrm{ch}(n\lambda \otimes k\lambda^* + n\lambda^*).$$

In particular

$$\mathrm{ch}(\Gamma_1) = \lim_{n\to\infty} \mathrm{ch}((\lambda + n\lambda) \otimes n\lambda^*), \qquad \mathrm{ch}(\Gamma_0) = \lim_{n\to\infty} \mathrm{ch}(n\lambda \otimes n\lambda^*),$$

$$\mathrm{ch}(\Gamma) = \sum_{k=0}^{\infty} \lim_{n\to\infty} \left(\mathrm{ch}((k\lambda + n\lambda) \otimes n\lambda^*) + \mathrm{ch}(n\lambda \otimes (k\lambda^* + n\lambda^*))\right).$$

The basic idea involved in establishing the above formulas may be referred to as the Borel-Weil realizations of elements of $\hat{G}$. For λ in $\hat{G} = \Lambda^+$, consider the systems of differential equations:

$$(R(X) - \lambda^*(X))f = 0, \tag{1}$$

$$(L(X) - \lambda(X))f = 0, \tag{2}$$

for X a positive root vector or an element of the Lie algebra of T. Let $\mathfrak{B}_\lambda$ denote the simultaneous solutions to (1); and $\mathfrak{B}_\lambda^0$ the simultaneous solutions to (1) and (2). Then, the subrepresentation of L in $\mathfrak{B}_\lambda$ is a representative of λ in $\hat{G}$ and $\mathfrak{B}_\lambda$ is the highest weight space; we take this statement to be the Borel-Weil Theorem, and refer to B_λ as the Borel-Weil realization of λ. One has the relations $B_{\lambda+\nu} = B_\lambda B_\nu$ (equality of sets, $B_\lambda B_\nu$ is the complex span of point-wise defined products fg with f in B_ν, g in B_ν). $\overline{B}_\nu$is isomorphic to B_{ν^*} ($\overline{B}_\nu$ is the set of $\bar{f}$ with f in B_ν) and the multiplication map $B_\lambda \otimes B_\nu \to B_\lambda \overline{B}_\nu$ a (nonunitary) G-module equivalence; thus $B_\lambda \overline{B}_{\nu^*}$ as a subrepresentation of L is isomorphic to the tensor product $\lambda \otimes \nu$.

Borel-Weil realizations are related to the original question by using the Stone-Weierstrass Theorem to show $\sum_{p,q \in \mathbf{Z}^+} \mathfrak{B}_{p\lambda} \overline{\mathfrak{B}}_{q\lambda}$ is dense in $\Gamma = L^2(P)$.

For certain special cases we determine $\mathrm{ch}(\Gamma_1)$ explicitly by working out the tensor product limits of our general expression for $\mathrm{ch}(\Gamma_1)$. There is a generally applicable theoretical formula for the Clebsch-Gordon series for the tensor product of two irreducible representations (Steinberg's formula). When λ is regular, multiplicity formulas, such as those of Kostant and of Freudenthal, are of practical use in computing $\mathrm{ch}(\Gamma_k)$.

The author expresses his appreciation to Joel W. Robbin for his lectures on the Souriau-Kostant approach to quantum mechanics and representation theory of Lie groups; these lectures and ensuing conversations made it possible to consider the questions of this paper.

2. **Compact connected Lie groups.** Throughout this paper G denotes a compact connected Lie group, $\mathcal{G}$ its Lie algebra and $\mathcal{G}_{\mathbf{C}}$ the complexification of $\mathcal{G}$.

2.1. *Background on completely continuous representations.* $C(G)$ ($C^\infty(G)$) denotes the complex vector space of continuous (smooth) functions on G. $L^2(G)$ denotes the Hilbert space of functions on G that are square-integrable with respect to the normalized bi-invariant Haar measure dx on G.

We denote by $\mathfrak{R}$ (respectively, $\hat{G}$) the set of unitary equivalence classes of completely continuous (respectively, irreducible) unitary representations of G. L and R, the left-regular and right-regular representations of G in $L^2(G)$, are (equal) elements of $\mathfrak{R}$ (see 2.6.2 and 2.8.2 of [17]).

If ρ is a unitary representation we may denote by V^ρ the Hilbert space in which ρ acts by unitary operators. For ρ in $\hat{G}$, V^ρ is a finite-dimensional vector space (see 2.6.3 of [17]) of dimension d_ρ.

For ρ in $\hat{G}$ we assume that an orthonormal basis $\{v_i^\rho\}_{i=1,\ldots,d_\rho}$ has been chosen for V^ρ; in (2.4) a certain choice of basis will be made. Define $\rho_{ij} \in C(G)$ by $\rho_{ij}(a) = \{\rho(a)v_j^\rho, v_i^\rho\}$, $a \in G$, $1 \leqslant i,j \leqslant d_\rho$. The Peter-Weyl Theorem [5, p. 203] asserts the density in $L^2(G)$ of $\{\rho_{ij} | \rho \in G, 1 \leqslant i,j \leqslant d_\rho\}$. Defining $E^{i,j,\rho}$ in $\mathrm{End}(V^\rho)$ (the set of linear maps from V^ρ to V^ρ) by

$$E^{i,j,\rho}v_k^\rho = \delta_{jk}v_i^\rho, \qquad 1 \leqslant i,j,k \leqslant d_\rho,$$

one has $\rho(a) = \Sigma_{i,j}\rho_{ij}(a)E^{i,j,\rho}$, $a \in G$.

Declaring $\{d_\rho^{-1/2}E^{i,j,\rho}: 1 \leqslant i,j \leqslant d_\rho\}$ to be an orthonormal set imposes a Hilbert space structure on $\mathrm{End}(V^\rho)$. One may form the Hilbert space direct sum $\mathfrak{S}$ of the $\mathrm{End}(V^\rho)$ for ρ in $\hat{G}$. For A in $\mathfrak{S}$ and ρ in $\hat{G}$, A_ρ denotes the component of A in $\mathrm{End}(V^\rho)$.

The Fourier transform $\mathcal{F}: L^2(G) \to \mathfrak{S}$ and the inverse Fourier transform $\mathcal{F}^*: \mathfrak{S} \to L^2(G)$ are mutually inverse isometries such that for f in $C(G)$, ρ in $\hat{G}$, A in $\mathrm{End}(V^\rho)$, y in G,

$$\mathcal{F}(f)_\rho = \int_G f(x)\rho(x)\,dx, \qquad \mathcal{F}^*(A)(y) = d_\rho \mathrm{tr}\big(A\rho(y^{-1})\big).$$

If ρ is in $\mathfrak{R}$ and λ is in $\hat{G}$, then the finite dimension of the space of all operators that intertwine λ and ρ we denote by $\mathrm{ch}(\rho)(\lambda)$. Thus, ch sets up a one-to-one correspondence between $\mathfrak{R}$ and the set $\mathfrak{S}$ of functions from $\hat{G}$ to $\mathbf{Z}^+$. We call $\mathrm{ch}(\rho)$ the formal character of the completely continuous representation ρ.

2.2. *Cartan semigroup*. Let ρ be a unitary representation of G. If v is a smooth vector we set

$$\rho(X)v = \lim_{t\to 0}\big(\rho(\exp(tX))v - v\big)/t, \quad X \in \mathcal{G}.$$

From now on in §2 *all representations are assumed to be finite-dimensional unitary*. For a representation ρ of G we denote also by ρ the associated Lie algebra homomorphism $\mathcal{G} \to \mathrm{End}(V^\rho)$ or its complex linear extension $\mathcal{G}_{\mathbf{C}} \to \mathrm{End}(V^\rho)$.

We assume fixed a maximal torus T of G with Lie algebra $\mathcal{T}$. $\mathfrak{Z}$ denotes the complexification of the Lie algebra of the center of G, $\mathfrak{L} = [\mathcal{G}_{\mathbf{C}}, \mathcal{G}_{\mathbf{C}}]$, $\mathcal{H} = \mathcal{T}_{\mathbf{C}} \cap \mathfrak{L}$.

The complexified adjoint representation of G in $\mathcal{G}_{\mathbf{C}}$ is denoted by ad, its contragredient by ad*. We assume fixed a positive definite inner product on $\mathcal{G}$ extended to Hermitian inner product on $\mathcal{G}_{\mathbf{C}}$ and dualized to one on $\mathcal{G}_{\mathbf{C}}^*$ which (see 5.6.1 of [17]) is $\mathrm{ad}(G)$ invariant, is equal to the negative Killing form on $\mathfrak{L} \cap \mathcal{G}$, and renders $\mathfrak{L}$ and $\mathfrak{Z}$ perpendicular. By means of the splittings $\mathcal{G}_{\mathbf{C}} = \mathcal{T}_{\mathbf{C}} \oplus \mathcal{T}_{\mathbf{C}}^\perp$, $\mathcal{T}_{\mathbf{C}} = \mathcal{H} \oplus \mathfrak{Z}$, we consider $\mathcal{T}_{\mathbf{C}}^*$ and $\mathcal{H}^*$ as subspaces of $\mathcal{G}_{\mathbf{C}}^*$.

