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PARAMETRIZATIONS OF TITCHMARSH'S $m(\lambda)$-FUNCTIONS IN THE LIMIT CIRCLE CASE[1]

BY

CHARLES T. FULTON

ABSTRACT. For limit-circle eigenvalue problems the so-called '$m(\lambda)$'-functions of Titchmarsh [15] are introduced in such a fashion that their parametrization is built into the definition.

1. **Introduction.** We consider the differential expression

$$\tau := -d^2/dx^2 + q(x) \quad \text{for } x \in [a, \infty) \tag{1.1}$$

with $-\infty < a < \infty$, and assume that $q(x)$ is real-valued and continuous in $[a, \infty)$ and belongs to the limit-circle case at ∞. Let L_0 and L_1 denote the minimal and maximal operators associated with τ defined as usual, cf. Dunford and Schwartz [2, p. 1291, Definition 8]. We consider the selfadjoint extensions of L_0 associated with boundary value problems of the form

$$\tau f = \lambda f \tag{1.2}$$

$$R_a^\alpha(f) := f(a)\cos\alpha + f'(a)\sin\alpha = 0, \qquad \alpha \in [0, \pi), \tag{1.3}$$

together with some suitably defined boundary condition at infinity. These self-adjoint extensions have been characterized in the literature by a variety of different brands of boundary conditions at the singular endpoint. In contrast to the boundary conditions originally given by H. Weyl in [18] and those given by M. H. Stone in [14], both of which depend on solutions of (1.2) for $\lambda = i$, or those used by K. Kodaira in [8], which depend on an element of the domain of L_1, E. C. Titchmarsh in his treatise on eigenfunction expansions [15] gave boundary conditions which depend on a certain function of the eigenvalue parameter, $m(\lambda)$. Introducing a fundamental system $\{\phi_\lambda, \theta_\lambda\}$ of (1.2) for each $\lambda \in \mathbf{C}$ ($\mathbf{C}$ = the complex numbers) by the initial conditions

$$\begin{pmatrix} \phi_\lambda(a) & \theta_\lambda(a) \\ \phi'_\lambda(a) & \theta'_\lambda(a) \end{pmatrix} = \begin{pmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{pmatrix}, \qquad \alpha \in [0, \pi), \tag{1.4}$$

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[1] This paper is based on the author's Ph.D. dissertation completed at Rheinisch-Westfälischen Technische Hochschule Aachen in 1973.

Titchmarsh formulates a boundary condition in terms of his $m(\lambda)$-function as follows (cf. [15, p. 31, equation 2.7.2]):

$$(1.5)(^2) \qquad \lim_{x\to\infty} W_x(\theta_\lambda + m(\lambda)\phi_\lambda, f) = 0 \quad \text{for all } \lambda, \operatorname{Im}\lambda \neq 0.$$

In the book of Titchmarsh attention is restricted to a single choice of $m(\lambda)$-function in (1.5), and there are no theorems giving a complete characterization of all $m(\lambda)$-functions admissible in (1.5). This contrasts with the books of Dunford and Schwartz [2] and M. H. Stone [14] where a strictly Hilbert-space approach is taken and the theory of selfadjoint extensions of symmetric operators is brought into play to give complete characterizations of the boundary conditions associated with 'all' selfadjoint extensions of L_0. The purpose of the present paper is to introduce a parametrization of the $m(\lambda)$-functions admissible in (1.5) in such a manner that (1.5) takes account of all boundary conditions at ∞ which complete (1.2) and (1.3) to a selfadjoint boundary value problem. This seems to provide a desirable connection between the function-theoretic methods of Titchmarsh, based on complex analysis, and the abstract operator-theoretic methods based on the theory of deficiency indices.

