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THE GENERALIZED GREEN'S FUNCTION FOR AN nTH ORDER LINEAR DIFFERENTIAL OPERATOR

BY

JOHN LOCKER

ABSTRACT. The generalized Green's function $K(t,s)$ for an nth order linear differential operator L is characterized in terms of the $2n$th order differential operators LL^* and L^*L. The development is operator oriented and takes place in the Hilbert space $L^2[a,b]$. Two features of the characterization are a determination of the jumps occurring in the derivatives of orders $n, n+1, \ldots, 2n-1$ at $t = s$ and a determination of the boundary conditions satisfied by the functions $K(a,\cdot)$ and $K(b,\cdot)$. Several examples are given to illustrate the properties of the generalized Green's function.

1. **Introduction.** The generalized Green's function for a linear differential operator was first systematically studied by Elliott [5] and Reid [12]. More recently it has been considered by Greub and Rheinboldt [6], Wyler [15], Loud [9], [10], Bradley [1], Reid [13], Chitwood [3] and Brown [2]. For the historical development of the generalized Green's function the reader is referred to the survey article by Reid [14].

The purpose of this paper is to present a new characterization of the generalized Green's function $K(t,s)$ for an nth order linear differential operator L, which is determined by a formal differential operator τ and linearly independent boundary conditions $B_i(f) = 0, i = 1, \ldots, k$. In our treatment the $2n$th order differential operators LL^* and L^*L play a major role, a role analogous to that played by L^* and L in characterizing the standard Green's function. Our development parallels that of Dunford and Schwartz [4, Chapter XIII] for the standard Green's function and is operator-theoretic in nature.

In §2 we define the various spaces and operators used in the paper, the setting being the Hilbert space $L^2[a,b]$. The generalized Green's function $K(t,s)$ is introduced by utilizing the generalized inverse $L^\dagger$ of L and the Riesz representation theorem. We derive in §3 the continuity and differentiability properties of $K(t,s)$. A new feature is a determination of the jumps in the derivatives of orders $n, n+1, \ldots, 2n-1$ at $t = s$ as well as the well-known

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jumps occurring in the derivatives of orders $0, 1, \ldots, n-1$. These jump conditions take a particularly simple form when specified in terms of the quasi-derivatives as is seen in §4. For each c with $a < c < b$ we show in §5 that $K(c, \cdot)$ satisfies the $2n$ boundary conditions which determine LL^*, included among these being the boundary conditions determining L^*.

In §6 we establish a representation theorem for $K(c, \cdot)$ in the case $a < c < b$ in terms of functions $\phi_1, \phi_2, \ldots, \phi_{2n}$ which form a basis for the solution space of $\tau\tau^*\phi = 0$. The representation is uniquely determined by the jump conditions, the boundary conditions determining LL^*, and the property that $K(c, \cdot)$ is orthogonal to the null space of L^*. The representation theorem is extended to the cases $c = a$ and $c = b$ in §7, and the boundary conditions satisfied by $K(a, \cdot)$ and $K(b, \cdot)$ are determined. It should be emphasized that these two functions do not satisfy the boundary conditions determining L^*.

In §8 we introduce the generalized Green's function $K_1(t,s)$ for the differential operator L^*, showing that $K(t,s) = K_1(s,t)$ for $t \neq s$. Letting $\psi_1, \psi_2, \ldots, \psi_{2n}$ be a basis for the solution space of $\tau^*\tau\psi = 0$, in §9 we represent $K(t,s)$ in terms of the functions $\psi_i(t)$ and $\phi_j(s)$. We relate $K(t,s)$ to the orthogonal projection operators on the null spaces of L and L^* in §10, and we conclude our discussion by giving several examples of generalized Green's functions in §11.

2. **Basic definitions.** For a closed interval $[a,b]$ let S be the real Hilbert space $L^2[a,b]$ with the standard inner product (f, g) and norm $\|f\|$. We denote the domain, range, and null space of any operator L in S by $\mathfrak{D}(L)$, $\mathfrak{R}(L)$, and $\mathfrak{N}(L)$, respectively. For each positive integer n let $H^n[a,b]$ denote the subspace of S consisting of all functions f in $C^{n-1}[a,b]$ with $f^{(n-1)}$ absolutely continuous on $[a,b]$ and $f^{(n)}$ in S. Also, let $H_0^n[a,b]$ denote the subspace consisting of all functions $f \in H^n[a,b]$ which are identically zero in neighborhoods of a and b (the neighborhoods can vary with the f). The space $H^n[a,b]$ becomes a Banach space under the norm

$$|f|_n = \sum_{i=0}^{n-1} \max_{a \leqslant t \leqslant b} |f^{(i)}(t)| + \|f^{(n)}\|, \qquad f \in H^n[a,b].$$

We will refer to this structure as the *H^n-structure* for $H^n[a,b]$.

Given an nth order formal differential operator

$$\tau = \sum_{i=0}^{n} a_i(t)\left(\frac{d}{dt}\right)^i,$$

where the coefficients $a_i(t)$ belong to $C^\infty[a,b]$ and $a_n(t) \neq 0$ on $[a,b]$, and given k $(0 \leqslant k \leqslant 2n)$ linearly independent boundary values

$$B_i(f) = \sum_{j=0}^{n-1} \alpha_{ij} f^{(j)}(a) + \sum_{j=0}^{n-1} \beta_{ij} f^{(j)}(b), \qquad i = 1, \ldots, k,$$

we define a linear differential operator L in S by

$$\mathfrak{D}(L) = \{f \in H^n[a,b] \mid B_i(f) = 0, i = 1, \ldots, k\}, \qquad Lf = \tau f.$$

Let

$$\tau^* = \sum_{i=0}^{n} b_i(t)\left(\frac{d}{dt}\right)^i$$

be the formal adjoint of τ where the coefficients are given by

$$b_i(t) = \sum_{j=i}^{n} (-1)^j \binom{j}{i} \left(\frac{d}{dt}\right)^{j-i} a_j(t), \qquad i = 0, 1, \ldots, n,$$

with $b_n(t) = (-1)^n a_n(t) \neq 0$ on $[a,b]$, and let

$$B_i^*(f) = \sum_{j=0}^{n-1} \alpha_{ij}^* f^{(j)}(a) + \sum_{j=0}^{n-1} \beta_{ij}^* f^{(j)}(b), \qquad i = 1, \ldots, 2n-k,$$

be a set of $2n-k$ linearly independent adjoint boundary values. We know that the adjoint operator L^* is the differential operator in S given by

$$\mathfrak{D}(L^*) = \{f \in H^n[a,b] \mid B_i^*(f) = 0, i = 1, \ldots, 2n-k\}, \qquad L^* f = \tau^* f.$$

The restriction of L to the subspace $\mathfrak{D}(L) \cap \mathfrak{N}(L)^\perp$ is a 1-1 closed linear operator, and its inverse $H = [L|\mathfrak{D}(L) \cap \mathfrak{N}(L)^\perp]^{-1}$ is a 1-1 linear operator with domain $\mathfrak{R}(L)$ and range $\mathfrak{D}(L) \cap \mathfrak{N}(L)^\perp$. By the open mapping theorem H is continuous from $\mathfrak{R}(L)$ under the L^2-topology to $\mathfrak{D}(L) \cap \mathfrak{N}(L)^\perp$ under the H^n-topology.

Let P and Q denote the orthogonal projections from S onto $\mathfrak{N}(L)$ and $\mathfrak{N}(L^*)$, respectively. Note that $I - P$ and $I - Q$ are the orthogonal projections from S onto the closed subspaces $\mathfrak{R}(L^*)$ and $\mathfrak{R}(L)$, respectively. Also,

$$LHf = f \quad \text{for all } f \in \mathfrak{R}(L),$$

and

$$HLf = f - Pf \quad \text{for all } f \in \mathfrak{D}(L).$$

Let $L^\dagger \colon S \to S$ be the linear operator defined by

$$L^\dagger f = H(I - Q)f, \qquad f \in S.$$

Clearly $L^\dagger$ is continuous from S under the L^2-topology to $\mathfrak{D}(L) \cap \mathfrak{N}(L)^\perp$

under the H^n-topology, and $L^\dagger|\mathfrak{R}(L) = H$. Also, it can be verified that $L^\dagger$ has the following properties:

(i) $LL^\dagger Lf = Lf$ for all $f \in \mathfrak{D}(L)$,
(ii) $L^\dagger LL^\dagger f = L^\dagger f$ for all $f \in S$,
(iii) $LL^\dagger f = f - Qf$ for all $f \in S$,
(iv) $L^\dagger Lf = f - Pf$ for all $f \in \mathfrak{D}(L)$.

The operator $L^\dagger$ is the *generalized inverse* of L, and it has been studied in [8].

Fix a point $t \in [a,b]$, and let $F\colon S \to \mathbf{R}$ be the linear functional defined by

$$F(f) = L^\dagger f(t) = H(I - Q)f(t) \quad \text{for } f \in S.$$

Since evaluation at a point is a continuous linear functional on $H^n[a,b]$ under its H^n-structure, it follows that F is a continuous linear functional on S. Therefore, by the Riesz representation theorem there exists a unique function $K(t,\cdot) \in S$ such that

$$L^\dagger f(t) = F(f) = \int_a^b K(t,s)f(s)\,ds \quad \text{for all } f \in S, \tag{1}$$

and equation (1) is valid for all $t \in [a,b]$. The function $K(t,s)$ is called the *generalized Green's function* for L. We are going to establish its properties which are analogous to the well-known properties which characterize the standard Green's function when L is invertible, i.e., when $P = Q = 0$ and $L^\dagger = H = L^{-1}$.