If ρ is a representation of G and λ is in $\mathcal{T}_{\mathbf{C}}^*$, set $V_\lambda^\rho = \{v \in V^\rho | \rho(\exp(H))v = e^{\lambda(H)}v \text{ if } H \in \mathcal{T}\}$. If $V_\lambda^\rho \neq (0)$ we say that λ is a *weight* of ρ, V_λ^ρ is the λ weight space of ρ and nonzero elements of V_λ^ρ are weight vectors of ρ of weight λ; $\Lambda(\rho)$ is the set of weights of ρ.

The root system Φ of G with respect to T may be defined as $\Lambda(\mathrm{ad}) - \{0\}$. We assume a fixed set Φ^+ of positive roots has been chosen, and set

$\Phi^- = -\Phi^+$ and denote by $\mathfrak{N}^+$ ($\mathfrak{N}^-$) the vector space sum of the positive (negative) root spaces; $\mathcal{G}_\alpha$ denotes the α weight space of ad, for $\alpha \in \Phi$.

Let Λ^+ denote the set of dominant weights of G (with respect to T, Φ^+). As a subset of the additive vector space $\mathfrak{T}_{\mathbf{C}}^*$, Λ^+ is an abelian semigroup with identity. Since identified with Λ^+ v ia the correspondence of ρ in $\hat{G}$ with its highest weight λ_ρ in Λ^+, $\hat{G}$ is an abelian semigroup, called the Cartan semigroup.

2.3. *Weyl group; opposition involution.* The Weyl group W of G (with respect to T) may be defined as $N(T)/T$, where $N(T) = \{a \in G | nTn^{-1} = T\}$, or as the subgroup of linear automorphisms of $\mathfrak{T}_{\mathbf{C}}^*$ generated by the reflections $\{\sigma_\alpha | \alpha \in \Phi\}$, where

$$\sigma_\alpha(\lambda) = \lambda - 2(\{\lambda, \alpha\}\alpha / \{\alpha, \alpha\}),$$

for α, λ in $\mathfrak{T}_{\mathbf{C}}^*$. If ad$^*(n)$, for n in $N(T)$, induces the automorphism w of $\mathfrak{T}_{\mathbf{C}}^*$, we may write $w = nT$.

There is a unique element w_0 in W with $w_0\Phi^+ = \Phi^-$; since w_0^2 is the identity, w_0 is called the opposition involution (with respect to T, Φ^+). We assume n_0 in $N(T)$ chosen with $w_0 = n_0T$.

2.4. *Lowest weight.* For each ρ in $\hat{G}$ there is unique λ_ρ^- in $-\Lambda^+$ so that $\lambda_\rho^- \in \Lambda(\rho)$ but, for $\alpha \in \Phi^-$, $\lambda_\rho^- + \alpha \notin \Lambda(\rho)$; λ_ρ^- is called the lowest weight of ρ and any nonzero element of the one-dimensional space $V_{\lambda_\rho^-}^\rho$ is called a lowest weight vector.

PROPOSITION (2.4.1). *For ρ in $\hat{G}$, $w_0(\lambda_\rho) = \lambda_\rho^- = -\lambda_{\rho^*}$ and $\rho(n_0^\varepsilon)V_{\lambda_\rho}^\rho = V_{\lambda_\rho^-}^\rho$, for $\varepsilon = \pm 1$. Here ρ^* is the representation of G in $(V^\rho)^*$ contragredient to ρ.*

PROOF. Choose α in Φ^+, $X_{-\alpha}$ in $\mathcal{G}_{-\alpha}$, and v in $V_{\lambda_\rho}^\rho$. As $\mathrm{ad}(n_0)\mathcal{G}_{-\alpha} \subset \mathcal{G}_{w_0(-\alpha)} \subset n^+$, one has $\rho(X_{-\alpha})\rho(n_0^{-1})v = \rho(n_0^{-1})\rho(\mathrm{ad}(n_0)X_{-\alpha})v = 0$. It follows that $V_{\lambda_\rho^-}^\rho = \rho(n_0^{-1})V_{\lambda_\rho}^\rho = V_{w_0(\lambda\rho)}^\rho$, so $\lambda_\rho^- = w_0(\lambda\rho)$. From $\Lambda(\rho^*) = -\Lambda(\rho)$ we see that $-\lambda_{\rho^*} - \alpha$ is not in $\Lambda(\rho)$ for α in Φ^+ and that $-\lambda_{\rho^*}$ is in $\Lambda(\rho)$. By the uniqueness of λ_ρ^-, $-\lambda_{\rho^*} = \lambda_\rho^-$. Q.E.D.

From now on in this paper we assume the Cartan identification of $\hat{G}$ with Λ^+. Frequently elements of $\hat{G}$ are denoted by λ in Λ^+; sometimes $\rho_\lambda(a)$ may be written in place of $\lambda(a)$, for a in G.

Suppose λ in $\Lambda^+ = \hat{G}$ is chosen. We choose once and for all, an orthonormal basis $\{v_i^\lambda\}_{i=1,\ldots,d(\lambda)}$ of V^λ consisting of weight vectors and enumerated in such a way that v_1^λ is of weight λ and $v_{d(\lambda)}^\lambda$ is of weight $-\lambda^*$, where λ^* is the element of Λ^+ corresponding to the contragredient of λ. We denote by $\{\phi_i^\lambda\}_{i=1,\ldots,d(\lambda)}$ the basis of V^{λ^*} dual to $\{v_i^\lambda\}_{i=1,\ldots,d(\lambda)}$; thus $\phi_{d(\lambda)}$ is of weight λ^* and ϕ_1^λ of weight $-\lambda$. We may further assume that $v_{d(\lambda)}^\lambda = \lambda(n_0)v_1^\lambda$ and define $\zeta_0 \in S^1$ by $\lambda(n_0^{-1})v_1^\lambda = \zeta_0 v_{d(\lambda)}^\lambda$.

For later use we introduce the notations f_i^λ for $\overline{\lambda}_{id(\lambda)}$ and $\mathfrak{B}_\lambda$ for the complex linear span of the independent set $\{f_1^\lambda, \ldots, f_{d(\lambda)}^\lambda\}$. One may readily verify the formulas $f_1^\lambda(n_0^{-1}) = 1$; $\lambda(a)v_{d(\lambda)}^\lambda = \sum \bar{f}_i^\lambda(a)v_i^\lambda$; $\lambda^*(a)\phi_{d(\lambda)}^\lambda = \sum f_i^\lambda(a)\phi_i^\lambda$; $\sum f_i^\lambda \bar{f}_i^\lambda = 1$ (the summations in these last three formulas are for $i = 1, \ldots, d(\lambda)$).

For λ in $\hat{G}$ we identify $\mathrm{End}(V^\lambda)$ and $V^\lambda \otimes V^{\lambda^*}$ by corresponding $v \otimes \phi$ to the endomorphism sending v' to $\phi(v')v$; in particular $v_i^\lambda \otimes \phi_j^\lambda$ corresponds to $E^{i,j,\lambda}$. Further equating $\mathrm{End}(V^\lambda)$ with $L^2(G)_\lambda = \mathfrak{F}^*(\mathrm{End}(V^\lambda))$ via the Fourier transform $\mathfrak{F}$, we see that $L(a)$ corresponds to $\lambda(a) \otimes 1$ and $R(a)$ to $1 \otimes \lambda^*(a)$, for a in G.

2.5. *Cyclic representations*. A representation ρ of G is called *cyclic* if there is v in V^ρ so that V^ρ equals the linear span of the orbit $\{\rho(a)v | a \in G\}$; a vector whose orbit spans V^ρ is called a cyclic vector.

PROPOSITION (2.5.1). (a) *ρ is cyclic if ρ is in $\hat{G}$.* (b) *If ρ_1 and ρ_2 are in $\hat{G}$, then $\rho_1 \otimes \rho_2$ is cyclic; in fact $v_1 \otimes v_2$ is a cyclic vector if v_1 is a highest weight vector for ρ_1 and v_2 a lowest weight vector for ρ_2, and $v_1 \neq 0$, $v_2 \neq 0$.*

PROOF. (a) The linear span of an orbit is an invariant subspace. Thus any nonzero vector in V^ρ is a cyclic vector for ρ in $\hat{G}$.

