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MODULI OF CONTINUITY FOR EXPONENTIAL LIPSCHITZ CLASSES

BY

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ABSTRACT. Let Ψ be a convex function, and let f be a real-valued function on [0, 1]. Let a modulus of continuity associated to Ψ be given as

$$Q_\Psi(\delta, f) = \inf\left\{\lambda: \frac{1}{\delta} \iint_{|x-y|<\delta} \Psi\left(\frac{|f(x)-f(y)|}{\lambda}\right) dx\, dy < \Psi(1)\right\}.$$

It is shown that $\int_0^1 Q_\Psi(\delta, f)\, d(-\Psi^{-1}(c/\delta)) < \infty$ guarantees the essential continuity of f, and, in fact, a uniform Lipschitz estimate is given. In the case that $\Psi(u) = \exp u^2$ the above condition reduces to

$$\int_0^1 Q_{\exp u^2}(\delta, f)\, \frac{d\delta}{\delta\sqrt{\log(c/\delta)}} < \infty.$$

This exponential square condition is satisfied almost surely by the random Fourier series $f_t(x) = \sum_{n=1}^\infty a_n R_n(t) e^{inx}$, where $\{R_n\}$ is the Rademacher system, as long as

$$\int_0^1 \sqrt{a_n^2 \sin^2(n\delta/2)}\, \frac{d\delta}{\delta\sqrt{\log(1/\delta)}} < \infty.$$

Hence, the random essential continuity of $f_t(x)$ is guaranteed by each of the above conditions.

1. **Introduction.** This study is concerned with a class of real valued functions defined on [0, 1] for which certain boundedness and smoothness properties can be characterized in terms of the following modulus of continuity which is associated with the convex function Ψ, namely:

$$Q_\Psi(\delta, f) = \inf\left\{\lambda: \frac{1}{\delta} \iint_{\substack{|x-y|<\delta \\ x,y\in[0,1]}} \Psi\left(\frac{|f(x)-f(y)|}{\lambda}\right) dx\, dy < \Psi(1)\right\}.$$

The case where $\Psi(u) = |u|^p$, $p > 1$, has been investigated [3] and motivates much of the more general case.

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If $f(x)$ is measurable on [0, 1] let $f^*(x)$ be the unique nonincreasing function on [0, 1] such that for all real λ,

$$m\{x: f(x) \geq \lambda\} = m\{x; f^*(x) \geq \lambda\}.$$

$f^*(x)$ is termed the "monotone rearrangement" of f. The first result is stated in terms of f^*, and it leads immediately to an estimate for the oscillation of a function. Let $L_\Psi[0, 1]$ denote the Orlicz space associated with Ψ on [0, 1].

THEOREM 1.1. *If $f \in L_\Psi[0, 1]$ and $0 < x \leq \frac{1}{2}$, then*

$$\left.\begin{matrix} f^*(x) - f^*(\frac{1}{2}) \\ f^*(\frac{1}{2}) - f^*(1-x) \end{matrix}\right\} \leq \frac{1}{\log(3/2)} \int_x^1 Q_\Psi(\delta, f)\Psi^{-1}\left(\frac{4\Psi(1)}{\delta}\right)\frac{d\delta}{\delta}. \tag{1.1}$$

This theorem is an immediate generalization of the results in [3] and can be used to give a sufficient condition for the essential continuity of a function f and also a uniform Lipschitz-type estimate. Namely:

THEOREM 1.2. *Let $f \in L_\Psi[0, 1]$ and suppose that*

$$\int_0^1 Q_\Psi(\delta, f)\Psi^{-1}\left(\frac{4\Psi(1)}{\delta}\right)\frac{d\delta}{\delta} < \infty. \tag{1.2}$$

Then f is essentially continuous and, for all x and y in the Lebesgue set of f,

$$|f(x) - f(y)| \leq \frac{2}{\log(3/2)} \int_0^{|x-y|} Q_\Psi(\delta, f)\Psi^{-1}\left(\frac{4\Psi(1)}{\delta}\right)\frac{d\delta}{\delta}. \tag{1.3}$$

Of particular interest is the case where $\Psi(u) = \exp u^2$. The above estimates are then in terms of $\int_0^1 Q_{\exp u^2}(\delta, f)\sqrt{\log(c/\delta)}\; d\delta/\delta$. Now there are heuristic arguments which strongly suggest that the best possible estimates involving $Q_{\exp u^2}(\delta, f)$ should be in terms of the quantity $\int_0^1 Q_{\exp u^2}(\delta, f)\, d\delta/\delta\sqrt{\log(c/\delta)}$. This suggests that a stronger result than Theorem 1.1 may hold in full generality. To this end observe that the above expression is the special case corresponding to $\Psi(u) = \exp u^2$ in $\int_0^1 Q_\Psi(\delta, f) d(-\Psi^{-1}(c/\delta))$, and indeed it turns out that the following improvement of Theorem 1.1 holds.