In studying the generalized Green's function we will utilize the $2n$th order selfadjoint differential operators LL^* and L^*L, which have been considered in [8]. They are given by

$$\mathfrak{D}(LL^*) = \{f \in H^{2n}[a,b] \mid B_i^*(f) = 0, i = 1,\dots,2n-k;$$
$$B_j(\tau^* f) = 0, j = 1,\dots,k\}, \quad LL^*f = \tau\tau^* f,$$

and

$$\mathfrak{D}(L^*L) = \{f \in H^{2n}[a,b] \mid B_i(f) = 0, i = 1,\dots,k;$$
$$B_j^*(\tau f) = 0, j = 1,\dots,2n-k\}, \quad L^*Lf = \tau^*\tau f,$$

where

$$\tau\tau^* = \sum_{i=0}^{2n} c_i(t)\left(\frac{d}{dt}\right)^i = \sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{l=0}^{i}\binom{i}{l} a_i(t) b_j^{(i-l)}(t)\left(\frac{d}{dt}\right)^{j+l},$$

from which we see that

$$c_{2n}(t) = a_n(t)b_n(t) = (-1)^n[b_n(t)]^2 \neq 0 \quad \text{on } [a,b].$$

If we set $B_i^+(f) = B_i(\tau^* f)$ for $f \in H^{2n}[a,b]$, then using Leibniz's rule we can express the boundary value B_i^+ as

$$
\begin{aligned}
B_i^+(f) = &\sum_{j=0}^{n-1} \alpha_{ij} \sum_{l=0}^{n} \sum_{p=l}^{l+j} \binom{j}{p-l} b_l^{(j-p+l)}(a) f^{(p)}(a) \\
&+ \sum_{j=0}^{n-1} \beta_{ij} \sum_{l=0}^{n} \sum_{p=l}^{l+j} \binom{j}{p-l} b_l^{(j-p+l)}(b) f^{(p)}(b)
\end{aligned} \tag{2}
$$

for all $f \in H^{2n}[a,b]$, $i = 1, \ldots, k$. Another expression for this higher order boundary value is given in [8, p. 178].

A tool that we will frequently use is Green's formula [4, p. 1288]:

$$
\int_a^b \tau f(t) g(t)\, dt = \int_a^b f(t) \tau^* g(t)\, dt + F_b(\tau)(f,g) - F_a(\tau)(f,g), \tag{3}
$$

for all $f, g \in H^n[a,b]$, where

$$
F_t(\tau)(f,g) = \sum_{j,l=0}^{n-1} F_t^{lj}(\tau) f^{(l)}(t) g^{(j)}(t)
$$

is the boundary form for τ at the point t with

$$
F_t^{lj}(\tau) = \sum_{i=j}^{n-l-1} (-1)^i \binom{i}{j} \left(\frac{d}{dt}\right)^{i-j} a_{l+i+1}(t), \quad j + l \leqslant n - 1,
$$

$$
F_t^{lj}(\tau) = 0, \qquad j + l > n - 1.
$$

For $j + l = n - 1$ we have

$$
F_t^{l,n-l-1}(\tau) = (-1)^{n-l-1} a_n(t),
$$

and hence, the $n \times n$ matrix $[F_t^{lj}(\tau)]$ is nonsingular for each $t \in [a,b]$. We have similar results for the boundary forms $F_t(\tau^*)$, $F_t(\tau\tau^*)$, and $F_t(\tau^*\tau)$.

3. Continuity properties and jump conditions. We begin our characterization of the generalized Green's function by establishing an orthogonality property.

THEOREM 1. *$K(t, \cdot) \in \mathfrak{N}(L^*)^\perp = \mathfrak{R}(L)$ for all $t \in [a,b]$.*

PROOF. Fix a point $t \in [a,b]$. Then for any function $f(s)$ in $\mathfrak{N}(L^*)$ we have $(I - Q)f = 0$, and hence, from (1)

$$
\int_a^b K(t,s) f(s)\, ds = H(I - Q) f(t) = 0,
$$

so $K(t, \cdot) \in \mathfrak{N}(L^*)^\perp$. Q.E.D.

In deriving the continuity and differentiability properties of $K(t,s)$, the following lemma is very useful.

LEMMA 1. *$\int_a^b K(t,s)\tau\tau^* f(s)\,ds = \tau^* f(t)$ for all $t \in [a,b]$ and for all $f \in \mathfrak{D}(LL^*)$.*

PROOF. Fix $t \in [a,b]$ and take any $f \in \mathfrak{D}(LL^*)$. Setting $g = L^* f = \tau^* f$, we have $g \in \mathfrak{D}(L) \cap \mathfrak{N}(L)^\perp$ and

$$\int_a^b K(t,s)\tau\tau^* f(s)\,ds = H(I-Q)Lg(t) = g(t). \quad \text{Q.E.D.}$$

THEOREM 2. *For each c with $a < c < b$, $K(c,\cdot) \in C^\infty[a,c]$ with $\tau\tau^* K(c,s) = 0$ for $a \leqslant s \leqslant c$, and $K(c,\cdot) \in C^\infty[c,b]$ with $\tau\tau^* K(c,s) = 0$ for $c \leqslant s \leqslant b$ (the left-hand and the right-hand derivatives are not required to be equal at $s = c$). Moreover, $K(a,\cdot) \in C^\infty[a,b]$ and $K(b,\cdot) \in C^\infty[a,b]$ with $\tau\tau^* K(a,s) = 0$ and $\tau\tau^* K(b,s) = 0$ for all $s \in [a,b]$.*

PROOF. For a fixed c with $a < c < b$, consider a function $f(t)$ in $H_0^{2n}[a,c]$. We can extend $f(t)$ to all of $[a,b]$ by taking $f(t) \equiv 0$ on $[b,c]$, in which case $f(t) \in \mathfrak{D}(LL^*)$. By Lemma 1 we have

$$\int_a^c K(c,s)\tau\tau^* f(s)\,ds = \int_a^b K(c,s)\tau\tau^* f(s)\,ds = \tau^* f(c) = 0.$$

By Lemma 9 in [4, p. 1291] we conclude that $K(c,\cdot) \in C^\infty[a,c]$ and

$$(\tau\tau^*)^* K(c,s) = \tau\tau^* K(c,s) = 0 \quad \text{for all } s \in [a,c].$$

The same proof establishes the other parts of the theorem. Q.E.D.

We want to study $K(c,s)$ as a function of s at the point $s = c$. It will be shown that $K(c,s)$ is continuous in its derivatives of orders $0, 1, \ldots, n-2$ at $s = c$, while its derivatives of orders $n-1, n, \ldots, 2n-1$ have prescribed jump discontinuities at $s = c$.

Fix a point c with $a < c < b$, and let $\eta_-(s) = K(c,s)$ for $a \leqslant s \leqslant c$ and $\eta_+(s) = K(c,s)$ for $c \leqslant s \leqslant b$. Take any $f \in H_0^{2n}[a,b]$. Clearly $f \in \mathfrak{D}(LL^*)$, and from Lemma 1 and Green's formula (3) we have

$$\begin{aligned}
\tau^* f(c) &= \int_a^b K(c,s)\tau\tau^* f(s)\,ds \\
&= F_c(\tau\tau^*)(f,\eta_-) - F_a(\tau\tau^*)(f,\eta_-) + F_b(\tau\tau^*)(f,\eta_+) - F_c(\tau\tau^*)(f,\eta_+) \\
&= F_c(\tau\tau^*)(f,\eta_-) - F_c(\tau\tau^*)(f,\eta_+)
\end{aligned}$$

or

$$(4) \qquad \sum_{l=0}^{n} b_l(c) f^{(l)}(c) = \sum_{l=0}^{2n-1} \sum_{j=0}^{2n-1} F_c^{lj}(\tau\tau^*)[\eta_-^{(j)}(c) - \eta_+^{(j)}(c)] f^{(l)}(c)$$

for all $f \in H_0^{2n}[a,b]$. By carefully choosing the function $f \in H_0^{2n}[a,b]$ it is possible to assign any value we wish to the quantities $f(c), f'(c), \ldots, f^{(2n-1)}(c)$. In particular, we can make one of them 1 and all the others 0, and hence, from (4) we conclude that

$$(5) \qquad \sum_{j=0}^{2n-1} F_c^{lj}(\tau\tau^*)[\eta_-^{(j)}(c) - \eta_+^{(j)}(c)] = b_l(c), \qquad l = 0, 1, \ldots, n,$$

$$(6) \qquad \sum_{j=0}^{2n-1} F_c^{lj}(\tau\tau^*)[\eta_-^{(j)}(c) - \eta_+^{(j)}(c)] = 0, \qquad l = n+1, \ldots, 2n-1.$$

Since the $2n \times 2n$ matrix $[F_c^{lj}(\tau\tau^*)]$ is nonsingular, equations (5) and (6) are uniquely solvable for the quantities $\eta_-^{(j)}(c) - \eta_+^{(j)}(c), j = 0, 1, \ldots, 2n-1$, in terms of the quantities $F_c^{lj}(\tau\tau^*)$ and $b_l(c)$.