To be more precise, we introduce a real-valued fundamental system $\{u, v\}$ of (1.2) for $\lambda = 0$ satisfying

$$(1.6) \qquad W_x(u, v) = 1$$

and a vector-valued transformation S defined on $D(L_1)$ by

$$(1.7) \qquad (Sf)(x) := \begin{pmatrix} u(x) & v(x) \\ u'(x) & v'(x) \end{pmatrix}^{-1} \begin{pmatrix} f(x) \\ f'(x) \end{pmatrix} = \begin{pmatrix} W_x(f, v) \\ -W_x(f, u) \end{pmatrix},$$

for $x \in [a, \infty)$ and

$$(1.8)(^3) \qquad (Sf)(\infty) := \lim_{x\to\infty} (Sf)(x),$$

and show in this paper that the collection of all limit-circle $m(\lambda)$-functions admissible in (1.5) can be represented in the form

$$(1.9)(^4) \qquad m^{\alpha,\gamma}(\lambda) = -\frac{(S\theta_\lambda)_1(\infty)\cot\gamma + (S\theta_\lambda)_2(\infty)}{(S\phi_\lambda)_1(\infty)\cot\gamma + (S\phi_\lambda)_2(\infty)}, \qquad \gamma \in [0, \pi),$$

where $(Sf)_1(x)$ and $(Sf)_2(x)$ denote the first and second components of $(Sf)(x)$. It does not seem that such a parametrization has occurred in the

(2) $W_x(f, g)$ denotes the Wronskian of f and g at x.

(3) The existence of this limit follows from Green's formula since u and $v \in L_2(a, \infty)$.

(4) Changing the choice of fundamental system $\{u, v\}$ in (1.6) does not produce more $m(\lambda)$-functions, but only a reparametrization of them, cf. Fulton [3, p. 49, Corollary 4.1 and p. 61, Remark 5.8]. The title of this paper derives from the fact that the γ-parametrization in (1.9) depends on the choice of $\{u, v\}$.

previous literature under general limit-circle conditions on $q(x)$.

In fact the only paper which was found to contain a representation of the type

$$(1.10) \qquad m(\lambda) = -\frac{a(\lambda)\cot K + b(\lambda)}{c(\lambda)\cot K + d(\lambda)}, \qquad K \in [0, \pi),$$

with $a(\lambda)$, $b(\lambda)$, $c(\lambda)$, $d(\lambda)$ entire in λ was the 1950 paper of D. B. Sears and E. C. Titchmarsh, *Some eigenfunction formulae* [13, §2]([5]). For the interval $[0, \infty)$, and under additional special assumptions on $q(x)$ (to guarantee the occurrence of the limit-circle case and to permit application of the Liouville Transformation) a representation of the type (1.10) is obtained where the entire functions $a(\lambda)$, $b(\lambda)$, $c(\lambda)$, $d(\lambda)$ arise as certain coefficients in the asymptotic representation of $\phi_\lambda(x)$ and $\theta_\lambda(x)$ near ∞. The special methods used by Sears and Titchmarsh are therefore quite different from the approach we take in this paper. Also of significance is the fact that Sears and Titchmarsh do not prove that the $m(\lambda)$-functions of (1.10) exhaust the class of all $m(\lambda)$-functions permissible in (1.5). This conclusion remains elusive in the paper of Sears and Titchmarsh, as well as in all the special examples of Titchmarsh's book, because of the nature of the limiting process used to define the limit-circle $m(\lambda)$-functions. We will, however, be able to draw this conclusion about our parametrization (1.9) as soon as we establish the connection between (1.5) and the boundary conditions arising from the abstract theory of selfadjoint extensions of symmetric operators.

The key to obtaining the above parametrization (1.9) is an application of a general theorem on differential systems which bears no a priori relation to spectral theory, but which enables us to introduce the limit-circle $m(\lambda)$-functions in a manner quite different from that of Titchmarsh. This theorem has not, to the author's knowledge, been applied in this connection before. (See Theorem 1 below.)