THEOREM 1.3. *Let $\frac{1}{2} < \alpha < 1$ with α fixed. Also let $c = \Psi(1)/2(1-\alpha)\alpha$ and $K = \Psi^{-1}(2c\alpha)/\Psi^{-1}(c)$. If $f(x) \in L_\Psi[0, 1]$ and $0 < x \leq \frac{1}{2}$, then*

$$\left.\begin{matrix} f^*(x) - f^*(\frac{1}{2}) \\ f^*(\frac{1}{2}) - f^*(1-x) \end{matrix}\right\} \leq \frac{K}{\log K} \int_x^1 Q_\Psi(\delta, f) d\left(-\Psi^{-1}\left(\frac{c}{\delta}\right)\right). \tag{1.4}$$

A change of scale argument gives Theorem 1.2 as a consequence of Theorem 1.1. This change of scale argument is unworkable in the present situation. However, this type of argument, necessary to obtain a continuity estimate, can be carried out when the right-hand side appears in an "integrated by parts" form. Let $Q_\Psi^*(\delta, f)$ denote the sup function, i.e.

$$Q_\Psi^*(\delta,f) = \sup_{0<t<\delta} Q_\Psi(t,f).$$

THEOREM 1.4. *Let* $f \in L_\Psi[0, 1]$ *and suppose that*

$$\int_0^1 Q_\Psi^*(\delta,f)\,d\left(-\Psi^{-1}\left(\frac{c}{\delta}\right)\right) < \infty. \tag{1.5}$$

Then

$$\lim_{t\to 0} Q_\Psi^*(t,f)\Psi^{-1}(c/t) = 0, \tag{1.6}$$

$$\int_0^1 \Psi^{-1}\left(\frac{c}{t}\right) dQ_\Psi^*(t,f) < \infty, \tag{1.7}$$

and

$$f^*(0^+) - f^*(1^-) < 8\int_0^{2/3} \Psi^{-1}\left(\frac{c}{\delta}\right) dQ_\Psi^*(\delta,f). \tag{1.8}$$

From (1.8) a continuity estimate now follows from a change of scale argument.

THEOREM 1.5. *Let* $f \in L_\Psi[0, 1]$. *If*

$$\int_0^1 Q_\Psi^*(\delta,f)\,d\left(-\Psi^{-1}\left(\frac{c}{\delta}\right)\right) < \infty, \tag{1.9}$$

then f is essentially continuous and, in fact, for x and y Lebesgue points of f, with $\Delta = |x - y|$,

$$|f(x) - f(y)| < 8\left[Q_\Psi^*\left(\frac{2\Delta}{3},f\right)\Psi^{-1}\left(\frac{3c}{2\Delta}\right) + \int_0^{2\Delta/3} Q_\Psi^*(\delta,f)\,d\left(-\Psi^{-1}\left(\frac{c}{\delta}\right)\right)\right]. \tag{1.10}$$

Theorem 1.5 gives (1.9) as a sufficient condition for the essential continuity of f. In [2] a long and detailed argument (which is omitted here) is given to show that for a large class of convex functions,

$$\int_0^1 Q_\Psi^*(\delta,f)\,d\left(-\Psi^{-1}\left(\frac{c}{\delta}\right)\right) < c\int_0^1 Q_\Psi(\delta,f)\,d\left(-\Psi^{-1}\left(\frac{c}{\delta}\right)\right).$$

Hence, $\int_0^1 Q_\Psi(\delta, f)d(-\Psi^{-1}(c/\delta)) < \infty$ can be used as the sufficient condition for essential continuity.

2. The rearrangement arguments. The fundamental inequality which serves as the cornerstone here is the "rearrangement result" (see [4]) which gives

$$\iint_{|x-y|<\delta} \Phi(|f^*(x) - f^*(y)|)\,dx\,dy < \iint_{|x-y|<\delta} \Phi(|f(x) - f(y)|)\,dx\,dy \tag{2.1}$$

whenever Φ is increasing and $0 < \delta \leqslant 1$. As will be shown, (2.1) is used directly to prove Theorems 1.1, 1.3 and 1.4.

To show Theorem 1.1, fix δ, $0 < \delta \leqslant 1$, and $\lambda_0 = Q_\Psi(\delta, f)$. Also, for $0 < \alpha < \beta < 1$, let the rectangle $R_\delta(\alpha, \beta)$ be given by

$$R_\delta(\alpha, \beta) = \{(x, y): 0 \leqslant y \leqslant \alpha\delta, \beta\delta \leqslant x \leqslant \delta\}.$$

Now using the definitions of $Q_\Psi(\delta, f)$ and $R_\delta(\alpha, \beta)$, (2.1) and the monotonicity of $f^*(x)$, it follows that

$$\begin{aligned}\Psi(1) &\geqslant \frac{1}{\delta} \iint_{|x-y|<\delta} \Psi\left(\frac{|f^*(x) - f^*(y)|}{\lambda_0} \right) dx\, dy \\ &\geqslant \frac{2}{\delta} \iint_{R_\delta(\alpha,\beta)} \Psi\left(\frac{f^*(x) - f^*(y)}{\lambda_0} \right) dx\, dy \\ &\geqslant 2\delta\, \alpha(1-\beta)\Psi\left(\frac{f^*(\alpha\delta) - f^*(\beta\delta)}{\lambda_0} \right).\end{aligned}$$

Thus one obtains

$$Q_\Psi(\delta, f)\Psi^{-1}(\Psi(1)/2\alpha(1-\beta)\delta) \geqslant f^*(\alpha\delta) - f^*(\beta\delta).$$

Now for $x \leqslant \alpha/\beta$,

$$\begin{aligned}\int_x^1 Q_\Psi(\delta, f)\Psi^{-1}\left(\frac{\Psi(1)}{2\alpha(1-\beta)\delta} \right) \frac{d\delta}{\delta} &\geqslant \int_x^1 [f^*(\alpha\delta) - f^*(\beta\delta)]\, \frac{d\delta}{\delta} \\ &= \int_{\alpha x}^{\beta x} f^*(t)\, \frac{dt}{t} - \int_\alpha^\beta f^*(t)\, \frac{dt}{t} \\ &\geqslant [f^*(\beta\alpha) - f^*(\alpha)]\log(\beta/\alpha).\end{aligned}$$

Now choosing $\alpha = \frac{1}{2}$ and $\beta = \frac{3}{4}$, the upper part of (1.1) follows immediately, and the lower part is obtained from observing that $(-f(x))^* = -f^*(1-x)$.