For each $t \in [a,b]$ consider the $2n \times 2n$ linear system

$$(7) \qquad \sum_{j=0}^{2n-1} F_t^{lj}(\tau\tau^*)\eta_j(t) = b_l(t), \qquad l = 0, 1, \ldots, n,$$

$$(8) \qquad \sum_{j=0}^{2n-1} F_t^{lj}(\tau\tau^*)\eta_j(t) = 0, \qquad l = n+1, \ldots, 2n-1,$$

for the $2n$ unknowns $\eta_0(t), \eta_1(t), \ldots, \eta_{2n-1}(t)$. From Cramer's rule it follows that (7)–(8) is uniquely solvable for the quantities $\eta_0(t), \eta_1(t), \ldots, \eta_{2n-1}(t)$ for each $t \in [a,b]$, and as functions of t these quantities each belong to $C^\infty[a,b]$.

In terms of these functions we can rewrite (4) as

$$(9) \qquad \tau^* f(c) = \sum_{j,l=0}^{2n-1} F_c^{lj}(\tau\tau^*) f^{(l)}(c)\eta_j(c)$$

for all $f \in H_0^{2n}[a,b]$ and for all c with $a < c < b$. If we take any function $f \in H^{2n}[a,b]$, then we can find a function $\bar{f} \in H_0^{2n}[a,b]$ with $\bar{f}^{(j)}(c) = f^{(j)}(c)$ for $j = 0, 1, \ldots, 2n-1$, and hence, we see that (9) is valid for all $f \in H^{2n}[a,b]$ and for all $a < c < b$. Finally, letting $c \to a$ and $c \to b$, we conclude that

$$(10) \qquad \tau^* f(t) = \sum_{j,l=0}^{2n-1} F_t^{lj}(\tau\tau^*) f^{(l)}(t)\eta_j(t)$$

for all $f \in H^{2n}[a,b]$ and for all $t \in [a,b]$. Let us summarize these results as a theorem.

THEOREM 3. *For each $t \in [a,b]$ the $2n \times 2n$ linear system* (7)–(8) *has a unique solution $\eta_0(t), \eta_1(t), \ldots, \eta_{2n-1}(t)$. Moreover, these functions belong to $C^\infty[a,b]$, and the following properties hold*:

(a) $\tau^* f(t) = \sum_{j,l=0}^{2n-1} F_t^{lj}(\tau\tau^*) f^{(l)}(t)\eta_j(t)$ *for all $f \in H^{2n}[a,b]$ and for all $t \in [a,b]$.*

(b) *For each point c with $a < c < b$,*

$$\lim_{s\to c^-} \frac{\partial^j K}{\partial s^j}(c,s) - \lim_{s\to c^+} \frac{\partial^j K}{\partial s^j}(c,s) = \eta_j(c),$$

for $j = 0, 1, \ldots, 2n-1$.

Fix a point c with $a < c < b$, and let us consider the quantities

$$\eta_j(c) = \eta_-^{(j)}(c) - \eta_+^{(j)}(c), \qquad j = 0, 1, \ldots, 2n-1,$$

which are determined by equations (7)–(8) with $t = c$. We know that $F_c^{lj}(\tau\tau^*) = 0$ for $j + l > 2n - 1$, and

$$F_c^{l,2n-l-1}(\tau\tau^*) = (-1)^{2n-l-1} c_{2n}(c) = (-1)^{n-l-1} b_n(c)^2$$

for $l = 0, 1, \ldots, 2n-1$, and hence, (7)–(8) is a triangular system. From equation (8) we obtain

$$\eta_0(c) = \eta_1(c) = \cdots = \eta_{n-2}(c) = 0, \tag{11}$$

and substituting this into the last equation in (7) yields

$$\eta_{n-1}(c) = -b_n(c)^{-1} = (-1)^{n-1} a_n(c)^{-1}. \tag{12}$$

In equations (11) and (12) we have obtained the well-known jump conditions satisfied by the standard Green's function.

At this point it is possible to determine the quantities $\eta_n(c), \eta_{n+1}(c), \ldots, \eta_{2n-1}(c)$ from the first n equations in (7) and from (11) and (12). Proceeding in this way leads to an $n \times n$ triangular system which can be solved for these quantities in terms of the quantities $F_c^{lj}(\tau\tau^*)$ and $b_l(c)$. By expressing the boundary form $F_c(\tau\tau^*)$ in terms of $F_c(\tau)$ and $F_c(\tau^*)$, it is possible to derive formulas for $\eta_n(c), \eta_{n+1}(c), \ldots, \eta_{2n-1}(c)$ in terms of $b_0(c), b_1(c), \ldots, b_n(c)$ and their derivatives. Instead of following this approach, we are going to reexamine the above discussion and derive a new set of equations for the quantities $\eta_n(c), \eta_{n+1}(c), \ldots, \eta_{2n-1}(c)$. This new method is much simpler, and the final system of equations is easily solved.

LEMMA 2. *Let c be a point with $a < c < b$, and let $\eta_-(s) = K(c,s)$ for $a \leqslant s \leqslant c$ and $\eta_+(s) = K(c,s)$ for $c \leqslant s \leqslant b$. Then*

$$F_c(\tau)(f,\eta_-) - F_c(\tau)(f,\eta_+) = f(c)$$

and

$$F_c(\tau^*)(\eta_-,f) - F_c(\tau^*)(\eta_+,f) = -f(c)$$

for all $f \in H^n[a,b]$.

PROOF. Take any $f \in H^n[a,b]$. Using (11) and (12) we have

$$\begin{aligned} &F_c(\tau)(f,\eta_-) - F_c(\tau)(f,\eta_+) \\ &\quad = \sum_{j,l=0}^{n-1} F_c^{lj}(\tau) f^{(l)}(c)\eta_j(c) = \sum_{l=0}^{n-1} F_c^{l,n-1}(\tau) f^{(l)}(c)\eta_{n-1}(c) \\ &\quad = (-1)^{n-1} a_n(c) f(c)(-1)^{n-1} a_n(c)^{-1} = f(c). \end{aligned}$$

The second equation is established similarly. Q.E.D.

Now proceeding as above, for each $f \in H_0^{2n}[a,b]$ we have

$$\begin{aligned} \tau^* f(c) &= \int_a^c \eta_-(s)\tau\tau^* f(s)\,ds + \int_c^b \eta_+(s)\tau\tau^* f(s)\,ds \\ &= \int_a^c \tau^*\eta_-(s)\tau^* f(s)\,ds + F_t(\tau)(\tau^* f,\eta_-)\Big|_a^c \\ &\qquad + \int_c^b \tau^*\eta_+(s)\tau^* f(s)\,ds + F_t(\tau)(\tau^* f,\eta_+)\Big|_c^b \\ &= F_c(\tau)(\tau^* f,\eta_-) - F_c(\tau)(\tau^* f,\eta_+) \\ &\qquad + F_c(\tau^*)(f,\tau^*\eta_-) - F_c(\tau^*)(f,\tau^*\eta_+). \end{aligned}$$

But by Lemma 2

$$\tau^* f(c) = F_c(\tau)(\tau^* f,\eta_-) - F_c(\tau)(\tau^* f,\eta_+),$$

and hence, comparing these last two equations we conclude that

$$F_c(\tau^*)(f,\tau^*\eta_-) - F_c(\tau^*)(f,\tau^*\eta_+) = 0 \quad \text{for all } f \in H_0^{2n}[a,b]. \tag{13}$$

We can rewrite (13) as

$$\sum_{j,l=0}^{n-1} F_c^{lj}(\tau^*) f^{(l)}(c)\left[\left(\frac{d}{ds}\right)^j [\tau^*\eta_-(s)]_{s=c} - \left(\frac{d}{ds}\right)^j [\tau^*\eta_+(s)]_{s=c}\right] = 0$$

for all $f \in H_0^{2n}[a,b]$, which implies that

$$\sum_{j=0}^{n-1} F_c^{lj}(\tau^*)\left[\left(\frac{d}{ds}\right)^j [\tau^*\eta_-(s)]_{s=c} - \left(\frac{d}{ds}\right)^j [\tau^*\eta_+(s)]_{s=c}\right] = 0$$

for $l = 0, 1, \ldots, n-1$. From this $n \times n$ triangular system we see that

$$(14)\quad \left(\frac{d}{ds}\right)^j [\tau^*\eta_-(s)]_{s=c} - \left(\frac{d}{ds}\right)^j [\tau^*\eta_+(s)]_{s=c} = 0, \qquad j = 0, 1, \ldots, n-1.$$

kobgpar div d den d s cbgpar sup j base lsquar tau supstar eta sub + base (s) $]_{s=c} = 0, \quad j = 0, 1, \ldots, n$ - 1.

REMARK 1. Equation (14) implies that the function $\tau^* K(c, \cdot)$ has a removable singularity at $s = c$ and that by assigning the right value we can consider $\tau^* K(c, \cdot)$ as a function in $H^n[a, b]$. We will examine this function in §10, relating it to the projection operator P.