In addition to its purely theoretical interest, formula (1.9) is also of some practical value. For problems involving special functions the four quantities in (1.9) are generally calculable and expressible in terms of the special functions arising as solutions of (1.2). Since the eigenvalues of the boundary value problem (1.2), (1.3), (1.5) are determined as the poles of $m^{\alpha,\gamma}(\lambda)$, they

([5]) Incidentally this paper was written to correct an erroneous theorem in the first edition of Titchmarsh's book [16, p. 108, Theorem 5.8], in which the occurrence of a continuous spectrum under certain special limit-circle conditions at ∞ is asserted. The theorem was corrected in the second edition of Titchmarsh's book [15, p. 125, Theorem 5.11] and a proof of the meromorphic character of the limit-circle $m(\lambda)$-functions (and hence the discreteness of limit-circle spectra) was also included, cf. [15, p. 125, Theorem 5.12]. The erroneous version of Titchmarsh's theorem (from the first edition) was recently (1969) repeated in E. Hille [6, p. 531, Theorem 10.3.9]. This fact was brought to the author's attention by Johann Walter.

are thus representable as the zeroes of certain special functions. For some cases involving Bessel functions the calculations have been carried out by the author in [3, p. 83, Examples 2 and 3]. Comparison with Titchmarsh's method of calculation [15, pp. 81–84] shows that (1.9) provides a considerably simpler formula for the calculation of the $m(\lambda)$-functions.

REMARK 1.1. The connection of (1.9) with the limit circles lies in the fact that when $\cot\gamma$ is replaced by a complex variable z, one has a linear fractional transformation mapping (for each λ, $\operatorname{Im}\lambda \neq 0$) the real z-axis onto the λ-limit circle, cf. Fulton [3, p. 52, Lemma 5.1]. This fact, however, will not be needed in the present paper.

2. A new definition of the limit-circle $m(\lambda)$-functions. Writing the equation

$$-f'' + qf = \lambda f \tag{2.1}$$

in the form

$$\frac{d}{dx}\begin{pmatrix} f \\ f' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -(\lambda - q) & 0 \end{pmatrix}\begin{pmatrix} f \\ f' \end{pmatrix} \tag{2.2}$$

we introduce the change of variable[6],

$$y = \begin{pmatrix} u & v \\ u' & v' \end{pmatrix}^{-1}\begin{pmatrix} f \\ f' \end{pmatrix} \quad (y = Sf), \tag{2.3}$$

where $\{u, v\}$ is as in (1.6). This yields the eigenvalue equation in the modified form

$$dy/dx = \lambda By \tag{2.4}$$

where

$$B(x) := \begin{bmatrix} u(x)v(x) & v^2(x) \\ -u^2(x) & -u(x)v(x) \end{bmatrix}.$$

The function $|B(x)|$ is integrable in $[a, \infty)$ because $u, v \in L_2(a, \infty)$. Hence the assumptions of Hartman [4, p. 273, Theorems 1.1 and 1.2] (in the linear case), or Coddington and Levinson [1, p. 99, Problem 6], are satisfied. It follows that solutions of (2.4) have limits at infinity and that solutions can be defined by specifying the limits. Denoting the solution of (2.4) defined by the "end conditions"

$$\lim_{x\to\infty} y_\lambda(x) = \begin{pmatrix} \sin\gamma \\ -\cos\gamma \end{pmatrix} \tag{2.5}$$

by $y_{\infty,\lambda}(x)$, we have

$$(y_{\infty,\lambda})_1(\infty)\cos\gamma + (y_{\infty,\lambda})_2(\infty)\sin\gamma = 0 \tag{2.6}$$

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[6] Compare Hartman [4, p. 330, equation 2.28].

much in analogy to the similar relation

$$R_a^\alpha(\phi_\lambda) = \phi_\lambda(a)\cos\alpha + \phi'_\lambda(a)\sin\alpha = 0 \tag{2.7}$$

at the regular endpoint.