To show that Theorem 1.2 is a consequence of Theorem 1.1, let $I = [\alpha, \beta] \subset [0, 1]$ and define

$$Q_\Psi(\delta, f, I) = \inf\left\{\lambda: \frac{1}{\delta} \iint_{\substack{|x-y|<\delta \\ x,y\in I}} \Psi\left(\frac{|f(x) - f(y)|}{\lambda} \right) dx\, dy \leqslant \Psi(1)\right\}.$$

Let $\Delta = \beta - \alpha$ and define $g(s)$ on $[0, 1]$ by $g(s) = f(\alpha + \Delta s)$. Let

$$\hat{Q}_\Psi(\delta, g) = \inf\left\{\lambda: \frac{1}{\delta} \iint_{|x-y|<\delta} \Psi\left(\frac{|g(x) - g(y)|}{\lambda} \right) dx\, dy \leqslant \frac{\Psi(1)}{\Delta}\right\}.$$

Observe that $Q_\Psi(\delta, f, I) \leqslant Q_\Psi(\delta, f)$ and $Q_\Psi(\Delta\delta, f, I) = \hat{Q}_\Psi(\delta, g)$. Let $O(I, f)$ denote the oscillation of f on I, i.e.

$$O(I, f) = \operatorname*{ess\,sup}_{t\in I} f(t) - \operatorname*{ess\,inf}_{t\in I} f(t).$$

Notice that $|f(\alpha) - f(\beta)| \leqslant O(I, f)$ whenever α and β are Lebesgue points of f.

Now applying Theorem 1.1 in this situation gives

$$O([0,1], g) = g^*(0^+) - g^*(1^-) \leqslant \frac{2}{\log(3/2)} \int_0^1 \hat{Q}_\Psi(\delta, g)\Psi^{-1}\left(\frac{4\Psi(1)}{\Delta\delta}\right)\frac{d\delta}{\delta}.$$

Clearly $O([0, 1], g) = O(I, f)$, so taking α and β to be Lebesgue points of f, one obtains

$$\begin{aligned} |f(\alpha) - f(\beta)| &\leqslant \frac{2}{\log(3/2)} \int_0^1 \hat{Q}_\Psi(\delta, g)\Psi^{-1}\left(\frac{4\Psi(1)}{\Delta\delta}\right)\frac{d\delta}{\delta} \\ &= \frac{2}{\log(3/2)} \int_0^1 Q_\Psi(\Delta\delta, f, I)\Psi^{-1}\left(\frac{4\Psi(1)}{\Delta\delta}\right)\frac{d\delta}{\delta} \\ &\leqslant \frac{2}{\log(3/2)} \int_0^1 Q_\Psi(\Delta\delta, f)\Psi^{-1}\left(\frac{4\Psi(1)}{\Delta\delta}\right)\frac{d\delta}{\delta} \\ &= \frac{2}{\log(3/2)} \int_0^\Delta Q_\Psi(\delta, f)\Psi^{-1}\left(\frac{4\Psi(1)}{\delta}\right)\frac{d\delta}{\delta}. \end{aligned}$$

Hence, the proof of Theorem 1.2 is complete.

Before giving the proof of Theorem 1.3, consider the function $\theta(t)$ on [0, 1] given by

$$\theta(t) = c\alpha/\Psi\left[K\Psi^{-1}(c\alpha/t)\right]$$

where $\frac{1}{2} < \alpha < 1$, $c = \Psi(1)/2(1-\alpha)\alpha$ and $K = \Psi^{-1}(2c\alpha)/\Psi^{-1}(c)$. Observe that

(1) $\theta(\alpha) = \frac{1}{2}$,

(2) $\theta^{-1}(s) = c\alpha/\Psi[(1/K)\Psi^{-1}(c\alpha/s)]$,

(3) $\Psi^{-1}(c\alpha/\theta(t)) = K\Psi^{-1}(c\alpha/t)$,

(4) $\Psi^{-1}(c\alpha/\theta^{-1}(t)) = (1/K)\Psi^{-1}(c\alpha/t)$.

The motivation for even considering $\theta(t)$ is that in the upcoming proof of Theorem 1.3, the solution to the following differential equation is needed:

$$\text{(2.2)}\quad \frac{d}{dt}\left[\Psi^{-1}\left(\frac{c\alpha}{\theta^{-1}(t)}\right)\right]\Big/\Psi^{-1}\left(\frac{c\alpha}{t}\right) = \frac{d}{dt}\left[\Psi^{-1}\left(\frac{c\alpha}{t}\right)\right]\Big/\Psi^{-1}\left(\frac{c\alpha}{\theta(t)}\right).$$

From the observations above it is clear that $\theta(t)$ is the desired solution since both the right- and left-hand sides of (2.2) are equal to

$$\frac{1}{K}\frac{d}{dt}\left(\Psi^{-1}\left(\frac{c\alpha}{t}\right)\right)\Big/\Psi^{-1}\left(\frac{c\alpha}{t}\right).$$

Now proceeding directly with the proof, fix δ, $0 < \delta < 1$, and let $\lambda_0 = Q_\Psi(\delta, f)$. Let $R_\Psi(\delta)$ denote the rectangle given by

$$R_\Psi(\delta) = \{(x,y): \alpha\delta \le x \le \delta, 0 \le y \le \theta(\alpha\delta)\}.$$