Now by Leibniz's rule

$$\begin{aligned}\left(\frac{d}{ds}\right)^j [\tau^*\eta_-(s)] &= \sum_{i=0}^{n}\sum_{l=0}^{j}\binom{j}{l} b_i^{(j-l)}(s)\eta_i^{(i+l)}(s)\\ &= \sum_{i=0}^{n}\sum_{p=i}^{i+j}\binom{j}{p-i} b_i^{(j-p+i)}(s)\eta_-^{(p)}(s),\end{aligned}$$

and hence, equation (14) can be rewritten as

$$(15)\qquad \sum_{i=0}^{n}\sum_{p=i}^{i+j}\binom{j}{p-i} b_i^{(j-p+i)}(c)\eta_p(c) = 0, \qquad j = 0, 1, \ldots, n-1.$$

In this last double sum the terms corresponding to $p = 0, 1, \ldots, n-2$ are all zero, so (15) reduces to

$$\begin{aligned}&\left[\sum_{i=n-j-1}^{n-1}\binom{j}{n-i-1} b_i^{(j-n+i+1)}(c)\right]\eta_{n-1}(c)\\ &\qquad + \sum_{p=n}^{n+j}\left[\sum_{i=p-j}^{n}\binom{j}{p-i} b_i^{(j-p+i)}(c)\right]\eta_p(c) = 0\end{aligned}$$

for $j = 0, 1, \ldots, n-1$. Finally, using (12) we obtain the equation

$$(16)\qquad \begin{aligned}\eta_{n+j}(c) = &\sum_{i=n-j-1}^{n-1}\binom{j}{n-i-1} b_i^{(j-n+i+1)}(c) b_n(c)^{-2}\\ &- \sum_{p=n}^{n+j-1}\left[\sum_{i=p-j}^{n}\binom{j}{p-i} b_i^{(j-p+i)}(c) b_n(c)^{-1}\right]\eta_p(c)\end{aligned}$$

for $j = 0, 1, \ldots, n-1$ (in case $j = 0$ the second sum does not occur). In this equation the quantity $\eta_{n+j}(c)$ is given in terms of the preceding ones

$\eta_n(c), \eta_{n+1}(c), \ldots, \eta_{n+j-1}(c)$ and the quantities $b_0(c), b_1(c), \ldots, b_n(c)$ and their derivatives. These equations are easily solved to yield the first few quantities, e.g.,

$$\eta_n(c) = b_{n-1}(c)b_n(c)^{-2} \tag{17}$$

and

$$\begin{aligned}\eta_{n+1}(c) = {} & b_{n-2}(c)b_n(c)^{-2} + b'_{n-1}(c)b_n(c)^{-2} \\ & -b_{n-1}(c)^2 b_n(c)^{-3} - b_{n-1}(c)b'_n(c)b_n(c)^{-3}.\end{aligned} \tag{18}$$

REMARK 2. In equations (11), (12), and (16) we have formulas for the quantities $\eta_0(c), \eta_1(c), \ldots, \eta_{2n-1}(c)$ which are valid for each c with $a < c < b$. By the continuity of these functions on $[a, b]$, it follows that these formulas are valid for each $c \in [a, b]$. Similarly, equation (15) holds for each $c \in [a, b]$.

4. **Quasi-derivatives.** The jump conditions can also be interpreted in terms of *quasi-derivatives* (see [11, Part II, pp. 48–50]). Since $\tau\tau^*$ is formally selfadjoint, it can be written in the form

$$\tau\tau^* = \sum_{i=0}^{n} (-1)^i \left(\frac{d}{dt}\right)^i A_i(t) \left(\frac{d}{dt}\right)^i$$

where $A_n(t) = (-1)^n c_{2n}(t) = b_n(t)^2$. For any function $f \in H^{2n}[a, b]$, the quasi-derivatives of f relative to $\tau\tau^*$ are defined by

$$\begin{aligned} f^{[i]}(t) &= f^{(i)}(t), \qquad i = 0, 1, \ldots, n-1, \\ f^{[n]}(t) &= A_n(t) f^{(n)}(t), \end{aligned}$$

and

$$f^{[n+i]}(t) = A_{n-i}(t) f^{(n-i)}(t) - \frac{d}{dt}(f^{[n+i-1]}(t)), \qquad i = 1, \ldots, n.$$

With these definitions $\tau\tau^* f = f^{[2n]}$, and Green's formula becomes

$$\begin{aligned}\int_a^b \tau\tau^* f(t) g(t)\, dt = {} & \int_a^b f(t) \tau\tau^* g(t)\, dt \\ & + \sum_{i=1}^{n} \{ f^{[i-1]}(t) g^{[2n-i]}(t) - f^{[2n-i]}(t) g^{[i-1]}(t) \}_{t=a}^{t=b}\end{aligned}$$

for all $f, g \in H^{2n}[a, b]$.

Fix a point c with $a < c < b$. We already know that

$$\eta_-^{[j]}(c) - \eta_+^{[j]}(c) = 0 \quad \text{for } j = 0, 1, \ldots, n-2. \tag{19}$$

Proceeding as in the last section, for each $f \in H_0^{2n}[a,b]$ we have

$$\begin{aligned}\sum_{i=0}^{n} b_i(c) f^{(i)}(c) &= \int_a^c \eta_-(s)\tau\tau^* f(s)\,ds + \int_c^b \eta_+(s)\tau\tau^* f(s)\,ds \\ &= \sum_{i=1}^{n} \{f^{[i-1]}(s)\eta_-^{[2n-i]}(s) - f^{[2n-i]}(s)\eta_-^{[i-1]}(s)\}_{s=a}^{s=c} \\ &\quad + \sum_{i=1}^{n} \{f^{[i-1]}(s)\eta_+^{[2n-i]}(s) - f^{[2n-i]}(s)\eta_+^{[i-1]}(s)\}_{s=c}^{s=b} \\ &= \sum_{i=1}^{n} f^{(i-1)}(c)\{\eta_-^{[2n-i]}(c) - \eta_+^{[2n-i]}(c)\} \\ &\quad - b_n(c)^2 f^{(n)}(c)\{\eta_-^{[n-1]}(c) - \eta_+^{[n-1]}(c)\},\end{aligned}$$

which implies that

$$\eta_-^{[n-1]}(c) - \eta_+^{[n-1]}(c) = -b_n(c)^{-1} \tag{20}$$

and

$$\eta_-^{[n+j]}(c) - \eta_+^{[n+j]}(c) = b_{n-j-1}(c) \quad \text{for } j = 0, 1, \ldots, n-1. \tag{21}$$

In equations (19), (20), and (21) we have formulas for the jumps in the quasi-derivatives at $s = c$. They are much simpler than the analogous formulas for the jumps in the derivatives.

5. **Boundary conditions.** Recall that the selfadjoint differential operator LL^* is determined by the boundary conditions $B_i^*(f) = 0$, $i = 1, \ldots, 2n-k$, and $B_j^+(f) = 0$, $j = 1, \ldots, k$ (see equation (2)).

THEOREM 4. *For each c with $a < c < b$ the function $K(c, \cdot)$ satisfies the boundary conditions $B_i^*(f) = 0$, $i = 1, \ldots, 2n-k$, and $B_j^+(f) = B_j(\tau^* f) = 0$, $j = 1, \ldots, k$.*

PROOF. Fix c with $a < c < b$, and choose a function g in $H^{2n}[a,b]$ which coincides with $K(c, \cdot)$ in neighborhoods of both a and b. Then for any $f \in \mathfrak{D}(LL^*)$ it follows by Green's formula that

$$\begin{aligned}\int_a^b \tau\tau^* f(s)[g(s) - K(c,s)]\,ds &= \int_a^b f(s)\tau\tau^* g(s)\,ds - F_c(\tau\tau^*)(f, \eta_- - \eta_+) \\ &= \int_a^b f(s)\tau\tau^* g(s)\,ds - \tau^* f(c) \quad \text{by Theorem 3(a)} \\ &= \int_a^b f(s)\tau\tau^* g(s)\,ds - \int_a^b K(c,s)\tau\tau^* f(s)\,ds \quad \text{by Lemma 1,}\end{aligned}$$

and hence,

$$\int_a^b \tau\tau^* f(s)g(s)\,ds = \int_a^b f(s)\tau\tau^* g(s)\,ds$$

for all $f \in \mathfrak{D}(LL^*)$. This implies that $g \in \mathfrak{D}(LL^*)$, so g satisfies the desired boundary conditions, and we conclude that $K(c, \cdot)$ also satisfies these boundary conditions. Q.E.D.

REMARK 3. We will show in §7 that the functions $K(a, \cdot)$ and $K(b, \cdot)$ satisfy the boundary conditions $B_j^+(f) = 0, j = 1, \ldots, k$, but they do not satisfy the boundary conditions $B_i^*(f) = 0, i = 1, \ldots, 2n - k$.

6. The representation theory. Let $\phi_1, \ldots, \phi_{2n}$ and $\psi_1, \ldots, \psi_{2n}$ be functions in $C^\infty[a,b]$ which form bases for the solution spaces of $\tau\tau^*\phi = 0$ and $\tau^*\tau\psi = 0$, respectively. Setting $p = \dim \mathfrak{N}(L)$ and $q = \dim \mathfrak{N}(L^*)$, it is well known [11, Part I, p. 12] that $n - k = p - q$. Also, with no loss of generality we can assume that $\mathfrak{N}(L^*) = \langle \phi_1, \ldots, \phi_q \rangle$ and that $\psi_1, \ldots, \psi_n$ form a basis for the solution space of $\tau\psi = 0$.

We are going to represent the generalized Green's function $K(t,s)$ in terms of the functions $\psi_i(t)$, $\phi_j(s)$. Before doing this we need to take a closer look at the boundary conditions $B_j^+(f) = 0, j = 1, \ldots, k$.