For the sake of later reference we take note of the following algebraic identity, a consequence of (1.6):

$$W_x(f, g) = D_x(Sf, Sg) \quad \text{for } x \in [a, \infty) \text{ and } f, g \in D(L_1) \tag{2.8}$$

where

$$D_x(Sf, Sg) := \begin{vmatrix} (Sf)_1(x) & (Sg)_1(x) \\ (Sf)_2(x) & (Sg)_2(x) \end{vmatrix}. \tag{2.9}$$

Because of the existence (for $f, g \in D(L_1)$) of the limit in (1.8), (2.9) may be used to define $D_\infty(Sf, Sg)$. Taking the limit on both sides in (2.8) we then have

(2.10)(7) $$W_\infty(f, g) = D_\infty(Sf, Sg) \quad \text{for } f, g \in D(L_1).$$

Application of the aforementioned theorem to (2.4) yields:

THEOREM 1. *Let $q(x)$ be continuous in $[a, \infty)$ and belong to the limit-circle case at ∞. Let $\lambda \in \mathbf{C}$. Then*:

(i) *If $y_\lambda(x)$ is a solution of (2.4), the limit $y_\lambda(\infty) := \lim_{x\to\infty} y_\lambda(x)$ exists. Moreover, for all $x, x_0 \in [a, \infty)$ we have*

$$\begin{aligned} &|y_\lambda(x_0)|\exp\left\{-|\lambda|\left|\int_{x_0}^x u^2 + v^2\, ds\right|\right\} \\ &\qquad \leqslant |y_\lambda(x)| \leqslant |y_\lambda(x_0)|\exp\left\{|\lambda|\left|\int_{x_0}^x u^2 + v^2\, ds\right|\right\} \end{aligned} \tag{2.11}$$

and

$$|y_\lambda(x) - y_\lambda(\infty)| \leqslant |y_\lambda(x_0)|\,|\lambda|\, M(x)\exp\{|\lambda|\, M(x_0)\} \tag{2.12}$$

where

$$M(x) := \int_x^\infty u^2 + v^2\, ds$$

and

$$|y_\lambda(x)| := \sqrt{|(y_\lambda)_1(x)|^2 + |(y_\lambda)_2(x)|^2}\,.$$

(ii) *Let $z_1, z_2 \in \mathbf{C}$. Then to each $\lambda \in \mathbf{C}$ there exists a unique solution $y_\lambda(x)$ of* (2.4) *such that*

(7) Compare K. Kodaira [8, p. 924, equation 1.5].

$$\lim_{x\to\infty} y_\lambda(x) = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

Note: For $\lambda = 0$ the solutions of (2.4) are constant vectors and we have equality in (2.11) and (2.12).

Theorem 1 (ii) justifies the definition of $y_{\infty,\lambda}(x)$ in (2.5). We denote the solution of (2.1) corresponding via (2.3) to $y_{\infty,\lambda}(x)$ by $\chi_\lambda^\gamma(x)$. From (2.10) and (2.5) it follows that

$$W_\infty(\chi_\lambda^{\gamma_1}, \chi_{\lambda'}^{\gamma_2}) = \sin(\gamma_2 - \gamma_1) \quad \text{for } \lambda, \lambda' \in \mathbf{C}. \tag{2.13}$$

Using $\chi_\lambda(x)$ we define

$$\text{(2.14)} \qquad \begin{aligned} &\text{(i)} \quad w^{\alpha,\gamma}(\lambda) := W_x(\phi_\lambda, \chi_\lambda), \quad x \in [a, \infty), \quad \text{and} \\ &\text{(ii)} \quad p^{\alpha,\gamma}(\lambda) := W_x(\chi_\lambda, \theta_\lambda), \quad x \in [a, \infty). \end{aligned}$$

The analyticity in λ of $(S\phi_\lambda)(\infty)$ and $(S\theta_\lambda)(\infty)$ (and hence the analyticity of $w(\lambda)$ and $\rho(\lambda)$) can be deduced from that of $\phi_\lambda(x)$ and $\theta_\lambda(x)$ by applying (2.12) and the Weierstrass theorem. The fact that $\chi_\lambda(x)$ is entire in λ for $x \in [a, \infty)$ then follows from the algebraic relation

$$\chi_\lambda(x) = w^{\alpha,\gamma}(\lambda)\theta_\lambda(x) + p^{\alpha,\gamma}(\lambda)\phi_\lambda(x). \tag{2.15}$$

Also, using the same type of argument as in the regular case (Titchmarsh [15, pp. 11–12]) it can be shown that $w^{\alpha,\gamma}(\lambda)$ and $p^{\alpha,\gamma}(\lambda)$ have only real, simple zeroes.