(b) Let v_1, v_2 be as enunciated, set $v = v_1 \otimes v_2$, and denote by V the span of the orbit of v under $\rho_1 \otimes \rho_2$. We must show $V = V^{\rho_1} \otimes V^{\rho_2}$. Choose $u_i \in V^{\rho_i}$, $i = 1, 2$. It suffices to show $u_1 \otimes u_2 \in v$. Now, u_1 is a linear combination of elements of the form v_1, Av_1, where $A = \rho_1(X_1) \ldots \rho_1(X_r)v_1$ ($\alpha(i) \in \Phi^+$, $X_i \in \mathcal{G}_{-\alpha(i)}$). But

$$(\rho_1 \otimes \rho_2)(X_{-\alpha})(w_1 \otimes v_2) = (\rho_1(X_{-\alpha})w_1) \otimes v_2 + w_1 \otimes \rho(X_{-\alpha})v_2$$
$$= (\rho_1(X_{-\alpha})w_1) \otimes v_2$$

for $\alpha \in \Phi^+$, $X_{-\alpha} \in \mathcal{G}_{-\alpha}$, $w_1 \in V^{\rho_1}$. As $(\rho_1 \otimes \rho_2)(\mathcal{G}_{\mathbf{C}})V \subset V$, one concludes $w_1 \otimes v_2 \in V$ for any $w_1 \in V^{\rho_1}$. Now by (a) there are $c_1, \ldots, c_r$ in $\mathbf{C}$, $a_1, \ldots, a_r$ in G with $u_2 = \sum_i c_i \rho_2(a_i)v_2$; hence

$$u_1 \otimes u_2 = \sum_i c_i u_1 \otimes \rho_2(a_i)v_2$$
$$= \sum_i c_i(\rho_1 \otimes \rho_2)(a_i)\big(\rho_1(a_i^{-1})u_1 \otimes v_2\big).$$

With the previous remark, one concludes that $u_1 \otimes u_2 \in V$. Q.E.D.

3. **The Borel-Weil Theorem and its consequences.** We retain the notations of the preceding sections.

3.1. *Borel-Weil Theorem*. For μ in $\mathfrak{T}_{\mathbf{C}}^*$ set

$$\mathfrak{M}(\mu) = \{f \in C^\infty(G) | (R(X) + \mu(X))f = 0 \text{ if } X \in \mathfrak{T}_{\mathbf{C}} + \mathfrak{N}^+\},$$
$$\mathfrak{M}'(\mu) = \{f \in C^\infty(G) | (L(X) + \mu(X))f = 0 \text{ if } X \in \mathfrak{T}_{\mathbf{C}} + \mathfrak{N}^+\}.$$

Let G_μ denote the isotropy subgroup of μ in G under the contragredient adjoint action, and let $\mathcal{G}_\mu$ denote the Lie algebra of G_μ.

The implication of the usual Borel-Weil Theorem (see [12]) from that stated below is detailed in [7].

THEOREM (3.1.1) (BOREL-WEIL THEOREM). (a) $\mathfrak{M}(\mu) = 0$ *unless* $-\mu$ *is in* Λ^+.

(b) *If* $-\mu = \lambda^*$ *with* λ *in* Λ^+, *then* $\mathfrak{M}(\mu) \subset L^2(G)_\lambda$, $L(G)\mathfrak{M}(\mu) \subset \mathfrak{M}(\mu)$, *and the subrepresentation of* L *in* $\mathfrak{M}(\mu)$ *is equivalent to* λ.

(c) *For* μ, μ' *in* $\mathfrak{T}_{\mathbf{C}}^*$, $\mathfrak{M}(\mu) \cap \mathfrak{M}'(\mu') \neq 0$ *if and only if there is* λ *in* Λ^+ *with* $-\mu = \lambda^*$, $-\mu' = \lambda$. *If* $\mathfrak{M}(\mu) \cap \mathfrak{M}'(\mu')$ *is not zero, then its dimension is* 1.

(d) $\mathfrak{M}(\mu) = \{f \in C^\infty(G) | (R(X) + \mu(X))f = 0 \text{ if } X \in (\mathcal{G}_\mu)_{\mathbf{C}} + \mathfrak{N}^+\}$.

PROOF. We assert that if f is in $\mathfrak{M}(\mu)$ then $f_\gamma = \mathfrak{F}^*((\mathfrak{F}(f)_\gamma))$ is in $\mathfrak{M}(\mu)$ for γ in $\hat{G}$. Writing $g_\gamma = f - f_\gamma$ we have $f = f_\gamma + g_\gamma, f_\gamma \in L^2(G)_\gamma, g_\gamma \in L^2(G)_\gamma^\perp$. Now as R is unitary and preserves $L^2(G)_\gamma$, it also preserves $L^2(G)_\gamma^\perp$; hence the same is true of $R(X) + \mu(X)$ for any X in $\mathcal{G}_{\mathbf{C}}$. Thus,

$$(R(X) + \mu(X))f_\gamma + (R(X) + \mu(X))g_\gamma$$

is the decomposition of $(R(X) + \mu(X))f$ into orthogonal components in $L^2(G)_\gamma$ and $L^2(G)_\gamma^\perp$. In particular, if $(R(X) + \mu(X))f = 0$, then

$$(R(X) + \mu(X))f_\gamma = 0 = (R(X) + \mu(X))G_\gamma,$$

from which the assertion follows.

We now show that for f in $\mathfrak{M}(\mu)$ and λ in Λ^+, $f = 0$ unless $-\mu = \lambda^*$. In the identification of $L^2(G)_\lambda$ with $V^\lambda \otimes V^{\lambda^*}$, $\mathfrak{M}(\mu) \cap L^2(G)_\lambda$ corresponds to $V^\lambda \otimes \{\phi \in V^{\lambda^*} | (\rho_{\lambda^*}(X) + \mu(X))\phi = 0 \text{ if } X \in \mathfrak{T}_{\mathbf{C}} + \mathfrak{N}^+\}$. But this latter set is zero unless $-\mu = \lambda^*$, by the uniqueness of highest weight of V^{λ^*}. This establishes (a). Now if $-\mu = \lambda^*$ with λ in Λ^+, then we have that $\mathfrak{M}(\mu) \subset L^2(G)_\lambda$, $L(G)\mathfrak{M}(\mu) \subset \mathfrak{M}(\mu)$ (as L and R commute), and the subrepresentation of L in $\mathfrak{M}(\mu)$ is isomorphic to that of $\lambda \otimes 1$ in $V^\lambda \otimes V^{\lambda^*}_{\lambda^*}$. As $\dim V^{\lambda^*}_{\lambda^*} = 1$, the representation $\lambda \otimes 1$ of G in $V^\lambda \otimes V^{\lambda^*}_{\lambda^*}$ is equivalent to that of λ. This establishes (b). Now if μ and μ' are in $\mathfrak{T}_{\mathbf{C}}^*$, we know by (a) that $\mathfrak{M}(\mu) \cap \mathfrak{M}'(\mu') = 0$ unless $\mu = -\lambda^*$ for some λ in Λ^+. If $\mu = -\lambda^*$, with λ in Λ^+, then, in the identification with $V^\lambda \otimes V^{\lambda^*}$, $\mathfrak{M}(\mu) \cap \mathfrak{M}'(\mu') = \{v \in V^\lambda | (\rho_\lambda(X) + \mu'(X))v = 0 \text{ if } X \in \mathfrak{T}_{\mathbf{C}} + \mathfrak{N}^+\} \otimes V^{\lambda^*}_{\lambda^*}$. The first factor is 0 unless $\mu' = -\lambda$, by the uniqueness of highest weight in V^λ. If $-\mu' = \lambda$, then $\mathfrak{M}(\mu) \cap \mathfrak{M}'(\mu')$ corresponds to $V^\lambda_\lambda \otimes V^{\lambda^*}_{\lambda^*}$, which is a one-dimensional space. This proves (c). To prove (d), we need only show that $\rho_{\lambda^*}(X)V^{\lambda^*}_{\lambda^*} \subset V^{\lambda^*}_{\lambda^*}$ if X is in $(\mathcal{G}_\mu)_{\mathbf{C}}$ where $-\mu = \lambda^*$. But $(\mathcal{G}_\mu)_{\mathbf{C}} = \mathfrak{T}_{\mathbf{C}} + \Sigma_{\alpha \in \Phi'} \mathcal{G}_\alpha$, where $\Phi' = \{\alpha \in \Phi | \{\alpha, \mu\} = 0\}$. Now if $\alpha \in \Phi' \cap \Phi^+$, then $V^{\lambda^*}_{\lambda^* + \alpha} = (0)$, as λ^* is the highest weight. But then $V^{\lambda^*}_{\lambda^* - \alpha} = V^{\lambda^*}_{\sigma_\alpha(\lambda^* + \alpha)} = \rho_{\lambda^*}(n_\alpha)V^{\lambda^*}_{\lambda^* + \alpha} = (0)$, where

$\sigma_\alpha = n_\alpha T$. Thus the highest weight space of λ^* is invariant under $(\mathcal{G}_\mu)_{\mathbf{C}}$. Q.E.D.