Let r be the rank of the $k \times n$ matrix having entries $B_i(\psi_j)$, $i = 1, \ldots, k; j = 1, \ldots, n$. Clearly $\psi = \sum_{j=1}^n x_j\psi_j$ belongs to $\mathfrak{N}(L)$ iff

$$\sum_{j=1}^n x_j B_i(\psi_j) = 0, \qquad i = 1, \ldots, k. \tag{22}$$

and hence, the mapping $(x_1, \ldots, x_n) \to \sum_{j=1}^n x_j\psi_j$ is an isomorphism between the solution space of (22) and $\mathfrak{N}(L)$, so $p = n - r$.

Let

$$\mathfrak{B} = \{B \in \langle B_1, \ldots, B_k \rangle \mid B(\psi_i) = 0 \text{ for } i = 1, \ldots, n\}.$$

We assert that $\dim \mathfrak{B} = q$. Note that $B = \sum_{i=1}^k y_i B_i \in \mathfrak{B}$ iff

$$\sum_{i=1}^k y_i B_i(\psi_j) = 0, \qquad j = 1, \ldots, n, \tag{23}$$

and it follows that the mapping $(y_1, \ldots, y_k) \to \sum_{i=1}^k y_i B_i$ is an isomorphism between the solution space of (23) and $\mathfrak{B}$, and hence,

$$\dim \mathfrak{B} = k - r = k - (n - p) = q.$$

Consequently, with no loss of generality we can assume that $\mathfrak{B} = \langle B_1, \ldots, B_q \rangle$, and this implies that

$$B_i(f) = 0 \quad \text{for all } f \in \langle \psi_1, \ldots, \psi_n \rangle, i = 1, \ldots, q. \tag{24}$$

Fix a point c with $a < c < b$, and let us examine the function $K(c, \cdot)$. We know it can be represented in the form

$$(25)\qquad \begin{aligned} K(c,s) &= \sum_{i=1}^{2n} \alpha_i(c)\phi_i(s), \qquad a \leqslant s < c, \\ K(c,s) &= \sum_{i=1}^{2n} \beta_i(c)\phi_i(s), \qquad c < s \leqslant b. \end{aligned}$$

We want to show that the $4n$ constants $\alpha_i(c)$, $\beta_i(c)$ can be calculated from (i) the $2n$ jump conditions (see Theorem 3), (ii) the $2n$ boundary conditions (see Theorem 4), and (iii) the q orthogonality conditions (see Theorem 1). This makes a total of $4n + q$ conditions for the $4n$ constants. At first glance it seems reasonable to discard the q orthogonality conditions. However, these conditions turn out to be necessary, and we will actually show that q of the boundary conditions $B_j^+(f) = 0, j = 1, \ldots, k$, can be deleted.

Suppose $\alpha_1(c), \ldots, \alpha_{2n}(c), \beta_1(c), \ldots, \beta_{2n}(c)$ are constants, and let

$$\eta_-(s) = \sum_{i=1}^{2n} \alpha_i(c)\phi_i(s), \qquad a \leqslant s \leqslant c,$$

$$\eta_+(s) = \sum_{i=1}^{2n} \beta_i(c)\phi_i(s) \qquad c \leqslant s \leqslant b.$$

We want to determine conditions on these constants that will make η_- and η_+ coincide with $K(c, \cdot)$ on the intervals $[a,c)$ and $(c,b]$, respectively. A first necessary condition is that the jump conditions be satisfied, i.e.,

$$(26)\qquad \sum_{i=1}^{2n} \alpha_i(c)\phi_i^{(j)}(c) - \sum_{i=1}^{2n} \beta_i(c)\phi_i^{(j)}(c) = \eta_j(c), \qquad j = 0, 1, \ldots, 2n-1.$$

Assume the constants $\alpha_i(c)$, $\beta_i(c)$ satisfy (26). Then

$$\eta_-^{(j)}(c) - \eta_+^{(j)}(c) = \eta_j(c) \quad \text{for } j = 0, 1, \ldots, 2n-1,$$

and hence, using Leibniz's rule and (15) we obtain

$$\left(\frac{d}{ds}\right)^j [\tau^* \eta_-(s)]_{s=c} - \left(\frac{d}{ds}\right)^j [\tau^* \eta_+(s)]_{s=c}$$
$$= \sum_{i=0}^{n} \sum_{l=i}^{i+j} \binom{j}{l-i} b_i^{(j-l+i)}(c)\eta_l(c) = 0$$

for $j = 0, 1, \ldots, n-1$. Setting $\phi(s) = \eta_-(s)$ for $a \leqslant s < c$ and $\phi(s) = \eta_+(s)$ for $c < s \leqslant b$, the above discussion shows that the function $\tau^*\phi$ has a removable singularity at $s = c$. By assigning the right value at $s = c$ we get $\tau^*\phi \in \langle \psi_1, \ldots, \psi_n \rangle$, and hence, from (24) we conclude that

$$B_j^+(\phi) = B_j(\tau^*\phi) = 0 \quad \text{for } j = 1, \ldots, q.$$

Let us summarize these results as a lemma.

LEMMA 3. *Let c be a point with $a < c < b$, let $\alpha_1(c), \ldots, \alpha_{2n}(c), \beta_1(c), \ldots, \beta_{2n}(c)$ be constants satisfying* (26), *and let*

$$\phi(s) = \sum_{i=1}^{2n} \alpha_i(c)\phi_i(s), \qquad a \leqslant s < c,$$

$$\phi(s) = \sum_{i=1}^{2n} \beta_i(c)\phi_i(s), \qquad c < s \leqslant b.$$

Then $\tau^*\phi$ has a removable singularity at $s = c$, $\tau^*\phi \in \langle \psi_1, \ldots, \psi_n \rangle$, and

$$B_j^+(\phi) = B_j(\tau^*\phi) = 0 \quad \textit{for } j = 1, \ldots, q.$$

A second necessary condition is that the boundary conditions in Theorem 4 be satisfied. Let

$$B_i^* = C_i^* + D_i^* \quad \text{for } i = 1, \ldots, 2n - k,$$

and let

$$B_i^+ = C_i^+ + D_i^+ \quad \text{for } i = 1, \ldots, k,$$

where C_i^* and C_i^+ are boundary values at a and D_i^* and D_i^+ are boundary values at b. Then the boundary conditions become

$$\sum_{i=1}^{2n} \alpha_i(c)C_j^*(\phi_i) + \sum_{i=1}^{2n} \beta_i(c)D_j^*(\phi_i) = 0, \qquad j = 1, \ldots, 2n - k, \tag{27}$$

and

$$\sum_{i=1}^{2n} \alpha_i(c)C_j^+(\phi_i) + \sum_{i=1}^{2n} \beta_i(c)D_j^+(\phi_i) = 0, \qquad j = q + 1, \ldots, k. \tag{28}$$

Notice that in (28) we have omitted the boundary conditions $B_j^+(f) = 0, j = 1, \ldots, q$, because of Lemma 3.

Finally, a third necessary condition is the orthogonality condition of Theorem 1:

$$\sum_{i=1}^{2n} \alpha_i(c) \int_a^c \phi_i(s)\phi_j(s)\,ds + \sum_{i=1}^{2n} \beta_i(c) \int_c^b \phi_i(s)\phi_j(s)\,ds = 0, \qquad j = 1, \ldots, q, \tag{29}$$

where we are assuming that $\mathcal{N}(L^*) = \langle \phi_1, \ldots, \phi_q \rangle$. In equations (26)–(29) we have obtained a $4n \times 4n$ linear system for the $4n$ unknowns $\alpha_i(c)$, $\beta_i(c)$. The

existence of the generalized Green's function $K(c, \cdot)$ guarantees this system has a solution.

THEOREM 5. *For each point c with $a < c < b$ the $4n \times 4n$ linear system* (26)–(29) *has a unique solution $\alpha_1(c), \ldots, \alpha_{2n}(c), \beta_1(c), \ldots, \beta_{2n}(c)$. Moreover, in terms of these quantities the generalized Green's function $K(c, \cdot)$ is given by*

$$K(c,s) = \sum_{i=1}^{2n} \alpha_i(c)\phi_i(s), \qquad a \leqslant s < c,$$

$$K(c,s) = \sum_{i=1}^{2n} \beta_i(c)\phi_i(s), \qquad c < s \leqslant b.$$

PROOF. Suppose α_i^0, β_i^0 is a solution to the homogeneous system associated with (26)–(29). Let

$$\phi(s) = \sum_{i=1}^{2n} \alpha_i^0 \phi_i(s), \qquad a \leqslant s \leqslant c,$$

$$\phi(s) = \sum_{i=1}^{2n} \beta_i^0 \phi_i(s), \qquad c < s \leqslant b.$$

From (26) we see that $\phi \in H^{2n}[a,b]$, and clearly $\tau\tau^*\phi(s) = 0$ for all $s \in [a,b]$. Also, by (24), (27), and (28) we have $\phi \in \mathfrak{D}(LL^*)$, which implies that $\phi \in \mathfrak{N}(LL^*) = \mathfrak{N}(L^*)$. On the other hand from (29) we obtain $\phi \in \mathfrak{N}(L^*)^\perp$. We conclude that $\phi(s) \equiv 0$ on $[a,b]$, so $\alpha_i^0 = \beta_i^0 = 0$ for $i = 1, \ldots, 2n$. Q.E.D.