We define our limit-circle $m(\lambda)$-functions by putting

$$\text{(2.16)(}^8\text{)} \qquad m^{\alpha,\gamma}(\lambda) := p^{\alpha,\gamma}(\lambda)/w^{\alpha,\gamma}(\lambda).$$

It follows that $m^{\alpha,\gamma}(\lambda)$ is a meromorphic function of λ with only simple poles on the real λ-axis. Replacing Titchmarsh's limit-circle $m(\lambda)$-function by the above function, the discussion in [15, pp. 28–41] then applies without change to give the spectral theory associated with the boundary value problem (1.2), (1.3), (1.5). We make only a few observations:

1. It follows from (2.15) that

$$\theta_\lambda(x) + m^{\alpha,\gamma}(\lambda)\phi_\lambda(x) = \chi_\lambda(x)/w^{\alpha,\gamma}(\lambda) \tag{2.17}$$

for all λ not zeroes of $w^{\alpha,\gamma}(\lambda)$. Titchmarsh's Lemma 2.3 [15, p. 26] (on which the rest of his discussion hinges) therefore follows from (2.13) with $\gamma_1 = \gamma_2$, since $w^{\alpha,\gamma}(\lambda)$ has only real zeroes.

2. Putting (2.16) in (1.5) and using (2.17) and (2.5) it is readily seen that (1.5) is equivalent to the boundary condition

(8) Formula (1.9) follows by putting $x = \infty$ in (2.14)(i) and (ii) and applying (2.10) and (2.5).

$$(2.18)(^9) \qquad \begin{aligned} W_\infty(\chi_\lambda, f) &= D_\infty(S\chi_\lambda, Sf) \\ &= (Sf)_1(\infty)\cos\gamma + (Sf)_2(\infty)\sin\gamma = 0. \end{aligned}$$

But with $\alpha \in [0, \pi)$ and $\gamma \in [0, \pi)$, the α- and γ-parametrizations in (1.3) and (2.18) take account of all possible (separated) symmetric boundary conditions(^{10}). For this reason we may conclude that the γ-parametrization of our $m(\lambda)$-functions given by (2.16) (and (1.9)) takes account of all limit-circle $m(\lambda)$-functions which can be used in (1.5) to yield a selfadjoint boundary value problem (1.2), (1.3), (1.5). This establishes the connection, referred to in the introduction, between Titchmarsh's boundary conditions and those arising out of the more abstract operator-theoretic approach taken in the book of Dunford and Schwartz(^{11}). Thus if $L^{\alpha,\gamma}$ is the restriction of L_1 defined by imposing the boundary conditions (1.3) and (2.18), it follows from the equivalence of (2.18) and (1.5) that the theory of Chapter 2 of Titchmarsh's book applies to give the spectral theory and expansion theorem associated with $L^{\alpha,\gamma}$. The eigenvalues of $L^{\alpha,\gamma}$ are therefore the poles of $m^{\alpha,\gamma}(\lambda)$ and hence the zeroes of our function, $w^{\alpha,\gamma}(\lambda)$.

3. The Green's function associated with the boundary value problem (1.2), (1.3), (2.18) may be written in the form

$$(2.19) \qquad G^{\alpha,\gamma}(x, y, \lambda) = \begin{cases} \chi_\lambda(x)\phi_\lambda(y)/w^{\alpha,\gamma}(\lambda), & a \leqslant y \leqslant x < \infty, \\ \phi_\lambda(x)\chi_\lambda(y)/w^{\alpha,\gamma}(\lambda), & a \leqslant x \leqslant y < \infty. \end{cases}$$

The fact that

$$(2.20) \qquad \Phi_\lambda(x; f) := \int_a^\infty G^{\alpha,\gamma}(x, y, \lambda) f(y)\, dy \quad (= R(\lambda; L^{\alpha,\gamma}))$$

satisfies (for $f \in L_2(a, \infty)$) the boundary condition (2.18) can then be readily demonstrated by applying S to $\Phi_\lambda(x; f)$ and making use of (2.5). Using [15, p.