In view of the Borel-Weil Theorem, we introduce the notation $\mathfrak{B}_\lambda = \mathfrak{M}(-\lambda^*)$, $\mathfrak{B}_\lambda^0 = \mathfrak{M}(-\lambda^*) \cap \mathfrak{M}'(-\lambda)$, for λ in Λ^+. The subrepresentation of L in $\mathfrak{B}_\lambda$ is equivalent to λ, and $\mathfrak{B}_\lambda^0$ is a one-dimensional subspace of $\mathfrak{B}_\lambda$, in fact being the λ weight-space of L in $\mathfrak{B}_\lambda$. We call $\mathfrak{B}_\lambda$ the *Borel-Weil* realization of λ in $\hat{G} = \Lambda^+$. Note that $\mathfrak{B}_\lambda$ as here defined agrees with the notation introduced in 2.4 and that f_1^λ is a spanning vector of $\mathfrak{B}_\lambda^0$. We call $\bigcup_{\lambda \in \Lambda^+} \mathfrak{B}_\lambda^0$ the set of *Borel-Weil functions*.

THEOREM (3.1.2). $\mathfrak{B}_\lambda^0 \mathfrak{B}_{\lambda'}^0 = \mathfrak{B}_{\lambda+\lambda'}^0$, *for* λ, λ' *in* Λ^+.

PROOF. The Leibnitz rule for differentiating a product of functions shows immediately that $\mathfrak{B}_\lambda^0 \mathfrak{B}_{\lambda'}^0 \subset \mathfrak{B}_{\lambda+\lambda'}^0$. Now, $(f_1^\lambda \cdot f_1^{\lambda'})(n_0^{-1}) = 1$ (see 2.4). Thus $\mathfrak{B}_\lambda^0 \mathfrak{B}_{\lambda'}^0 \neq 0$ and as $\mathfrak{B}_{\lambda+\lambda'}^0$ is one-dimensional, the result holds. Q.E.D.

Explicit formulas for Borel-Weil functions of the four classical series of compact simple Lie groups are detailed in [7].

3.2. *Cartan product and tensor product*. The results 3.2.1 and 3.2.4 below show how Borel-Weil realizations interact under point-wise multiplication and complex conjugation to yield realizations of the Cartan semigroup and tensor product operations.

THEOREM (3.2.1). $\mathfrak{B}_\lambda \mathfrak{B}_\nu = \mathfrak{B}_{\lambda+\nu}$.

PROOF. The Leibnitz rule and the definition of the $\mathfrak{B}_\lambda$'s shows at once that $\mathfrak{B}_\lambda \mathfrak{B}_\nu \subseteq \mathfrak{B}_{\lambda+\nu}$. $\mathfrak{B}_\lambda \mathfrak{B}_\nu$ is invariant under L and $\mathfrak{B}_{\lambda+\nu}$ is irreducible under L. Thus either $\mathfrak{B}_\lambda \mathfrak{B}_\nu = (0)$ or $\mathfrak{B}_\lambda \mathfrak{B}_\nu = \mathfrak{B}_{\lambda+\nu}$. But $0 \neq \mathfrak{B}_{\lambda+\nu}^0 \subseteq \mathfrak{B}_\lambda \mathfrak{B}_\nu$, by (3.1.2). Thus, $\mathfrak{B}_\lambda \mathfrak{B}_\nu = \mathfrak{B}_{\lambda+\nu}$. Q.E.D.

The next two results are preparatory to 3.2.4, but of interest in their own right.

THEOREM (3.2.2). *The multiplication map* M: $\mathfrak{B}_\lambda \otimes \overline{\mathfrak{B}}_\nu \to \mathfrak{B}_\lambda \overline{\mathfrak{B}}_\nu$ *is a bijection, for* λ, ν *in* Λ^+.

PROOF. M is clearly $\mathbf{C}$-linear and surjective. To show M is injective, choose T with $M(T) = 0$. We may write $T = \sum_{i,j} T_{ij} f_i^\lambda \otimes \bar{f}_j^\nu$, where the T_{ij} are certain numbers. Now set $\tau = \sum_{i,j} T_{ij} \tau_{ij}$, where $\{\tau_{ij}\}_{i,j}$ is the basis of $(V^{\lambda^*} \otimes V^\nu)^*$ dual to the basis $\{\phi_i^\lambda \otimes v_j^\nu\}_{i,j}$ of $V^{\lambda^*} \otimes V^\nu$. The relations of 2.4 may be used to see that, for a in G,

$$\tau\big((\lambda^* \otimes \nu)(a)(\phi_{d(\lambda)}^\lambda \otimes v_{d(\nu)}^\nu)\big) = M(T)(a).$$

Since $M(T) = 0$, we conclude, with the assistance of 2.5.1(b), that $\tau = 0$. But then $T_{ij=0}$ for all i, j which shows $T = 0$. Q.E.D.

PROPOSITION (3.2.3). *For* λ *in* Λ^+, $R(\underline{n_0})\mathfrak{B}_{\lambda^*} = \overline{\mathfrak{B}}_\lambda$. *Thus* $R(n_0)$ *provides a* G*-module equivalence between* $\mathfrak{B}_{\lambda^*}$ *and* $\overline{\mathfrak{B}}_\lambda$.

PROOF. Choose $\underline{f}$ in $\mathfrak{B}_{\lambda^*}$ and X in $\mathfrak{T}_{\mathbf{C}} + \mathfrak{N}^+$. Note that $\mathrm{Ad}(n_0^{-1})X$ is also in $\underline{\mathfrak{T}_{\mathbf{C}} + \mathfrak{N}^+}$ (here $\bar{X} = X_1 - iX_2$ if $X = X_1 + iX_2$, with $X_1, X_2 \in \mathcal{G}$) and that $\lambda(\bar{X}) = -\lambda(X)$ (as $\lambda(\mathcal{G}) \subset i\mathbf{R}$). Then,

$$R(X)R(n_0)\bar{f} = R(n_0)\overline{R\left(\overline{\mathrm{Ad}(n_0^{-1})X}\right) f} = R(n_0)\overline{\lambda^*\left(\overline{\mathrm{Ad}(n_0^{-1})X}\right) f}$$

$$= \overline{(w_0\lambda^*)(\bar{X})}\, R(n_0)\bar{f} = -\overline{\lambda(\bar{X})}\, R(n_0)\bar{f} = \lambda(X)R(n_0)\bar{f}.$$

Thus, $R(n_0)\bar{f}$ is in $\mathfrak{B}_\lambda$, so $R(n_0)f$ is in $\overline{\mathfrak{B}}_\lambda$. Q.E.D.

COROLLARY (3.2.4). *The map* $f \otimes g \to f.\ R(n_0)g$ *yields an isomorphism of* G*-modules between* $\mathfrak{B}_\lambda \otimes \mathfrak{B}_\nu$ *and* $\mathfrak{B}_\lambda\overline{\mathfrak{B}}_{\nu^*}$ *for any* λ, ν *in* Λ^+.

Of course, this isomorphism is nonunitary, in general; but still $\mathrm{ch}(\lambda \otimes \nu) = \mathrm{ch}(\mathfrak{B}_\lambda\overline{\mathfrak{B}}_{\nu^*})$.

4. G-module structure of certain infinite-dimensional representations. Throughout §4, λ in Λ^+ is fixed and for convenience usually assumed nonzero.

4.1. *The Hopf bundle for* λ. Set $K = G_{\lambda^*} = \{a \in G \mid \mathrm{ad}^*(a)\lambda^* = \lambda^*\}$, $M = G/K$, $\mathcal{K}$ the Lie algebra of K. K is a compact connected subgroup of G (see 6.6.2 of [16]).

PROPOSITION (4.1.1); *There is a unique one-dimensional unitary representation* χ: $K \to S^1$ *so that* $\chi(\exp X) = e^{\lambda^*(X)}$ *for* $X \in \mathcal{K}$.

PROOF. Since K is compact connected, the given formula ensures the uniqueness of χ. Since $\lambda^*(\mathcal{I}) \subset 2\pi i\mathbf{Z}$ and T is abelian, there does exist a character χ on T so that $\chi(\exp H) = e^{\lambda^*(H)}$ for H in $\mathfrak{T}$. If $aTa^{-1} = T$ for a in K, then the element a represents an element of the Weyl group W_K of T in K; but W_K is generated by the reflections σ_α, where α is in Φ with $\mathcal{G}_\alpha \subset \mathcal{K}_{\mathbf{C}}$, and $\sigma_\alpha\lambda^* = \lambda^*$ for such α; therefore $\chi(ata^{-1}) = \chi(t)$. Thus, there is a unique continuous class function χ on K extending χ on T (see 4.32 of [1]). It is immediate that $\chi(\exp X) = e^{\lambda^*(X)}$ for X in $\mathcal{K}$, and it remains to see that χ is a group homomorphism. For this we write $\mathcal{K} = \mathfrak{Z}(\mathcal{K}) \oplus [\mathcal{K}, \mathcal{K}]$ and note that $\lambda^*([\mathcal{K}, \mathcal{K}]) = 0$, $\exp \mathfrak{Z}(\mathcal{K}) \subset Z(K)$, and the connected Lie subgroup with Lie algebra $[\mathcal{K}, \mathcal{K}]$ is compact connected. Choose (see [15, p. 234]) a, b in K, X, Y in $\mathcal{K}$ with $\exp X = a$, $\exp Y = b$, and write $X = X_1 + X_2$, $Y = Y_1 + Y_2$ according to the preceding decomposition of $\mathcal{K}$, and choose Z_2 in $[\mathcal{K}, \mathcal{K}]$ with $\exp X_2 \exp Y_2 = \exp Z_2$. Then one sees that $ab = \exp(X_1 + Y_1 + Z_2)$, from which it follows that $\chi(ab) = \chi(a)\chi(b)$. Thus χ is a continuous homomorphism from K to S^1. It follows that χ is also smooth. Q.E.D.