7. **The end point theory.** Up to this point we have been restricting our attention to the case $a < c < b$. Note that the linear system (26)–(29) makes perfectly good sense in case $c = a$ or $c = b$, and the coefficients in this system are infinitely differentiable functions of c on the closed interval $[a,b]$.

THEOREM 6. *For each $c \in [a,b]$ the $4n \times 4n$ linear system* (26)–(29) *has a unique solution $\alpha_1(c), \ldots, \alpha_{2n}(c), \beta_1(c), \ldots, \beta_{2n}(c)$. Moreover, these quantities are infinitely differentiable functions of c on the interval $[a,b]$.*

PROOF. Consider the system (26)–(29) in the special case $c = a$. Suppose α_i^0, β_i^0 is a solution of the associated homogeneous system, and set

$$\phi(s) = \sum_{i=1}^{2n} \beta_i^0 \phi_i(s), \qquad a \leqslant s \leqslant b.$$

Clearly $\phi \in C^\infty[a,b]$ and $\tau\tau^*\phi(s) \equiv 0$ on $[a,b]$. Now

$$\begin{aligned} B_j^*(\phi) &= C_j^*(\phi) + D_j^*(\phi) \\ &= \sum_{l=0}^{n-1} \alpha_{jl}^* \left[\sum_{i=1}^{2n} \beta_i^0 \phi_i^{(l)}(a) \right] + \sum_{i=1}^{2n} \beta_i^0 D_j^*(\phi_i) \\ &= \sum_{i=1}^{2n} \alpha_i^0 C_j^*(\phi_i) + \sum_{i=1}^{2n} \beta_i^0 D_j^*(\phi_i) \quad \text{by (26)} \\ &= 0 \quad \text{by (27) for } j = 1, \ldots, 2n-k. \end{aligned}$$

By (24) we have $B_j^+(\phi) = 0$ for $j = 1, \ldots, q$, while for $j = q+1, \ldots, k$ the above argument yields

$$B_j^+(\phi) = C_j^+(\phi) + D_j^+(\phi) = 0 \quad \text{by (26) and (28)}.$$

Therefore, $\phi \in \mathfrak{N}(LL^*) = \mathfrak{N}(L^*)$. Since $\phi \in \mathfrak{N}(L^*)^\perp$ by (29), we conclude that $\phi(s) \equiv 0$ on $[a,b]$. This certainly implies that $\beta_i^0 = 0$ for $i = 1, \ldots, 2n$. Note that (26) becomes

$$\sum_{i=1}^{2n} \alpha_i^0 \phi_i^{(j)}(a) = 0, \qquad j = 0, 1, \ldots, 2n-1.$$

In this $2n \times 2n$ linear system the determinant of the coefficient matrix is the Wronskian $W(\phi_1, \ldots, \phi_{2n})$ evaluated at a, and hence, $\alpha_i^0 = 0$ for $i = 1, \ldots, 2n$.

We have shown that the linear system (26)–(29) is uniquely solvable for $c = a$, and similarly, we have unique solvability for $c = b$. By Cramer's rule the $\alpha_i(c)$ and $\beta_i(c)$ are infinitely differentiable functions of c on $[a,b]$. Q.E.D.

We are now in a position to be able to characterize the functions $K(a, \cdot)$ and $K(b, \cdot)$.

THEOREM 7. *The function $K(a, \cdot)$ satisfies the boundary conditions $B_i^*(f) = \alpha_{i,n-1}^* b_n(a)^{-1}$, $i = 1, \ldots, 2n-k$, and $B_j^+(f) = 0, j = 1, \ldots, k$, and*

$$K(a,s) = \sum_{i=1}^{2n} \beta_i(a)\phi_i(s) \quad \textit{for } a \leqslant s \leqslant b.$$

The function $K(b, \cdot)$ satisfies the boundary conditions $B_i^*(f) = -\beta_{i,n-1}^* b_n(b)^{-1}$, $i = 1, \ldots, 2n-k$, and $B_j^+(f) = 0, j = 1, \ldots, k$, and

$$K(b,s) = \sum_{i=1}^{2n} \alpha_i(b)\phi_i(s) \quad \textit{for } a \leqslant s \leqslant b.$$

PROOF. Let $\phi(s) = \sum_{i=1}^{2n} \beta_i(a)\phi_i(s)$, $a \leqslant s \leqslant b$. Clearly $\phi \in C^\infty[a,b]$ and $\tau\tau^*\phi(s) \equiv 0$ on $[a,b]$. We want to show that $\phi(s) = K(a,s)$ for all $s \in [a,b]$ and check the boundary conditions. For $j = 1, \ldots, 2n-k$ we have

$$B_j^*(\phi) = \sum_{l=0}^{n-1} \alpha_{jl}^* \left[\sum_{i=1}^{2n} \alpha_i(a)\phi_i^{(l)}(a) - \eta_l(a) \right] + \sum_{i=1}^{2n} \beta_i(a) D_j^*(\phi_i) \quad \text{by (26)}$$

$$= \sum_{i=1}^{2n} \alpha_i(a) C_j^*(\phi_i) + \sum_{i=1}^{2n} \beta_i(a) D_j^*(\phi_i) + \alpha_{j,n-1}^* b_n(a)^{-1} \quad \text{by Remark 2}$$

$$= \alpha_{j,n-1}^* b_n(a)^{-1} \quad \text{by (27);}$$

for $j = 1, \ldots, q$ we have

$$B_j^+(\phi) = B_j(\tau^*\phi) = 0 \quad \text{by (24);}$$

and for $j = q + 1, \ldots, k$ we have

$$B_j^+(\phi) = \sum_{l=0}^{2n-1} \alpha_{jl}^+ \left[\sum_{i=1}^{2n} \beta_i(a)\phi_i^{(l)}(a) \right] + \sum_{i=1}^{2n} \beta_i(a) D_j^+(\phi_i)$$

$$= -\sum_{l=0}^{2n-1} \alpha_{jl}^+ \eta_l(a) \quad \text{by (26) and (28)}$$

$$= -\sum_{i=0}^{n-1} \alpha_{ji} \sum_{l=0}^{n} \sum_{m=l}^{l+i} \binom{i}{m-l} b_l^{(i-m+l)}(a)\eta_m(a) \quad \text{by (2)}$$

$$= 0 \quad \text{by (15) and Remark 2 .}$$

Finally, let us show that $\phi = K(a, \cdot)$. We know that $\phi \in \mathfrak{R}(L)$ by (29) and $K(a, \cdot) \in \mathfrak{R}(L)$ by Theorem 1. Take any $f \in \mathfrak{D}(LL^*)$ and set $g = L^*f = \tau^*f$. Clearly $f \in \mathfrak{D}(L^*)$ and $g = \tau^*f \in \mathfrak{D}(L) \cap \mathfrak{N}(L)^\perp$, and by the above $\tau^*\phi \in \mathfrak{N}(L)$. Thus,

$$\int_a^b \phi(s)\tau\tau^*f(s)\,ds = \int_a^b \tau^*\phi(s)\tau^*f(s)\,ds + F_t(\tau)(g,\phi)\Big|_a^b$$

$$= F_b(\tau)(g,\phi) - F_a(\tau)(g,\phi).$$

Choose a function $\psi \in H^n[a,b]$ such that $\psi^{(j)}(a) = \psi^{(j)}(b) = 0$ for $j = 0, 1, \cdots, n-1$, except $\psi^{(n-1)}(a) = b_n(a)^{-1}$. Then

$$\phi - \psi \in H^n[a,b] \quad \text{and} \quad B_i^*(\phi - \psi) = 0 \quad \text{for } i = 1, \ldots, 2n-k,$$

so $\phi - \psi \in \mathfrak{D}(L^*)$. Thus, we can rewrite the above equation as

$$\int_a^b \phi(s)\tau\tau^*f(s)ds = F_b(\tau)(g,\phi-\psi) + F_b(\tau)(g,\psi) - F_a(\tau)(g,\phi-\psi)$$

$$- F_a(\tau)(g,\psi)$$

$$= -F_a(\tau)(g,\psi) = -\sum_{l=0}^{n-1} F_a^{l,n-1}(\tau) g^{(l)}(a) b_n(a)^{-1}$$

$$= \tau^*f(a) = \int_a^b K(a,s)\tau\tau^*f(s)\,ds \quad \text{by Lemma 1,}$$

or

$$\int_a^b [\phi(s) - K(a,s)]\tau\tau^* f(s)\,ds = 0 \quad \text{for all } f \in \mathfrak{D}(LL^*).$$

Since $\mathfrak{R}(LL^*) = \mathfrak{R}(L)$, we must have $\phi(s) = K(a,s)$ for $a \leqslant s \leqslant b$. This same argument can be used at the end point b to study the function $K(b, \cdot)$. Q.E.D.

Combining Theorems 5, 6, and 7 we obtain the following result.

THEOREM 8. *In terms of the functions $\alpha_i(t)$, $\beta_i(t)$, $i = 1, \ldots, 2n$, which belong to $C^\infty[a,b]$, the generalized Green's function $K(t,s)$ has the representation*

$$K(t,s) = \sum_{i=1}^{2n} \alpha_i(t)\phi_i(s), \qquad a \leqslant s < t \leqslant b,$$

$$K(t,s) = \sum_{i=1}^{2n} \beta_i(t)\phi_i(s), \qquad a \leqslant t < s \leqslant b.$$

Moreover, $K(t,s)$ is infinitely differentiable in both variables for $t \neq s$.