(^{9}) Boundary conditions at a singular limit-circle endpoint were first written in this form by F. Rellich in [10, p. 354, equation 9]. For $f \in D(L_1)$ the (complex) numbers $(Sf)_1(\infty)$ and $(Sf)_2(\infty)$ are the so-called Rellich Initial Numbers for f at ∞ 'with respect to $\lambda = 0$, $u(x)$ and $v(x)$'. Compare also K. Jörgens [7, p. 9.10, Corollary to Theorem 4]. In the case when L_0 is bounded below (which may happen only at a finite limit-circle endpoint), we may, with Rellich, take $v(x)$, $u(x)$ to be principal and nonprincipal solutions at the limit-circle end. Rellich's characterization of the Friedrich's extension then corresponds to putting $\gamma = 0$.

(^{10}) This follows from Dunford and Schwartz [2, pp. 1306–1309, Theorem 30 and Corollary 31, Case (i)] since $(S(\cdot))_1(\infty)$ and $(S(\cdot))_2(\infty)$ are real linearly independent 'boundary values for τ at ∞' (cf. [2, p. 1297, Definition 17, and p. 1302, Theorem 27]), and $b_1(f) := f(a)$ and $b_2(f) := f'(a)$ are likewise real linearly independent 'boundary values for τ at a' (cf. [2, p. 1301, Corollary 23]). These results of Dunford and Schwartz rely, of course, on the abstract theory of deficiency indices [2, p. 1238, Theorem 30], which is not exploited in Titchmarsh's book.

(^{11}) In [3] we have given a more detailed discussion of the equivalence of various types of boundary conditions, based on the abstract notion of boundary values. The boundary conditions of H. Weyl [18], M. H. Stone [14], F. Rellich [10], Titchmarsh [15], K. Kodaira [8], and Coddington and Levinson [1], are discussed and compared there.

29, equation 2.6.1] and Titchmarsh's definition of $m(\lambda)$, the proof that $\Phi_\lambda(x; f)$ satisfies the boundary condition (1.5) is, on the other hand, somewhat more complicated and is actually not contained in his book, cf. [15, pp. 29–30].

4. Using (2.19) the residue of the Resolvent Operator can be readily calculated yielding

$$\text{(2.21)}(^{12}) \qquad \operatorname*{Res}_{\lambda=\lambda_n} R(\lambda; L^{\alpha,\gamma}) = \frac{k_n}{w'(\lambda_n)} \phi_{\lambda_n}(x) \int_a^\infty \phi_{\lambda_n}(y) f(y)\, dy$$

where λ_n is a zero of $w^{\alpha,\gamma}(\lambda)$ and k_n is the real number defined by

$$\text{(2.22)} \qquad \chi_{\lambda_n}(x) =: k_n \phi_{\lambda_n}(x).$$

Using (2.21) the expansion formula (cf. [15, p. 31, equation 2.7.3]) assumes the form

$$\text{(2.23)} \qquad f(x) = \sum_{n=1}^{\infty} \frac{k_n}{w'(\lambda_n)} \phi_{\lambda_n}(x) \int_a^\infty \phi_{\lambda_n}(y) f(y)\, dy$$

in strong analogy to the regular case, cf. [15, p. 8, equation 1.6.5].Because of (2.21) Titchmarsh's contour integration in [15, §2.12 or §2.15], giving the proof of the expansion theorem, may be carried out using only the function $w^{\alpha,\gamma}(\lambda)$, so that there is actually no need to introduce the function $m^{\alpha,\gamma}(\lambda)$. This brings the spectral theory of limit-circle eigenvalue problems more in line with the analogous theory of regular Sturm-Liouville eigenvalue problems on a finite closed interval.