Using (2.1) and the monotonicity of $f^*(x)$, one obtains

$$\Psi(1) \geq \frac{1}{\delta} \iint_{|x-y| \le \delta} \Psi\left(\frac{|f^*(x) - f^*(y)|}{\lambda_0} \right) dx\, dy$$

$$\geq \frac{2}{\delta} \iint_{R_\Psi(\delta)} \Psi\left(\frac{f^*(y) - f^*(x)}{\lambda_0} \right) dx\, dy$$

$$\geq 2(1-\alpha)\theta(\alpha\delta)\Psi\left(\frac{f^*(\theta(\alpha\delta)) - f^*(\alpha\delta)}{\lambda_0} \right).$$

Hence

$$Q_\Psi(\delta, f)\Psi^{-1}(c\alpha/\theta(\alpha\delta)) \geq f^*(\theta(\alpha\delta)) - f^*(\alpha\delta).$$

Now observe for $0 < x \le \frac{1}{2}$,

$$\int_x^1 Q_\Psi(\delta, f) d\left(-\Psi^{-1}\left(\frac{c}{\delta}\right)\right) \geq \int_x^1 \left[f^*(\theta(\alpha\delta)) - f^*(\alpha\delta) \right] \frac{d(-\Psi^{-1}(c/\delta))}{\Psi^{-1}(c\alpha/\theta(\alpha\delta))}$$

$$= \int_{\theta(\alpha x)}^{\theta(\alpha)} f^*(t) \frac{d(-\Psi^{-1}(c\alpha/\theta^{-1}(t)))}{\Psi^{-1}(c\alpha/t)} - \int_{\alpha x}^{\alpha} f^*(t) \frac{d(-\Psi^{-1}(c\alpha/t))}{\Psi^{-1}(c\alpha/\theta(t))}$$

$$= \int_{\theta(\alpha x)}^{\theta(\alpha)} f^*(t) \frac{d(-\Psi^{-1}(c\alpha/t))}{\Psi^{-1}(c\alpha/\theta(t))} - \int_{\alpha x}^{\alpha} f^*(t) \frac{d(-\Psi^{-1}(c\alpha/t))}{\Psi^{-1}(c\alpha/\theta(t))}$$

$$= \int_{\theta(\alpha x)}^{\alpha x} f^*(t) \frac{d(-\Psi^{-1}(c\alpha/t))}{\Psi^{-1}(c\alpha/\theta(t))} - \int_{\theta(\alpha)}^{\alpha} f^*(t) \frac{d(-\Psi^{-1}(c\alpha/t))}{\Psi^{-1}(c\alpha/\theta(t))}$$

$$\geq \left[f^*(\alpha x) - f^*\left(\frac{1}{2}\right) \right] \int_{\theta(\beta)}^{\beta} \frac{d(-\Psi^{-1}(c\alpha/t))}{\Psi^{-1}(c\alpha/\theta(t))}$$

$$\geq (K^{-1}\log K)\left[f^*(x) - f^*\left(\tfrac{1}{2}\right) \right].$$

So the top part of (1.4) is immediate and the lower part follows from the fact that $(-f(x))^* = -f^*(1-x)$. This completes the proof of Theorem 1.3.

Of the three conclusions in Theorem 1.4 only the proof of (1.8) will be presented. The arguments involved in (1.6) and (1.7) are straightforward and can be seen in [2].

As a preliminary to this proof consider the function

$$\theta(t) = c/\Psi\left[K\Psi^{-1}(c\alpha/t) \right]$$

where α is fixed, $0 < \alpha < 1$, $c = \Psi(1)/2(1-\alpha)\alpha$ and $K > 1$. Let δ also be fixed, $0 < \delta < 1$, and define the following sequence $\{\delta_n\}$:

$$\delta_0 = \alpha\delta, \qquad \delta_n = \alpha\theta(\delta_{n-1}), \qquad n = 1, 2, 3, \ldots .$$

Observe that

(1) $$\delta_n = c\alpha/\Psi\left[K^n\Psi^{-1}(c/\delta)\right];$$

(2) $$\Psi^{-1}\left(\frac{c\alpha}{\delta_{n+1}}\right) = \frac{K}{K-1}\left[\Psi^{-1}\left(\frac{c\alpha}{\delta_{n+1}}\right) - \Psi^{-1}\left(\frac{c\alpha}{\delta_n}\right)\right].$$

From the first observation it is clear that δ_n decreases monotonically to zero.

This proof begins as before with δ fixed and $\lambda_0 = Q_\Psi(\delta, f)$. Let

$$R_\Psi(\delta) = \{(x,y)\colon \alpha\delta \leqslant x \leqslant \delta, 0 \leqslant y \leqslant \alpha\theta(\alpha\delta)\}.$$

Again invoking (2.1), one obtains

$$\Psi(1) \geqslant \frac{1}{\delta}\iint_{|x-y|\leqslant\delta}\Psi\left(\frac{|f^*(x) - f^*(y)|}{\lambda_0}\right)dx\,dy$$
$$\geqslant \frac{2}{\delta}\iint_{R_\Psi(\delta)}\Psi\left(\frac{f^*(y) - f^*(x)}{\lambda_0}\right)dx\,dy$$
$$\geqslant 2\alpha(1-\alpha)\theta(\alpha\delta)\Psi\left(\frac{f^*(\alpha\theta(\alpha\delta)) - f^*(\alpha\delta)}{\lambda_0}\right).$$

Hence,

$$Q_\Psi(\delta, f)\Psi^{-1}(c\alpha/\alpha\theta(\alpha\delta)) \geqslant f^*(\alpha\theta(\alpha\delta)) - f^*(\alpha\delta).$$

Using the above reasoning with δ replaced by $\theta(\delta_{n-1})$, one obtains

$$f^*(\delta_{n+1}) - f^*(\delta_n) \leqslant Q_\Psi(\theta(\delta_{n-1}), f)\Psi^{-1}(c\alpha/\delta_{n+1}).$$