8. **The adjoint relationship.** In a series of theorems we have characterized the function $K(t, \cdot)$, with the differential operators L^* and LL^* playing a major role. We now want to study the function $K(\cdot, s)$, and we expect the differential operators L and L^*L to play an analogous role. Indeed, this will turn out to be the situation.

If we set

$$H_1 = [L^* | \mathfrak{D}(L^*) \cap \mathfrak{N}(L^*)^\perp]^{-1},$$

then

$$\begin{aligned} L^* H_1 f &= f && \text{for all } f \in \mathfrak{R}(L^*), \\ H_1 L^* f &= f - Qf && \text{for all } f \in \mathfrak{D}(L^*), \end{aligned}$$

and the generalized inverse of L^* is the operator $(L^*)^\dagger : S \to S$ given by

$$(L^*)^\dagger f = H_1(I - P)f, \qquad f \in S.$$

THEOREM 9. $(L^*)^\dagger = (L^\dagger)^*$

PROOF. Take $f \in S$ and $g \in S$, and set $u = L^\dagger f$ and $v = (L^*)^\dagger g$. Clearly $u \in \mathfrak{D}(L) \cap \mathfrak{N}(L)^\perp$, $v \in \mathfrak{D}(L^*) \cap \mathfrak{N}(L^*)^\perp$, and $Lu = f - Qf$ and $L^* v = g - Pg$ by property (iii) of the generalized inverse. Thus,

$$(L^\dagger f, g) = (u, L^* v + Pg) = (Lu, v) = (Lu + Qf, v) = (f, (L^*)^\dagger g).$$

This proves that $(L^*)^\dagger = (L^\dagger)^*$. Q.E.D.

Proceeding as in §2, the generalized Green's function for L^* is the unique function $K_1(t,s)$ determined by the equation

$$(L^*)^\dagger f(t) = H_1(I-P)f(t) = \int_a^b K_1(t,s)f(s)\,ds, \tag{30}$$

which is valid for all $f \in S$ and for all $t \in [a,b]$. Theorems 1–8 can now be applied to establish the basic properties of $K_1(t,\cdot)$. We will discuss these properties after we relate $K_1(t,s)$ to $K(t,s)$.

THEOREM 10. *$K(t,s) = K_1(s,t)$ for all $t \neq s$.*

PROOF. Take functions $f \in S$ and $g \in S$. We know that $K(t,s)$ and $K_1(s,t)$ are very smooth functions for $t \neq s$, and both are bounded on the square $[a,b] \times [a,b]$. Thus, by Fubini's theorem

$$\int_a^b \int_a^b K(t,s)f(s)g(t)\,ds\,dt = \int_a^b L^\dagger f(t)g(t)\,dt$$
$$= \int_a^b f(t)(L^*)^\dagger g(t)\,dt = \int_a^b \int_a^b K_1(t,s)f(t)g(s)\,ds\,dt,$$

or

$$\int_a^b \int_a^b K(t,s)f(s)g(t)\,ds\,dt = \int_a^b \int_a^b K_1(s,t)f(s)g(t)\,ds\,dt.$$

Since the set of functions $f(s)g(t)$ with f, $g \in S$ is fundamental in $L^2([a,b] \times [a,b])$, we conclude that $K(t,s) = K_1(s,t)$ a.e. on $[a,b] \times [a,b]$. But these functions are continuous for $t \neq s$, so $K(t,s) = K_1(s,t)$ for all $t \neq s$. Q.E.D.

If we fix a point $s \in [a,b]$ and consider $K_1(s,\cdot)$ as a function of the variable t, then all our earlier results can be restated for $K_1(s,\cdot)$. In view of Theorem 10 each of these results can be interpreted as a statement about $K(\cdot,s)$ as a function of t. Let us summarize our earlier work, stating it in terms of the function $K(\cdot,s)$.

THEOREM 1′. *$K(\cdot,s) \in \mathfrak{N}(L)^\perp = \mathfrak{R}(L^*)$ for all $s \in [a,b]$.*

THEOREM 2′. *For each c with $a < c < b$, $K(\cdot,c) \in C^\infty[a,c]$ with $\tau^*\tau K(t,c) = 0$ for $a \leqslant t \leqslant c$, and $K(\cdot,c) \in C^\infty[c,b]$ with $\tau^*\tau K(t,c) = 0$ for $c \leqslant t \leqslant b$ (the left-hand and the right-hand derivatives are not required to be equal at $t = c$). Moreover, $K(\cdot,a) \in C^\infty[a,b]$ and $K(\cdot,b) \in C^\infty[a,b]$ with $\tau^*\tau K(t,a) = 0$ and $\tau^*\tau K(t,b) = 0$ for all $t \in [a,b]$.*

For each $s \in [a,b]$ we form the $2n \times 2n$ linear system

$$\sum_{j=0}^{2n-1} F_s^{lj}(\tau^*\tau)v_j(s) = a_l(s), \qquad l = 0, 1, \ldots, n, \tag{31}$$

$$\sum_{j=0}^{2n-1} F_s^{lj}(\tau^*\tau)v_j(s) = 0, \qquad l = n+1, \ldots, 2n-1, \tag{32}$$

for the $2n$ unknowns $v_0(s), v_1(s), \ldots, v_{2n-1}(s)$.

THEOREM 3′. *For each $s \in [a,b]$ the $2n \times 2n$ linear system* (31)–(32) *has a unique solution $v_0(s), v_1(s), \ldots, v_{2n-1}(s)$. Moreover, these functions belong to $C^\infty[a,b]$, and the following properties hold*:

(a) $\tau f(s) = \sum_{j,l=0}^{2n-1} F_s^{lj}(\tau^*\tau)f^{(l)}(s)v_j(s)$ *for all $f \in H^{2n}[a,b]$ and for all $s \in [a,b]$.*

(b) *For each point c with $a < c < b$,*

$$\lim_{t\to c^-} \frac{\partial^j K}{\partial t^j}(t,c) - \lim_{t\to c^+} \frac{\partial^j K}{\partial t^j}(t,c) = v_j(c),$$

for $j = 0, 1, \ldots, 2n-1$.

The functions $v_i(s)$, $i = 0, 1, \ldots, n-1$, are given by

$$v_0(s) = v_1(s) = \cdots = v_{n-2}(s) = 0 \tag{33}$$

and

$$v_{n-1}(s) = -a_n(s)^{-1} = (-1)^{n-1} b_n(s)^{-1}, \tag{34}$$

while the functions $v_i(s)$, $i = n, n+1, \ldots, 2n-1$, are given by

$$\begin{aligned} v_{n+j}(s) = {} & \sum_{i=n-j-1}^{n-1} \binom{j}{n-i-1} a_i^{(j-n+i+1)}(s) a_n(s)^{-2} \\ & - \sum_{p=n}^{n+j-1} \left[\sum_{i=p-j}^{n} \binom{j}{p-i} a_i^{(j-p+i)}(s) a_n(s)^{-1} \right] v_p(s) \end{aligned} \tag{35}$$

for $j = 0, 1, \ldots, n-1$ (in case $j = 0$ the second sum does not occur). Explicit formulas can be given for $v_n(s)$ and $v_{n+1}(s)$, and the jump conditions for $K(\cdot,c)$ in case $a < c < b$ can also be characterized in terms of quasi-derivatives (see equations (19)–(21)).

THEOREM 4′. *For each c with $a < c < b$ the function $K(\cdot,c)$ satisfies the boundary conditions $B_i(f) = 0$, $i = 1, \ldots, k$, and $B_j^{*+}(f) = B_j^*(\tau f) = 0$, $j = 1, \ldots, 2n-k$.*

THEOREM 7′. *The function $K(\cdot,a)$ satisfies the boundary conditions $B_i(f) = \alpha_{i,n-1} a_n(a)^{-1}$, $i = 1, \ldots, k$, and $B_j^{*+}(f) = B_j^*(\tau f) = 0$, $j = 1, \ldots, 2n$*

$-k$. *The function* $K(\cdot, b)$ *satisfies the boundary conditions* $B_i(f) = -\beta_{i,n-1} a_n(b)^{-1}$, $i = 1, \ldots, k$, *and* $B_j^{*+}(f) = B_j^*(\tau f) = 0, j = 1, \ldots, 2n - k$.

9. **A representation theorem.** In our next theorem we obtain a representation of $K(t,s)$ in terms of the functions $\phi_1, \ldots, \phi_{2n}$ and $\psi_1, \ldots, \psi_{2n}$, which form bases for the solution spaces of $\tau\tau^*\phi = 0$ and $\tau^*\tau\psi = 0$, respectively.