Set $K_0 = \ker \chi$; K_0 is a closed normal subgroup of K, and a closed subgroup of G. If $\lambda^* = 0$, then $K_0 = K = G$. Since $\lambda^* \neq 0$, the induced homomorphism $\tilde{\chi}$: $K/K_0 \to S^1$ is a Lie group isomorphism, by means of which we identify K/K_0 and S^1. If $K = G$, then most of our theory is, while valid, trivial; as examples $\lambda^* = w_1 + \cdots + w_n$, $G = U(n)$, $\chi = \det$; and $\chi = \lambda^*$ if G is abelian.

Set $P = G/K_0$ and let π: $P \to M$ send aK_0 to aK. Using χ, we may define $P \times S^1 \to P$ by $(aK_0, \zeta) \to (ax)K_0$, where $x \in K$ is chosen so that $\chi(x) = \zeta$. Then π is the projection for a principal S^1 bundle, which we refer to as the Hopf bundle for λ. We show first how to describe this bundle in terms of the element λ^* of $\hat{G}$.

Proposition (4.1.2). *The map from G to V^{λ^*} sending a to $\lambda^*(a)\phi^{\lambda}_{d_\lambda}$ determines an S^1-bundle equivalence of π with $P_0 \to^{\pi_0} M_0$, where P_0 is the orbit of $\phi^{\lambda}_{d_\lambda}$ in V^{λ^*}, M_0 is the orbit of $V^{\lambda^*}_{\lambda^*}$ in the projective representation determined by λ^*, and π_0 is the restriction to P_0 of the natural projection. The S^1 action on P_0 is induced by scalar multiplication in V^{λ^*}.*

Proof. Several steps are required in the proof. Abbreviate $\phi^{\lambda}_{d_\lambda}$ by ϕ, V^{λ^*} by V.

Step 1. $K = \{a \in G | \lambda^*(a)V_{\lambda^*} = V_{\lambda^*}\}$.

Proof of Step 1. Set $K' = \{a \in G|\lambda^*(a)V_{\lambda^*} = V_{\lambda^*}\}$, $\mathfrak{K}'$ the Lie algebra of K'. There is χ' in $\hat{K}'$, defined by $\lambda^*(a)\phi = \chi'(a)\phi$, $a \in K'$. Let $\theta \in \mathfrak{K}'^*_{\mathbf{C}}$ be the infinitesimal representation for χ'.

The proof of 3.1.1(d) establishes that $\mathfrak{K} \subset \mathfrak{K}'$. Since K is connected, $\mathfrak{K} \subset \mathfrak{K}'$.

We claim θ is the restriction to $\mathfrak{K}'_{\mathbf{C}}$ of λ^* in $\mathfrak{G}^*_{\mathbf{C}}$. Clearly λ^* and θ agree on $\mathfrak{T}_{\mathbf{C}}$. Now $\mathfrak{K}'_{\mathbf{C}}$ is the sum of $\mathfrak{T}_{\mathbf{C}}$ and the span of certain root vectors; this latter span being contained in the derived algebra $[\mathfrak{K}'_{\mathbf{C}}, \mathfrak{K}'_{\mathbf{C}}]$ of $\mathfrak{K}'_{\mathbf{C}}$, θ vanishes on it, as does λ^*. Thus λ^* agrees with θ on $\mathfrak{K}'_{\mathbf{C}}$.

Next we show $K' \subset K$. Fix X in $\mathfrak{K}'$, a in K'. Then

$$\begin{aligned}\lambda^*(\mathrm{ad}(a)X)\cdot\phi &= \theta(\mathrm{ad}(a)X)\cdot\phi = \rho_{\lambda^*}(\mathrm{ad}(a)X)\cdot\phi\\ &= \rho_{\lambda^*}(a)\rho_{\lambda^*}(X)\rho_{\lambda^*}(a^{-1})\cdot\phi = \chi'(a)\theta(X)\chi'(a^{-1})\cdot\phi\\ &= \theta(X)\cdot\phi = \lambda^*(X)\cdot\phi.\end{aligned}$$

Thus, $(\mathrm{ad}^*(a)\lambda^*)|_{\mathfrak{K}'} = \lambda^*|_{\mathfrak{K}'}$. Since $\mathrm{ad}(a)\mathfrak{K}' \subset \mathfrak{K}'$, also $\mathrm{ad}(a)\mathfrak{K}'^{\perp} \subset \mathfrak{K}'^{\perp}$; and one has $(\mathrm{ad}^*(a)\lambda^*)|_{\mathfrak{K}'^{\perp}} = 0 = \lambda^*|_{\mathfrak{K}'^{\perp}}$. We conclude that $\mathrm{ad}^*(a)\lambda^* = \lambda^*$, i.e., a is in K. This concludes Step 1.

Step 2. $K_0 = \{a \in G|\lambda^*(a)\cdot\phi = \phi\}$.

This is trivial when Step 1 is used.

Step 3. Conclusion of proof of proposition. We may, by Steps 1 and 2, define $\tilde{F}$: $P \to P_0$ and F: $M \to M_0$ by $\tilde{F}(aK_0) = \lambda^*(a)\phi$, $F(aK) = \lambda^*(a)V_{\lambda^*}^{\lambda^*}$, and moreover $\tilde{F}$ and F are bijections. $\tilde{F}$ is easily seen to be an equivalence of S^1 bundles, with $\pi_0\tilde{F} = F\pi$. Q.E.D.

4.2. *Main theorem.* Recall from §1 the subrepresentations Γ and Γ_k ($k \in \mathbf{Z}$) of the left-regular representation of G.

We relate our discussion here to that of §3 by introducing

$$\Gamma_{p,q} = \mathfrak{B}_{p\lambda}\overline{\mathfrak{B}}_{q\lambda}, \qquad p, q \in \mathbf{Z}^+.$$

PROPOSITION (4.2.1). (a) $\Gamma_k \subseteq \Gamma$ *for* k *in* $\mathbf{Z}$.

(b) $\Gamma_{p,q} \subseteq \Gamma_{p-q}$ *for* p, q *in* $\mathbf{Z}^+$.

PROOF. (a) is clear. For (b), choose $f \in \mathfrak{B}_{p\lambda}$, $g \in \mathfrak{B}_{q\lambda}$, and $x \in K$. Then

$$R(x)(f\bar{g}) = (R(x)f)(R(x)\bar{g}) = (R(x)f)\,\overline{(R(x)g)}$$

$$= \chi(x)^p\,\overline{\chi(x)^q}\,f\bar{g} = \chi(x)^{p-q}(f\bar{g}).$$

Thus $f\bar{g} \in \Gamma_{p-q}$, as desired. Q.E.D.

PROPOSITION (4.2.2). *The algebraic sum* $\Sigma_{p,q\in\mathbf{Z}^+}\Gamma_{p,q}$ *is dense in* Γ.

PROOF. The continuous functions Γ' in Γ are dense in Γ. Γ' may be regarded as $C(P)$, the continuous functions on P (see (4.1.2)). It will suffice to show $\tilde{\Gamma} = \Sigma_{p,q}\Gamma_{p,q}$ is dense in Γ' with respect to the sup-norm. As P is compact, this density will follow from the Stone-Weierstrass Theorem, provided we show that $\tilde{\Gamma}$ has the following properties:

(i) If $f, g \in \tilde{\Gamma}$, and $z \in \mathbf{C}$ then $f + zg$, fg, and $\bar{f} \in \tilde{\Gamma}$.

(ii) For all $u \in P$, there is f in $\tilde{\Gamma}$ with $f(u) \neq 0$.

(iii) For all u_1, u_2 in P with $u_1 \neq u_2$, there is f in $\tilde{\Gamma}$ with $f(u_1) \neq f(u_2)$.

We show $\tilde{\Gamma}$ has these three properties.