Now let $A(t) = Q_\Psi^\sharp(t/\alpha, f)$, so that $Q_\Psi(t, f) \leqslant A(\alpha t)$. Combining the above and using the second observation gives

$$f^*(0^+) - f^*(\delta_0) = \sum_{n=0}^{\infty}\left[f^*(\delta_{n+1}) - f^*(\delta_n)\right]$$
$$\leqslant \sum_{n=0}^{\infty} A(\delta_n)\Psi^{-1}\left(\frac{c\alpha}{\delta_{n+1}}\right)$$
$$= \frac{K}{K-1}\sum_{n=0}^{\infty}\left[\Psi^{-1}\left(\frac{c\alpha}{\delta_{n+1}}\right) - \Psi^{-1}\left(\frac{c\alpha}{\delta_n}\right)\right]A(\delta_n).$$

Applying partial summation yields

$$\sum_{n=0}^{N}\left[\Psi^{-1}\left(\frac{c\alpha}{\delta_{n+1}}\right) - \Psi^{-1}\left(\frac{c\alpha}{\delta_n}\right)\right]A(\delta_n) = \sum_{n=1}^{N}\Psi^{-1}\left(\frac{c\alpha}{\delta_n}\right)\left[A(\delta_{n-1}) - A(\delta_n)\right]$$
$$+ \Psi^{-1}(c\alpha/\delta_{n+1})A(\delta_n) - \Psi^{-1}(c/\delta)A(\alpha\delta).$$

Observing that (1.6) applies to the second term and combining this with the above gives

$$\begin{aligned} f^*(0^+) - f^*(\alpha\delta) &\leqslant \frac{K}{K-1} \sum_{n=1}^{\infty} \Psi^{-1}\left(\frac{c\alpha}{\delta_{n-1}}\right)\left[A(\delta_{n-1}) - A(\delta_n)\right] \\ &\leqslant \frac{K^2}{K-1} \sum_{n=1}^{\infty} \int_{\delta_n}^{\delta_{n-1}} \Psi^{-1}\left(\frac{c\alpha}{\sigma}\right) dA(\sigma) \\ &= \frac{K^2}{K-1} \int_0^{\alpha\delta} \Psi^{-1}\left(\frac{c\alpha}{\sigma}\right) dA(\sigma) \\ &= \frac{K^2}{K-1} \int_0^{\delta} \Psi^{-1}\left(\frac{c}{t}\right) dQ_\Psi^*(t,f). \end{aligned}$$

Choosing $\alpha = \frac{3}{4}$, $\delta = \frac{2}{3}$ and $K = 2$ gives

$$f^*(0^+) - f^*\left(\frac{1}{2}\right) \leqslant 4\int_0^{2/3} \Psi^{-1}\left(\frac{c}{t}\right) dQ_\Psi^*(t,f).$$

So (1.8) follows since $(-f(x))^* = -f^*(1-x)$.

With the proof of Theorem 1.4 now complete, a change of scale argument and integration by parts can be used to prove Theorem 1.5. The notation $g(s)$, $\hat{Q}_\Psi(\delta, g)$ and $Q_\Psi(\delta, f, I)$ where $I = [\alpha, \beta] \subset [0, 1]$ is consistent with that used in the proof of Theorem 1.2. Applying the above argument to g and $\hat{Q}_\Psi(\delta, g)$ gives

$$g^*(0^+) - g^*(1^-) \leqslant 8\int_0^{2/3} \Psi^{-1}\left(\frac{c}{\Delta\delta}\right) d\hat{Q}_\Psi^*(\delta, g).$$

Combining this with the previous inequalities and identities yields for α and β Lebesgue points of f and $\Delta = \beta - \alpha$:

$$\begin{aligned} |f(\alpha) - f(\beta)| \leqslant O(I,f) = O([0,1],g) &= g^*(0^+) - g^*(1^-) \\ &\leqslant 8\int_0^{2/3} \Psi^{-1}\left(\frac{c}{\Delta\delta}\right) d\hat{Q}_\Psi^*(\delta, g) \\ &= 8\int_0^{2/3} \Psi^{-1}\left(\frac{c}{\Delta\delta}\right) dQ_\Psi^*(\Delta\delta, f, I) \\ &= 8\int_0^{2\Delta/3} \Psi^{-1}\left(\frac{c}{\delta}\right) dQ_\Psi^*(\delta, f, I). \end{aligned}$$

Continuing with the integration by parts one obtains

$$\begin{aligned} |f(\alpha) - f(\beta)| &\leqslant 8\left[\lim_{\varepsilon \to 0} \Psi^{-1}\left(\frac{c}{\delta}\right) Q_\Psi^*(\delta, f, I)\Big|_{\varepsilon}^{2\Delta/3} + \int_0^{2\Delta/3} Q_\Psi^*(\delta, f, I)\, d\left(-\Psi^{-1}\left(\frac{c}{\delta}\right)\right)\right] \\ &\leqslant 8\left[\Psi^{-1}\left(\frac{3c}{2\Delta}\right) Q_\Psi^*\left(\frac{2\Delta}{3}, f\right) + \int_0^{2\Delta/3} Q_\Psi^*(\delta, f)\, d\left(-\Psi^{-1}\left(\frac{c}{\delta}\right)\right)\right] \end{aligned}$$

and so the proof is complete.