THEOREM 11. *There exist unique* $2n \times 2n$ *scalar matrices* $\Gamma = [\gamma_{ij}]$ *and* $\Gamma' = [\gamma'_{ij}]$ *such that*

$$K(t,s) = \sum_{i,j=1}^{2n} \gamma_{ij}\psi_i(t)\phi_j(s), \qquad a \leqslant s < t \leqslant b,$$

$$K(t,s) = \sum_{i,j=1}^{2n} \gamma'_{ij}\psi_i(t)\phi_j(s), \qquad a \leqslant t < s \leqslant b.$$

PROOF. By Theorem 8 we have

$$(*) \qquad \begin{aligned} K(t,s) &= \sum_{j=1}^{2n} \alpha_j(t)\phi_j(s), \qquad a \leqslant s < t \leqslant b, \\ K(t,s) &= \sum_{j=1}^{2n} \beta_j(t)\phi_j(s), \qquad a \leqslant t < s \leqslant b, \end{aligned}$$

where the $\alpha_j(t)$, $\beta_j(t)$ belong to $C^\infty[a,b]$. If we fix $s \in [a,b]$ in these equations and apply $\tau^*\tau$ to the resulting functions of t, then

$$0 = \sum_{j=1}^{2n} \tau^*\tau\alpha_j(t)\phi_j(s), \qquad a \leqslant s < t \leqslant b,$$

$$0 = \sum_{j=1}^{2n} \tau^*\tau\beta_j(t)\phi_j(s), \qquad a \leqslant t < s \leqslant b.$$

Fixing $t \in (a,b)$ and considering these last equations as functions of s on the intervals $a \leqslant s < t$ and $t < s \leqslant b$, by the independence of the functions $\phi_1, \ldots, \phi_{2n}$ we get

$$\tau^*\tau\alpha_j(t) = \tau^*\tau\beta_j(t) = 0 \quad \text{for } j = 1, \ldots, 2n.$$

From the continuity we conclude that these equations hold for all $t \in [a,b]$, and consequently, there exist unique $2n \times 2n$ scalar matrices $\Gamma = [\gamma_{ij}]$ and $\Gamma' = [\gamma'_{ij}]$ such that

$$(**)\qquad \begin{aligned} \alpha_j(t) &= \sum_{i=1}^{2n} \gamma_{ij}\psi_i(t), \qquad a \leqslant t \leqslant b, \\ \beta_j(t) &= \sum_{i=1}^{2n} \gamma'_{ij}\psi_i(t), \qquad a \leqslant t \leqslant b. \end{aligned}$$

Combine (*) and (**). Q.E.D.

10. **The projections on the null spaces.** In this section we relate the generalized Green's function $K(t,s)$ to the two projection operators P and Q. Suppose the functions $\psi_1, \ldots, \psi_p$ and $\phi_1, \ldots, \phi_q$ form orthonormal bases for $\mathfrak{N}(L)$ and $\mathfrak{N}(L^*)$, respectively. Then

$$(36)\qquad Pf(t) = \int_a^b \left[\sum_{i=1}^{p} \psi_i(t)\psi_i(s)\right] f(s)\,ds \quad \text{for all } t \in [a,b], f \in S,$$

and

$$(37)\qquad Qf(s) = \int_a^b \left[\sum_{i=1}^{q} \phi_i(t)\phi_i(s)\right] f(t)\,dt \quad \text{for all } s \in [a,b], f \in S.$$

LEMMA 4. *For each $c \in [a,b]$ the function $\tau^* K(c,s)$ has a removable singularity at $s = c$ and $\tau^* K(c,s) \in \mathfrak{N}(L)$.*

PROOF. In case $a < c < b$ the proof is contained in equation (14) and Remark 1, Theorem 4, and Theorem 2. For $c = a$ or $c = b$ we use Theorem 2 and Theorem 7. Q.E.D.

THEOREM 12. *For each $c \in [a,b]$,*

$$\tau^* K(c,s) = -\sum_{i=1}^{p} \psi_i(c)\psi_i(s) \quad \textit{for all } s \in [a,b],\ s \neq c.$$

PROOF. Fix a point c with $a < c < b$, and take any function $f \in S$. Setting $u = (I - P)f \in \mathfrak{N}(L)^{\perp}$ and $v = Pf \in \mathfrak{N}(L)$, we have

$$\begin{aligned} \int_a^b \tau^* K(c,s) f(s)\,ds &= \int_a^b \tau^* K(c,s) u(s)\,ds + \int_a^b \tau^* K(c,s) v(s)\,ds \\ &= \int_a^b \tau^* K(c,s) v(s)\,ds \quad \text{by Lemma 4} \\ &= F_c(\tau^*)(\eta_-, v) - F_c(\tau^*)(\eta_+, v) \quad \text{by Theorem 4} \\ &= -Pf(c) \quad \text{by Lemma 2 .} \end{aligned}$$

Comparing this to (36) we conclude that

$$\tau^* K(c,s) = -\sum_{i=1}^{p} \psi_i(c)\psi_i(s) \quad \text{for all } s \in [a,b], \quad s \neq c.$$

The proof is completed by using the continuity in the $c = t$ variable. Q.E.D.

We conclude with the analogue of Theorem 12 when the roles of the t and s variables are interchanged.

THEOREM 12′. *For each* $c \in [a,b]$,

$$\tau K(t,c) = -\sum_{i=1}^{q} \phi_i(t)\phi_i(c) \quad \textit{for all } t \in [a,b],\ t \neq c.$$

PROOF. We can use the proof of Theorem 12, or we can combine Theorem 10 and Theorem 12. Q.E.D.

11. **Examples.** We conclude our discussion by presenting two examples of generalized Green's functions. The actual construction of these functions was carried out using a method developed in [7]. This method requires having bases $\phi_1, \ldots, \phi_n$ and $\psi_1, \ldots, \psi_n$ for the solution spaces of $\tau^*\phi = 0$ and $\tau\psi = 0$, respectively (see Theorem 2 of [7, p. 203] with $m = q$).

EXAMPLE 1. In $S = L^2[0,2\pi]$ consider the differential operator L given by

$$\mathfrak{D}(L) = \{f \in H^2[0,2\pi] \mid f(0) - f(2\pi) = 0, f'(0) - f'(2\pi) = 0\},$$
$$Lf = -f''.$$

Clearly L is determined by the formal differential operator τ having coefficients $a_0(t) = a_1(t) = 0$ and $a_2(t) = -1$ and by the boundary conditions $B_1(f) = f(0) - f(2\pi) = 0$, $B_2(f) = f'(0) - f'(2\pi) = 0$. Also, it is well known that $L^* = L$, so LL^* is the differential operator determined by $\tau\tau^* f = f^{(4)}$ and the four boundary conditions

$$B_1^*(f) = B_1(f) = 0,$$
$$B_2^*(f) = B_2(f) = 0,$$
$$B_1^+(f) = -f''(0) + f''(2\pi) = 0,$$
$$B_2^+(f) = -f'''(0) + f'''(2\pi) = 0.$$

Proceeding as in [7], the generalized Green's function for L is given by

$$K(t,s) = \frac{1}{4\pi}s^2 + \frac{1}{2}s - \frac{1}{2\pi}st - \frac{1}{2}t + \frac{1}{4\pi}t^2 + \frac{\pi}{6}, \qquad 0 \leqslant s \leqslant t \leqslant 2\pi,$$
$$K(t,s) = \frac{1}{4\pi}t^2 + \frac{1}{2}t - \frac{1}{2\pi}ts - \frac{1}{2}s + \frac{1}{4\pi}s^2 + \frac{\pi}{6}, \qquad 0 \leqslant t \leqslant s \leqslant 2\pi.$$

Notice that $K(t,s)$ is symmetric, which we expect in view of Theorem 10 and the selfadjointness of L. Also, for $0 < c < 2\pi$ the functions $(\partial^2 K/\partial s^2)(c,s)$ and $(\partial^3 K/\partial s^3)(c,s)$ have zero jumps at $s = c$, which agrees with the facts that $\eta_2(c) = 0$ and $\eta_3(c) = 0$ by equations (17) and (18). Finally, the function $K(0,s) = -(1/2)s + (1/4\pi)s^2 + \pi/6$ satisfies the boundary conditions

$$B_1^*(f) = \alpha_{11}^* b_2(0)^{-1} = 0,$$

$$B_2^*(f) = \alpha_{21}^* b_2(0)^{-1} = -1, \quad B_1^+(f) = B_2^+(f) = 0,$$

in agreement with Theorem 7.

EXAMPLE 2. The differential operator L defined by

$$\mathfrak{D}(L) = \{f \in H^2[0,2\pi] \mid f(0) = 0\}, \qquad Lf = f'' + f,$$

is a nonselfadjoint operator in $S = L^2[0,2\pi]$. Its adjoint is given by

$$\mathfrak{D}(L^*) = \{f \in H^2[0,2\pi] \mid f(0) = f(2\pi) = f'(2\pi) = 0\}, \qquad L^*f = f'' + f,$$

and hence, the 4th order differential operator LL^* is given by

$$\mathfrak{D}(LL^*) = \{f \in H^4[0,2\pi] \mid f(0) = f(2\pi) = f'(2\pi) = f''(0) + f(0) = 0\}, \qquad Lf = f^{(4)} + 2f'' + f.$$

The generalized Green's function for L is

$$K(t,s) = -\cos t \sin s - \frac{1}{2\pi} \sin t \sin s + \frac{s}{2\pi} \sin t \cos s, \qquad 0 \leqslant s \leqslant t \leqslant 2\pi,$$

$$K(t,s) = -\sin t \cos s - \frac{1}{2\pi} \sin t \sin s + \frac{s}{2\pi} \sin t \cos s, \qquad 0 \leqslant t \leqslant s \leqslant 2\pi.$$

It is easy to verify that $K(t,s)$ satisfies the various properties established in our theorems.

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Department of Mathematics, Colorado State University, Fort Collins, Colorado 80523