PROOF OF (i). $\Gamma_{p,q}$ is a complex subspace and $\overline{\Gamma}_{p,q} = \Gamma_{q,p}$, so $f + zg$ and $\bar{f}$ are in $\tilde{\Gamma}$. From (3.6),

$$\Gamma_{p,q}\Gamma_{p',q'} = \mathfrak{B}_{p\lambda}\overline{\mathfrak{B}}_{q\lambda}\mathfrak{B}_{p'\lambda}\overline{\mathfrak{B}}_{q'\lambda} = \mathfrak{B}_{(p+p')\lambda}\overline{\mathfrak{B}}_{(q+q')\lambda} = \Gamma_{p+p',q+q'}.$$

Thus fg is in $\tilde{\Gamma}$.

PROOF OF (ii). We will in fact find the desired functions for (ii) and (iii) in $\Gamma_{1,0} = \mathfrak{B}_\lambda$. As $\mathfrak{B}_\lambda \neq 0$, choose f_0 in $\mathfrak{B}_\lambda$, a_0 in G with $f_0(a_0) \neq 0$. Then for a in G, $0 \neq f_0(a_0) = f_0(a_0a^{-1}a) = (L_{aa_0^{-1}}f_0)(a)$. But $L_{aa_0^{-1}}f_0$ is in $\mathfrak{B}_\lambda$. This proves (ii).

PROOF OF (iii). As in (ii), the homogeneity of P and the result $L(G)\mathfrak{B}_\lambda \subset \mathfrak{B}_\lambda$ reduces the question to demonstrating the validity of the following statement:

$$\forall a \notin K_0 \quad \exists f \in \mathfrak{B}_\lambda \quad f(a) \neq f(e).$$

Now the statement in question is false if and only if

$$\exists a \notin K_0 \quad \forall f \in \mathfrak{B}_\lambda \quad f(a) = f(e).$$

Thus, we need to show that

$$K_0 \supset \{a \in G | \forall f \in \mathfrak{B}_\lambda, \ \ f(a) = f(e)\}.$$

Let S be the set we want K_0 to contain. Then, using again $L(G)\mathfrak{B}_\lambda \subset \mathfrak{B}_\lambda$, one shows

$$S = \{a \in G | \forall f \in \mathfrak{B}_\lambda, \ \ R(a) f = f\}.$$

Letting $\mathfrak{B}'_\lambda = \mathfrak{F}(\mathfrak{B}_\lambda) \subset V^\lambda \otimes V^{\lambda^*}$, we recall that $\mathfrak{B}'_\lambda = V^\lambda \otimes \{\phi\}$ ($\phi = \phi^\lambda_{d_\lambda}$) so $S = \{a \in G | \lambda^*(a) \cdot \phi = \phi\}$. Thus $S = K_0$ by Step 2 of the proof of (4.1.2). Q.E.D.

Proposition (4.2.3). $\Gamma = \bigoplus_{k \in \mathbf{Z}} \Gamma_k$, *a Hilbert space direct sum.*

Proof. Choose distinct integers k and l; since $\lambda^* \neq 0$, we may choose x in K with $\chi(x)^{k-l} \neq 1$. Then for f in Γ_k and g in Γ_l, $\{f, g\} = \chi(x)^{k-l}\{f, g\} = 0$; thus $\Gamma_k \perp \Gamma_l$. By the previous two propositions $\Sigma_{k \in \mathbf{Z}} \Gamma_k$ is dense in Γ; (4.2.3) follows. Q.E.D.

Proposition (4.2.4). $\Gamma_{p,q} \subset \Gamma_{p+1,q+1}$, *for p, q in $\mathbf{Z}^+$.*

Proof.

$$\Gamma_{p+1,q+1} = \mathfrak{B}_{(p+1)\lambda} \overline{\mathfrak{B}}_{(q+1)\lambda} = \mathfrak{B}_{p\lambda} \overline{\mathfrak{B}}_{q\lambda} \mathfrak{B}_\lambda \overline{\mathfrak{B}}_\lambda.$$

Thus it suffices to show $1 \in \mathfrak{B}_\lambda \overline{\mathfrak{B}}_\lambda$. But from 2.4, $1 = \Sigma_i f_i^\lambda \bar{f}_i^\lambda \in \mathfrak{B}_\lambda \overline{\mathfrak{B}}_\lambda$. Q.E.D.

The same proof shows that $\mathfrak{B}_{\nu_1} \overline{\mathfrak{B}}_{\nu_2} \subseteq \mathfrak{B}_{\nu_1+\nu_3} \overline{\mathfrak{B}}_{\nu_1+\nu_3}$, for any ν_1, ν_2, ν_3 in Λ^+.

Proposition (4.2.5). *Let ν_1, ν_2 be in Λ^+. Then*

$$\{\mathrm{ch}(\nu_1 + n\nu_2 \otimes n\nu_2^*)\}_{n=0}^{\infty}, \qquad \{\mathrm{ch}(n\nu_2 \otimes \nu_1 + n\nu_2^*)\}_{n=0}^{\infty}$$

are increasing sequences in $\mathbb{S}$, bounded above by d. Thus, their limits exist in $\mathbb{S}$.

Proof. By (3.1.1), (3.2.4) and (4.2.4),

$$\mathrm{ch}((\nu_1 + n\nu_2) \otimes n\nu_2^*) = \mathrm{ch}(\mathfrak{B}_{\nu_1+n\nu_2} \otimes \mathfrak{B}_{n\nu_2^*}) = \mathrm{ch}\left(\mathfrak{B}_{\nu_1} \mathfrak{B}_{n\nu_2} \overline{\mathfrak{B}}_{n\nu_2}\right)$$

$$\leqslant \mathrm{ch}\left(\mathfrak{B}_{\nu_1} \mathfrak{B}_{n\nu_2} \overline{\mathfrak{B}}_{n\nu_2} \mathfrak{B}_{\nu_2} \overline{\mathfrak{B}}_{\nu_2}\right) = \mathrm{ch}(\nu_1 + (n+1)\nu_2 \otimes (n+1)\nu_2^*),$$

so the first sequence is increasing. Since $\mathrm{ch}(\mathfrak{B}_{\mu_1} \overline{\mathfrak{B}}_{\mu_2}) < d$ for μ_1, μ_2 in Λ^+, the above expression assures the first sequence bounded above by d. This proves the proposition for the first sequence; the proof for the second sequence is entirely analogous. Q.E.D.

THEOREM (4.2.6).

$$\mathrm{ch}(\Gamma_k) = \lim_{n\to\infty} \mathrm{ch}(k\lambda + n\lambda \otimes n\lambda^*), \qquad \mathrm{ch}(\Gamma_{-k}) = \lim_{n\to\infty} \mathrm{ch}(n\lambda \otimes k\lambda^* + n\lambda^*)$$

for k in $\mathbf{Z}^+$. The sequences in $\mathbb{S}$ involved are increasing and bounded above.

PROOF. By (4.2.5) we know the limits involved exist. The argument being similar in both cases, consider Γ_k; call the limit in question f. From 3.2.4, $\mathrm{ch}(\Gamma_{k+n,n}) = \mathrm{ch}(k\lambda + n\lambda \otimes n\lambda^*)$; now application of (4.2.1)–(4.2.4) shows that $f = \mathrm{ch}(\Gamma_k)$. Q.E.D.

4.3. *Some formulas holding for Γ_k in general.* Let $\lambda \neq 0$ in Λ^+ be chosen. For f in $\mathbb{S}$, define f^* in $\mathbb{S}$ by $f^*(\mu) = f(\mu^*)$, for μ in Λ^+.

PROPOSITION (4.3.1). *For k in $\mathbf{Z}^+$,*
(a) $\mathrm{ch}(\Gamma_k(\lambda)) \geqslant \Sigma_{n=0}^{\infty} \mathrm{ch}(k\lambda + n\lambda + n\lambda^*)$,
(b) $\mathrm{ch}(\Gamma_{-k}(\lambda)) = \mathrm{ch}(\Gamma_k(\lambda))^* = \mathrm{ch}(\Gamma_k(\lambda^*))$,
(c) $\mathrm{ch}(\Gamma_k(\lambda)) = \mathrm{ch}(\Gamma_1(k\lambda))$, *if $k \neq 0$.*

PROOF. (a) $\mathrm{ch}(\Gamma_k) = \lim_{n\to\infty}\mathrm{ch}(k\lambda + n\lambda \otimes n\lambda^*) \geqslant \mathrm{ch}(k\lambda + N\lambda \otimes N\lambda^*)$, for fixed $N \in \mathbf{Z}^+$. But $\mathrm{ch}(k\lambda + N\lambda \otimes N\lambda^*) \geqslant \mathrm{ch}(k\lambda + N\lambda + N\lambda^*)$, by the usual realization of the Cartan product [11, p. 111]. (a) follows.