3. **The periodic case.** Let f be periodic of period 2π, and let $\omega_\Psi(\delta, f)$ denote the following modulus of continuity:

$$\omega_\Psi(\delta, f) = \inf\left\{\lambda: \int_0^{2\pi} \Psi\left(\frac{|f(x+\delta) - f(x)|}{\lambda}\right) dx \leqslant \Psi(1)\right\}.$$

In the previous sections, functions on [0, 1] have been considered, however, by a change of scale the previous results remain valid for functions defined on any interval $[a, b]$ as long as $Q_\Psi(\delta, f, [a, b])$ is used. In this section consider functions on $[0, 2\pi]$ and let $Q_\Psi(\delta, f) = Q_\Psi(\delta, f, [0, 2\pi])$.

Theorem 1.5 gives (1.9) as a sufficient condition for the essential continuity of f. The goal of this section is to show that

$$\int_0^1 \omega_\Psi(\delta, f) d\left(-\Psi^{-1}\left(\frac{c}{\delta}\right)\right) < \infty \tag{3.1}$$

is also a sufficient condition in the periodic case for a large class of convex functions. The following lemma is an essential preliminary.

LEMMA 3.1. *Let $f \in L_\Psi[0, 2\pi]$ be periodic of period 2π. Let $0 \leqslant \sigma \leqslant \delta \leqslant 2\pi$. Then*

$$\omega_\Psi(\delta, f) \leqslant \frac{2}{\delta} \int_0^\delta \omega_\Psi(t, f)\, dt; \tag{3.2}$$

and, in fact,

$$\omega_\Psi(\sigma, f) \leqslant \frac{10}{\delta} \int_0^\delta \omega_\Psi(t, f)\, dt, \tag{3.3}$$

and

$$Q_\Psi(\sigma, f) \leqslant \frac{200}{\delta} \int_0^\delta \omega_\Psi(t, f)\, dt. \tag{3.4}$$

Assuming the validity of (3.4) for the moment, the main result of this section can be presented.

THEOREM 3.2. *Suppose Ψ has the property that there exists an $a > 0$ so that $\varphi(t) = (d/dt)(-\Psi^{-1}(c/t))$ is monotone decreasing on $(0, a]$. Then for all $f \in L_\Psi[0, 2\pi]$ there exists a universal constant C_Ψ such that for $a_0 = \min\{a, 1\}$,*

$$\int_0^{a_0} Q_\Psi^*(\delta, f) d\left(-\Psi^{-1}\left(\frac{c}{\delta}\right)\right) \leqslant C_\Psi \int_0^{a_0} \omega_\Psi(\delta, f) d\left(-\Psi^{-1}\left(\frac{c}{\delta}\right)\right). \tag{3.5}$$

Note. The above condition on Ψ is equivalent to $\Psi(t)\Psi''(t)/(\Psi'(t))^2 < 2$ for $t > t_0$ where t_0 is to be specified (see [2]). Also observe that convex functions which behave like $|u|^p$, $e^{|u|}$, $\exp|u|^p$ and others satisfy this condition.

PROOF. First observe that (3.4) gives

$$Q_\Psi^*(\delta, f) \leq \frac{200}{\delta} \int_0^\delta \omega_\Psi(t, f)\, dt.$$

Let $\gamma(u)$ be the inverse to $\varphi(t)$ on $(0, a]$, i.e.,

$$\gamma(u) = \begin{cases} a & \text{if } 0 \leq u \leq \varphi(a), \\ \varphi^{-1}(u) & \text{if } u \geq \varphi(a). \end{cases}$$

Now observe that

$$\int_0^a \omega_\Psi(t, f)\varphi(t)\, dt = \int_0^a \omega_\Psi(t, f)\left(\int_0^{\varphi(t)} du\right) dt$$

$$= \int_0^\infty \int_0^{\gamma(u)} \omega_\Psi(t, f)\, dt\, du.$$

Utilizing (3.4) yields

$$200 \int_0^\infty \gamma(u)\left(\frac{1}{\gamma(u)} \int_0^{\gamma(u)} \omega_\Psi(t, f)\, dt\right) du \geq \int_0^\infty Q_\Psi^*(\gamma(u), f)\gamma(u)\, du$$

$$= a\varphi(a)Q_\Psi^*(a, f) + \int_{\varphi(a)}^\infty Q_\Psi^*(t, f)(-t\varphi'(t))\, dt.$$

Integration by parts gives (3.5) as follows:

$$200 \int_0^a \omega_\Psi(t, f)\varphi(t)\, dt \geq a\varphi(a)Q_\Psi^*(a, f) - \lim_{\varepsilon \to 0} Q_\Psi^*(t, f)t\varphi(t)\Big|_\varepsilon^a$$

$$+ \int_0^a \varphi(t)\left[t \frac{d}{dt} Q_\Psi^*(t, f) + Q_\Psi^*(t, f)\right] dt$$

$$= \lim_{\varepsilon \to 0} Q_\Psi^*(\varepsilon, f)\varepsilon\, \varphi(\varepsilon)$$

$$+ \int_0^a t\varphi(t) \frac{d}{dt} Q_\Psi^*(t, f)\, dt + \int_0^a Q_\Psi^*(t, f)\varphi(t)\, dt$$

$$\geq \int_0^a Q_\Psi^*(t, f)\varphi(t)\, dt.$$

The argument will be complete once Lemma 3.1 is shown to be valid. To show (3.2), first observe that

$$\omega_\Psi(\delta + \sigma, f) \leq \omega_\Psi(\delta, f) + \omega_\Psi(\sigma, f),$$

so that for $0 < t \leq \delta$,

$$\omega_\Psi(\delta, f) \leq \omega_\Psi(t, f) + \omega_\Psi(\delta - t, f).$$

This implies that

$$\omega_\Psi(\delta, f) \leq \frac{1}{\delta} \int_0^\delta \omega_\Psi(t, f)\, dt + \frac{1}{\delta} \int_0^\delta \omega_\Psi(\delta - t, f)\, dt$$

$$= \frac{2}{\delta} \int_0^\delta \omega_\Psi(t, f)\, dt.$$

To show (3.3) the triangle inequality along with the evenness of $\omega_\Psi(\delta, f)$ is utilized to give for $0 < \sigma \leq \delta$,

$$\omega_\Psi(\sigma, f) \leq \omega_\Psi(\sigma + \delta, f) + \omega_\Psi(\delta, f).$$

From (3.2) one obtains

$$\omega_\Psi(\sigma, f) \leq \frac{2}{\sigma + \delta} \int_0^{\sigma+\delta} \omega_\Psi(t, f)\, dt + \frac{2}{\delta} \int_0^{\delta} \omega_\Psi(t, f)\, dt.$$

(3.3) follows since

$$\frac{2}{\sigma + \delta} \int_0^{\sigma+\delta} \omega_\Psi(t, f)\, dt \leq \frac{8}{\delta} \int_0^{\delta} \omega_\Psi(t, f)\, dt.$$

Finally, to show (3.4) let $0 \leq \sigma \leq \delta$, and let $\lambda_\sigma = \omega_\Psi(\sigma, f)$ and $\lambda = (10/\delta)\int_0^\delta \omega_\Psi(t, f)\, dt$. From the definition of $\omega_\Psi(\sigma, f)$, one has

$$\Psi(1) \geq \int_0^{2\pi} \Psi\left(\frac{|f(x + \sigma) - f(x)|}{\lambda_0} \right) dx \geq \int_0^{2\pi - \sigma} \Psi\left(\frac{|f(x + \sigma) - f(x)|}{\lambda} \right) dx.$$

Integrating gives

$$\Psi(1) \geq \frac{1}{\delta} \int_0^{\delta} \int_0^{2\pi - \sigma} \Psi\left(\frac{|f(x + \sigma) - f(x)|}{\lambda} \right) dx d\sigma$$

$$\geq \frac{1}{\delta} \iint_{\substack{|x-y| \leq \delta \\ x, y \in [0, 2\pi]}} \Psi\left(\frac{|f(x) - f(y)|}{2\lambda} \right) dx dy.$$

So

$$Q_\Psi(\delta, f) \leq \frac{20}{\delta} \int_0^{\delta} \omega_\Psi(t, f)\, dt.$$

Now let $0 < \sigma \leq \delta$; then

$$Q_\Psi(\sigma, f) \leq \frac{20}{\sigma} \int_0^{\sigma} \omega_\Psi(u, f)\, du$$

$$\leq \frac{20}{\sigma} \int_0^{\sigma} \left(\frac{10}{\delta} \int_0^{\delta} \omega_\Psi(t, f)\, dt \right) du$$

$$\leq \frac{200}{\delta} \int_0^{\delta} \omega_\Psi(t, f)\, dt.$$

4. Application to random Fourier series. Let

$$f_t(x) = \sum_{n=1}^{\infty} a_n R_n(t) e^{inx},$$

where $\{R_n\}$ is the Rademacher system, denote a random Fourier series. The goal in this section is to apply the tools developed in earlier sections to give sufficient conditions for the random continuity of $f_t(x)$.

To facilitate doing this, let

$$Q_\Psi(\delta,f) = \inf\left\{\lambda: \frac{1}{\delta} \iint_{\substack{|x-y|<\delta \\ x,y\in[0,2\pi]}} \Psi\left(\frac{|f(x)-f(y)|}{\lambda}\right) dxdy \leqslant 4\pi\right\},$$

$$\omega_\Psi(\delta,f) = \inf\left\{\lambda: \int_0^{2\pi} \Psi\left(\frac{|f(x+\delta)-f(x)|}{\lambda}\right) dx \leqslant 4\pi\right\},$$

and

$$\Omega_\Psi(\delta,f) = \inf\left\{\lambda: \int_0^1\int_0^{2\pi} \Psi\left(\frac{|f_t(x+\delta)-f_t(x)|}{\lambda}\right) dx\, dt \leqslant 4\pi\right\}.$$

Observe that all previous results remain valid with these altered moduli of continuity as long as the constants are appropriately adjusted. The following result is the tool needed in the actual application.

THEOREM 4.1. *Let* $f_t(x) = \sum_{n=1}^{\infty} a_n R_n(t) e^{inx}$. *Then there exists a constant* K_Ψ *depending only on* Ψ *such that*

$$\int_0^1 \omega_\Psi(\delta,f_t)\, dt \leqslant K_\Psi \Omega_\Psi(\delta,f). \tag{4.1}$$

Hence, if Ψ *satisfies the condition in Theorem* 3.2, *then*

$$\int_0^1 \Omega_\Psi(\delta,f)\, d\left(-\Psi^{-1}\left(\frac{c}{\delta}\right)\right) < \infty \tag{4.2}$$

implies that for almost all t,

$$\int_0^1 \omega_\Psi(\delta,f_t)\, d\left(-\Psi^{-1}\left(\frac{c}{\delta}\right)\right) < \infty.$$

So for almost all t, $f_t(x)$ *is essentially continuous in* x.