(b)
$$\begin{aligned} \mathrm{ch}(\Gamma_k(\lambda))^* &= \left(\lim_{n\to\infty} \mathrm{ch}(k\lambda + n\lambda \otimes n\lambda^*)\right)^* \\ &= \lim_{n\to\infty} \mathrm{ch}(n\lambda \otimes k\lambda^* + n\lambda^*) = \mathrm{ch}(\Gamma_{-k}(\lambda)). \end{aligned}$$

Similarly $\mathrm{ch}(\Gamma_k(\lambda^*)) = \lim_{n\to\infty}\mathrm{ch}(k\lambda^* + n\lambda^* \otimes n\lambda) = \mathrm{ch}(\Gamma_{-k}(\lambda))$.

(c)
$$\begin{aligned} \Gamma_k(\lambda) &= \left\{ f \in L^2(G) \,|\, R(\exp X) f = (e^{\lambda^*(X)})^k f,\ X \in \mathcal{G}_\lambda \right\} \\ &= \left\{ f \in L^2(G) \,|\, R(\exp X) f = e^{k\lambda^*(X)} f,\ X \in \mathcal{G}_\lambda \right\} \\ &= \Gamma_1(k\lambda), \quad \text{as } \mathcal{G}_\lambda = \mathcal{G}_{k\lambda} \text{ since } \lambda \neq 0. \quad \text{Q.E.D.} \end{aligned}$$

One might be better able to think about the infinite series in part (a) of the proposition if it were summed in a closed form. This idea may be formalized as follows. In order to avoid an additional symbol, let $\mathbb{S}$ now represent the set of all $\mathbf{Z}$-valued functions defined on Λ^+ (earlier, we restricted to $\mathbf{Z}^+$-valued functions). The set $\{\mathrm{ch}(\lambda) | \lambda \in \Lambda^+\}$ is a $\mathbf{Z}$-independent subset and we may write $f = \Sigma_{\lambda\in\Lambda^+} f(\lambda)\mathrm{ch}(\lambda)$ for f in $\mathbb{S}$. Define $\mathrm{ch}(\lambda)\mathrm{ch}(\lambda') = \mathrm{ch}(\lambda + \lambda')$, and extend this definition to elements f, g in $\mathbb{S}$ by

$$fg = \sum_{\lambda,\lambda'} f(\lambda) g(\lambda')\mathrm{ch}(\lambda + \lambda'),$$

whenever the summation converges absolutely to an element of $\mathbb{S}$. In particular fg is defined if either f or g is 0 off some finite set. Notice that $\mathrm{ch}(0)f = f$ for f in $\mathbb{S}$. By introducing the formal identity $1/(1 - x) = \Sigma_{n=0}^{\infty} x^n$, for elements x in $\mathbb{S}$ whose powers are defined, we obtain the expression

$$\sum_{n=0}^{\infty} \operatorname{ch}(k\lambda + n\lambda + n\lambda^*) = \frac{\operatorname{ch}(k\lambda)}{\operatorname{ch}(0) - \operatorname{ch}(\lambda + \lambda^*)}.$$

We will use such formalism without comment in the sequel.

4.4. *The usual representation of $SU(n+1)$, $n \geqslant 1$.* Investigation of the present example was suggested to the author J. W. Robbin and led to the main theorem when correctly viewed. Namely, we consider $G = SU(n+1)$, $n \geqslant 1$, in its natural representation on $\mathbf{C}^{n+1}$. If $\{e_i\}_{i=1,\ldots,n+1}$ is the usual basis of $\mathbf{C}^{n+1}$, then $ae_i = \Sigma_j a_{ji} e_j$, for $a \in G$. The highest weight of this representation is Λ_1 with weight vector e_1. In view of (4.1.1) we should take $\lambda^* = \Lambda_1$, $\lambda = \Lambda_n$, $e_1 = \phi = \phi^\lambda_{d_\lambda}$. The basis $\{f^\lambda_k\}_{i=1,\ldots,n+1}$ of $\mathfrak{B}_\lambda$ (see proof of (3.2.2)) may be regarded as the restrictions $\{z_i\}_{i=1,\ldots,n+1}$ to $Ge_1 = S^{2n+1} = P_0$ of the complex coordinate functions on $\mathbf{C}^{n+1}$. Employing the usual multinomial notation $Z^I\bar{Z}^J = Z_1^{I_1} \ldots \bar{Z}_{n+1}^{J_{n+1}}$, we see that $\Gamma_{p,q} = \mathfrak{B}_{p\lambda}\overline{\mathfrak{B}}_{q\lambda} = \mathfrak{B}^p_\lambda \overline{\mathfrak{B}}^q_\lambda$ is spanned by $\{Z^I\bar{Z}^J \mid |I| = I_1 + \cdots + I_{n+1} = p, |J| = J_1 + \cdots + J_{n+1} = q\}$. The inclusions $\Gamma_{p,q} \subset \Gamma_{p+1,q+1}$ of (4.2.4) result from the fact that $\Sigma_{i=1}^{n+1} z_i \bar{z}_i = 1$ on P_0. M_0 is $\mathbf{C}P^n$ and $\pi_0 \colon P_0 \to M_0$ is the (usual) Hopf map.

We see that $\mathfrak{B}_\lambda$ may be regarded as the restriction to P_0 of the linear functions on $\mathbf{C}^{n+1}$. Thus, $\mathfrak{B}_\lambda$ may be regarded as sections of the line bundle dual to the Hopf bundle; this dual Hopf bundle is associated with the principal S^1 bundle $\pi_0^* \colon P_0^* \to M_0$, where $P_0^* = P_0, \pi_0^* = \pi_0$, but $\psi \cdot \zeta = \zeta^{-1} \cdot \psi$ for $\psi \in P_0^*$, $\zeta \in S^1$.

PROPOSITION (4.4.1).

(a) $$\operatorname{ch}(\Gamma_k) = \frac{\operatorname{ch}(\lambda)^k}{\operatorname{ch}(0) - \operatorname{ch}(\lambda + \lambda^*)}, \qquad k \in \mathbf{Z}^+.$$

(b) $$\operatorname{ch}(\Gamma) = \frac{\operatorname{ch}(0)}{(\operatorname{ch}(0) - \operatorname{ch}(\lambda))(\operatorname{ch}(0) - \operatorname{ch}(\lambda^*))}.$$

PROOF. (a) The point here is that

$$\operatorname{ch}((p+1)\lambda + (q+1)\lambda^*) = \operatorname{ch}(\Gamma_{p+1,q+1}) - \operatorname{ch}(\Gamma_{p,q});$$

this formula is proved using the Weyl dimension formula. The result then follows from (4.3).

(b) $\operatorname{ch}(\Gamma) = \Sigma_{k=-\infty}^{\infty} \operatorname{ch}(\Gamma_k)$. Setting $x = \operatorname{ch}(\lambda)$, $y = \operatorname{ch}(\lambda^*)$ and using (a) and (4.3), one has (setting $\operatorname{ch}(0) = 1$)

$$\begin{aligned}\operatorname{ch}(\Gamma) &= \left[(1-x)^{-1} + (1-y)^{-1} - 1\right](1-xy)^{-1} \\ &= (1-x)^{-1}(1-y)^{-1}. \quad \text{Q.E.D.}\end{aligned}$$

4.5. *Use of Steinberg's formula.* The expression for $\operatorname{ch}(\Gamma_k)$ may be said to completely solve the question of Γ_k's G-module structure, as it expresses

$\mathrm{ch}(\Gamma_k)$ in terms of certain $\mathrm{ch}(\nu_1 \otimes \nu_2)$, ν_1, ν_2 in $\hat{G}$. There is a certain closed expression for a general 'outer' multiplicity $\mathrm{ch}(\nu_1 \otimes \nu_2)(\nu_3)$, ν_1, ν_2, $\nu_3 \in \Lambda^+$, namely Steinberg's formula [8, p. 141].

Steinberg's formula has certain drawbacks as a computational device, requiring as it does a double summation over the Weyl group and a knowledge of Kostant's partition function. The interested reader may refer to the references in [3, p. 120] for examples.

One question arising in the computation of

$$\mathrm{ch}(\Gamma_1(\lambda))(\nu) = \lim_{n\to\infty} \mathrm{ch}(\lambda + n\lambda \otimes n\lambda^*)(\nu)$$

is the determination of the lowest value $n(\lambda, \nu)$ of n at which

$$\mathrm{ch}(k\lambda + n\lambda \otimes n\lambda^*)(\nu) = \mathrm{ch}(\Gamma_1(\lambda))(\nu).$$

At present the author has no general information regarding this function $n(\lambda, \nu)$.

4.6. *Frobenius reciprocity*; $SU(2)$. Our representation Γ_k is an induced representation. Namely, it is the unitary representation of G induced from the unitary representation χ^{-k} of the closed subgroup $K = G_{\lambda^*}$, χ being the character for λ^* on K. As with all unitarily induced representations of G, we may analyze Γ_k by means of the Frobenius reciprocity relation. When used in coordination with such formulas as those of Kostant and Freudenthal [8, pp. 122, 138], the following Frobenius reciprocity statement is very useful computationally for regular λ.