PROOF. Only (4.1) needs to be argued. All else is immediate from it and the prior development. (4.1) follows from the more general fact that for $F(t, x)$ defined on $[0, 1] \times [0, 2\pi]$ and positive there exists a constant K_Ψ so that

$$\int_0^1 \|F(t,x)\|_\Psi\, dt \leqslant K_\Psi |||F|||_\Psi \tag{4.3}$$

where $\|\ \|_\Psi$ and $|||\ |||_\Psi$ denote the Orlicz norms on $[0, 2\pi]$ and $[0, 1] \times [0, 2\pi]$, respectively. (4.3) follows from a monotone class theorem argument [1]. Observe that if $F_n \uparrow F$ and F_n satisfies (4.3) then F does also. Further, observe that linear combinations of indicator functions of dyadic rectangles also satisfy (4.3) (see [2]). So the monotone class theorem gives the validity of (4.3) for all positive Borel functions.

The main result of this section can now be presented.

THEOREM 4.2. *Let* $f_t(x) = \sum_{n=1}^{\infty} a_n R_n(t) e^{inx}$. *If*

$$\int_0^1 \left(\sum_{n=1}^{\infty} a_n^2 \sin^2 \frac{n\delta}{2} \right)^{1/2} \frac{d\delta}{\delta\sqrt{\log(c/\delta)}} < \infty, \tag{4.4}$$

then

$$\int_0^1 \Omega_\Psi(\delta, f) \frac{d\delta}{\delta\sqrt{\log(c/\delta)}} < \infty, \tag{4.5}$$

where $\Psi(u) = \exp(u^2/4) - 1$. *Hence, almost surely*

$$\int_0^1 \omega_\Psi(\delta, f_t) \frac{d\delta}{\delta\sqrt{\log(c/\delta)}} < \infty$$

and f_t *is essentially continuous. In particular,* (4.4) *is true if* $\{a_n\}$ *satisfies the Salem-Zygmund condition* [7],

$$\sum_{k=1}^{\infty} \left(\sum_{n > 2^k} a_n^2 / k \right)^{1/2} < \infty. \tag{4.6}$$

Note. In [5] Marcus and Jain have also shown by entirely different methods that (4.4) is a sufficient condition for the a.s. uniform convergence of $f_t(x)$.

PROOF. Recall that for a random series $Y(t) = \sum_{n=1}^{\infty} c_n R_n(t)$ where $c_n = a_n + ib_n$, $\sum_{n=1}^{\infty} a_n^2 \leq \frac{1}{4}$ and $\sum_{n=1}^{\infty} b_n^2 \leq \frac{1}{4}$, that Khinchine's inequality is valid (see [2] or [8]): $m\{t: |Y(t)| \geq \lambda\} \leq 2\exp(-\lambda^2/2)$. Using this result, one calculates that $\int_0^1 (\exp(|Y(t)|^2/4) - 1)\, dt \leq 2$. This can be applied to

$$Y(t) = (f_t(x+\delta) - f_t(x))/4\left(\sum_{n=1}^{\infty} a_n^2 \sin^2 \frac{n\delta}{2} \right)^{1/2}.$$

This gives that

$$\int_0^1 \int_0^{2\pi} \exp\left[\frac{1}{4} \left[\frac{f_t(x+\delta) - f_t(x)}{4\left(\sum_{n=1}^{\infty} a_n^2 \sin^2(n\delta/2)\right)^{1/2}} \right]^2 \right] - 1\, dx\, dt \leq 4\pi.$$

So from the definition of $\Omega_\Psi(\delta, f)$, one has

$$\Omega_\Psi(\delta, f) \leq 4\left(\sum_{n=1}^{\infty} a_n^2 \sin^2 \frac{n\delta}{2} \right)^{1/2}.$$

So (4.5) follows immediately.

To show (4.6) implies (4.4) observe that

$$\left(\sum_{n=1}^{\infty} a_n^2 \sin^2 \frac{n\delta}{2}\right)^{1/2} \leq \frac{\delta}{2}\left(\sum_{n<1/\delta} a_n^2 n^2\right)^{1/2} + \left(\sum_{n>1/\delta} a_n^2\right)^{1/2}$$

So it is enough to show that (4.6) implies each of the following:

(a) $$\int_0^1 \left(\sum_{n<1/\delta} a_n^2 n^2\right)^{1/2} \frac{d\delta}{\sqrt{\log(c/\delta)}} < \infty;$$

(b) $$\int_0^1 \left(\sum_{n>1/\delta} a_n^2\right)^{1/2} \frac{d\delta}{\delta\sqrt{\log(c/\delta)}} < \infty.$$

Part (b) is verified by the following calculations:

$$\begin{aligned}
\int_0^1 \left(\sum_{n>1/\delta} a_n^2\right)^{1/2} \frac{d\delta}{\delta\sqrt{\log(c/\delta)}} &= \sum_{k=0}^{\infty} \int_{2^{-k-1}}^{2^{-k}} \left(\sum_{n>1/\delta} a_n^2\right)^{1/2} \frac{d\delta}{\delta\sqrt{\log(c/\delta)}} \\
&\leq \sum_{k=0}^{\infty} \int_{2^{-k-1}}^{2^{-k}} \left(\sum_{n>2^k} a_n^2\right)^{1/2} \frac{d\delta}{\delta\sqrt{\log(c/\delta)}} \\
&\leq \sum_{k=0}^{\infty} \left(\sum_{n>2^k} a_n^2\right)^{1/2} \frac{2^{k+1}}{\sqrt{\log c2^k}} \left(\frac{1}{2^k} - \frac{1}{2^{k+1}}\right) \\
&\leq \frac{1}{\sqrt{\log 2}} \sum_{k=1}^{\infty} \left(\frac{\sum_{n>2^k} a_n^2}{k}\right)^{1/2} + \left(\frac{\sum_{n=1}^{\infty} a_n^2}{\log c}\right)^{1/2}.
\end{aligned}$$

Hence (b) follows from (4.6). A similar but somewhat more delicate calculation also gives (a) as a consequence of (4.6) [2].

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