PROPOSITION (4.6.1). $\mathrm{ch}(\Gamma_k(\lambda))(\nu) = \dim\{v \in V^\nu | \nu(x)v = \chi^{-k}(x)v$ *if* $x \in K\}$. *In particular*, $\mathrm{ch}(\Gamma_k(\lambda))(\nu) \leqslant \dim V^\nu_{k\lambda}$, *with equality occurring if* λ *is regular*.

PROOF. $\mathrm{ch}(\Gamma_k)(\nu) = \dim \mathrm{Hom}_G(V^\nu, \Gamma_k)$, so by 5.3.6 of [16], $\mathrm{ch}(\Gamma_k)(\nu) = \dim \mathrm{Hom}_K(V^\nu, \chi^{-k})$, where $\mathrm{Hom}_K(V^\nu, \chi^{-k}) = \{\psi \in V^{\nu *} | \psi(\nu(x)v) = \chi^{-k}(x)\psi(v)$ if $x \in K$ and $v \in V\} = \{\psi \in V^{\nu *} | \nu^*(x)(\psi) = \chi^k(x)\psi$ if $x \in K\}$. Along with (4.3) one concludes $\mathrm{ch}(\Gamma_k)(\nu) = \mathrm{ch}(\Gamma_k)^*(\nu^*) = \mathrm{ch}(\Gamma_{-k})(\nu^*) = \dim\{v \in V^\nu | \nu(x)(v) = \chi^{-k}(x)v$ if $x \in K\}$. This shows the first formula of the proposition. Now since $\chi^{-k}(\exp H) = e^{-k\lambda^*(H)}$ for H in $\mathfrak{T}$, the set whose dimension equals $\mathrm{ch}(\Gamma_k(\lambda))(\nu)$ is contained in $V^\nu_{-\lambda^*}$, with the containment being equality when λ is regular. Applying (2.3), one gets $\mathrm{ch}(\Gamma_k(\lambda))(\nu) \leqslant \dim V^\nu_{-k\lambda^*} = \dim V^\nu_{w_0(k\lambda)} = \dim V^\nu_{k\lambda}$, with equality if λ is regular. Q.E.D.

The proposition yields $\mathrm{ch}(\Lambda_k)$ at once for $G = SU(2)$. Namely, suppose $\lambda = m\Lambda_1$, $m \in \mathbf{Z}^+$, $m \neq 0$. Suppose $\nu = n\Lambda_1$, $n \in \mathbf{Z}^+$. Then $\dim V^\nu_{k\lambda} = \dim V^{n\Lambda_1}_{km\Lambda_1} = 1$ if $-n \leqslant km \leqslant n$ and $km \equiv n \pmod 2$, 0 otherwise. Thus, if $x = \mathrm{ch}(\Lambda_1)$,

$$\mathrm{ch}(\Lambda_k) = \sum_{n=0}^{\infty} \mathrm{ch}((km + 2n)\Lambda_1) = \frac{x^{km}}{\mathrm{ch}(0) - x^2}.$$

When $m = 1$, we recover a formula of (4.3) as here $\lambda = \lambda^*$.

4.7. $SU(3)$. We introduce notation which is convenient for the algebraic expressions for $\mathrm{ch}(\Lambda_1(\lambda))$, $\mathrm{ch}(\Gamma_0(\lambda))$, $\lambda \neq 0 \in \Lambda^+$. Recalling the usual weight-root notations for $SU(3)$ (see [8]), set $\alpha = 2\Lambda_1 - \Lambda_2$, $\beta = -\Lambda_1 + 2\Lambda_2$, $\delta = \alpha + \beta = \Lambda_1 + \Lambda_2$; where Λ_1, Λ_2 are usual fundamental weights; $\Phi^+ = \{\alpha, \beta, \delta\}$. In $\mathcal{S}$ set $x_i = \mathrm{ch}(\Lambda_i)$, $i = 1, 2$, $A = x_1^2 X_2^{-1}$, $B = x_1^{-1}x_2^2$; $\mathrm{ch}(0) = 1$ in $\mathcal{S}$.

PROPOSITION (4.7.1). *Let* $\lambda = m_1\Lambda_1 + m_2\Lambda_2 \neq 0 \in \Lambda^+$. *Then*:

(a) *If* $m_1 m_2 = 0$, *then*

$$\mathrm{ch}(\Gamma_1(\lambda)) = \frac{x_1^{m_1}x_2^{m_2}}{(1 - x_1x_2)}, \qquad \mathrm{ch}(\Gamma_0(\lambda)) = \frac{1}{(1 - x_1x_2)}.$$

(b) *If* $m_1 m_2 \neq 0$, *then*

$$\mathrm{ch}(\Gamma_1(\lambda)) = \frac{x_1^{m_1}x_2^{m_2}}{(1 - x_1x_2)^2}\left\{1 + \frac{A}{1 - A}(1 - A^{m_2}) + \frac{B}{1 - B}(1 - B^{m_1}) + \frac{x_1^3}{1 - x_1^3}A^{m_2} + \frac{x_2^3}{1 - x_2^3}B^{m_1}\right\}$$

and

$$\mathrm{ch}(\Gamma_0(\lambda)) = \frac{1}{(1 - x_1x_2)^2}\left\{-1 + \frac{x_1^3}{1 - x_1^3} + \frac{x_2^3}{1 - x_2^3}\right\}.$$

PROOF. (a) When $\lambda = \Lambda_2$, we have found $\mathrm{ch}(\Gamma_k(\lambda))$, for k in $\mathbf{Z}$, in (4.4). The cases in (a) follow by applying (4.3).

(b) In this case λ is regular. To prove the formula for $\mathrm{ch}(\Gamma_1(\lambda))$ one need only show, by Frobenius reciprocity, that dim V_ν^ν is given by the right-hand side of the desired equation. This may be accomplished by using the Kostant multiplicity formula (see [12, p. 131]; for the partition function for $SU(3)$ see [16, Table I]). For details, see [7]; the main theorem serves as a heuristic device to suggest the result and its method of proof (since an algorithm for tensor products of irreducibles is known (see [16])). Q.E.D.

BIBLIOGRAPHY

1. J. F. Adams, *Lectures on Lie groups*, Benjamin, New York, 1969. MR **40** #5780.

2. L. Auslander and B. Kostant, *Polarization and unitary representations of solvable Lie groups*, Invent. Math. **14** (1971), 255–354. MR **45** #2092.

3. J. G. T. Belinfante and B. Kolman, *A survey of Lie groups and Lie algebras with applications and computational methods*, SIAM, Philadelphia, 1973.

4. A. Borel and F. Hirzbruch, *Characteristic classes and homogeneous spaces*. I, Amer. J. Math. **80** (1958), 458–538. MR **21** #1586.

5. C. Chevalley, *Theory of Lie groups*. I, Princeton Univ. Press, Princeton, N.J., 1946. MR **7**, 412.

6. L. Fonda and G. C. Ghirardi, *Symmetry principles in quantum physics*, Dekker, New York, 1970.

7. J. Funderburk, *Module structure of certain induced representations of compact Lie groups*, Ph.D. Thesis, Univ. of Wisconsin, Madison, May, 1975.

8. J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Springer-Verlag, New York, 1972. MR **48** #2197.

9. S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Vol. II, Interscience, New York, 1969. MR **38** #6501.

10. B. Kostant, *Lie algebra cohomology and the generalized Borel-Weil theorem*, Ann. of Math. (2) **74** (1961), 329–387. MR **26** #265.

11. ———, *Quantization and unitary representations*, I. *Prequantization*, Lectures in Modern Analysis and Applications, III, Lecture Notes in Math., vol. 170, Springer-Verlag, Berlin, 1970, pp. 87–208. MR **45** #3638.

12. H. Samelson, *Notes on Lie algebras*, Mathematical Studies, no. 23, Van Nostrand Reinhold, New York, 1969. MR **40** #7322.

13. J.-P. Serre, *Représentations linéaires et espaces homogènes kählériens des groupes de Lie compacts*, Séminaire Bourbaki: 1953/54, Exposé 100, 2nd corr. ed., Secrétariat mathématique, Paris, 1959. MR **28** #1087.

14. J.-M. Souriau, *Structure des systèmes dynamiques*, Maîtrises de Mathématiques, Dunod, Paris, 1970. MR **41** #4866.

15. S. Sternberg, *Lectures on differential geometry*, Prentice-Hall, Englewood Cliffs, N.J., 1964. MR **33** #1797.

16. J. Tarski, *Partition function for certain simple Lie algebras*, J. Mathematical Phys. **4** (1963), 569–574. MR **26** #5099.

17. Nolan R. Wallach, *Harmonic analysis on homogeneous spaces*, Dekker, New York, 1973.

DEPARTMENT OF MATHEMATICAL SCIENCES, NORTHERN ILLINOIS UNIVERSITY, DEKALB, ILLINOIS 60115