In view of the fact that our approach to the limit-circle theory differs from well-known methods of approach, a few remarks are in order:

REMARK 2.1. The fact that our definition (2.16) is in agreement with Titchmarsh's definition of the limit-circle $m(\lambda)$-functions(13) can be shown as follows:

For fixed $b \in (0, \infty)$

$$\text{(2.24)} \quad l_b^\beta(\lambda) := -\frac{\theta_\lambda(b)\cot\beta + \theta'_\lambda(b)}{\phi_\lambda(b)\cot\beta + \phi'_\lambda(b)} = -\frac{(S\theta_\lambda)_1(b)\cot\gamma + (S\theta_\lambda)_2(b)}{(S\phi_\lambda)_1(b)\cot\gamma + (S\phi_\lambda)_2(b)}$$

for all λ, $\operatorname{Im}\lambda \neq 0$, if and only if

$$\text{(2.25)} \quad \beta = \operatorname{Arc}\cot\left(\frac{-v'(b)\cos\gamma + u'(b)\sin\gamma}{v(b)\cos\gamma - u(b)\sin\gamma}\right), \qquad \gamma \in [0, \pi)(^{14}).$$

(12) Compare Titchmarsh [15, p. 30, last line].

(13) Titchmarsh [17, p. 40]; K. Yosida [19, pp. 171–172]; Coddington and Levinson [1, p. 242, Theorem 4.1] or D. B. Sears [12, p. 51].

(14) We need the convention $\operatorname{Arc}\cot(\pm\infty) = 0$ here to get a one-to-one correspondence between γ and $\beta \in [0, \pi)$.

It follows that if (b_n, β_n) is a sequence for which the limit of $l_{b_n}^{\beta_n}(\lambda)$ exists (existence for one λ_0, $\operatorname{Im} \lambda_0 \neq 0$, implies existence for all λ, $\operatorname{Im} \lambda \neq 0$), we have

$$(2.26) \quad \lim_{n\to\infty} l_{b_n}^{\beta_n}(\lambda) = -\frac{(S\theta_\lambda)_1(\infty) \cot \gamma^* + (S\theta_\lambda)_2(\infty)}{(S\phi_\lambda)_1(\infty) \cot \gamma^* + (S\phi_\lambda)_2(\infty)} = m^{\alpha,\gamma^*}(\lambda)$$

where

$$(2.27) \qquad \gamma^* := \operatorname{Arc} \cot\left(\lim_{n\to\infty} \frac{u(b_n) \cos \beta_n + u'(b_n) \sin \beta_n}{v(b_n) \cos \beta_n + v'(b_n) \sin \beta_n} \right).$$

For more details Fulton [3, pp. 67–79]. The existence of the limit in (2.27) (with finite or infinite value) is necessary and sufficient for the sequence (b_n, β_n) to be admissible in Titchmarsh's definition of $m(\lambda)$. If functions $\beta(b)$ are used instead of sequences, cf. Titchmarsh [15, p. 26, lines 6–8] or Sears and Titchmarsh [13, §2], there is a similar formula for γ^*.

Remark 2.2. In his well-known paper [8], K. Kodaira defines a limit-circle boundary condition at ∞, and a 'characteristic function $m(\lambda)$', by taking a fixed $w(x) \in D(L_1)$(15) and putting

$$(2.28) \qquad R_\infty(f) := W_\infty(w, f) = 0,$$

$$(2.29) \qquad m(\lambda) := -W_\infty(w, \theta_\lambda)/W_\infty(w, \phi_\lambda).$$

It can be shown that for

$$(2.30) \qquad \gamma^* = \operatorname{Arc} \cot(-(Sw)_2(\infty)/(Sw)_1(\infty))$$

we get

$$(2.31) \qquad m(\lambda) = m^{\alpha,\gamma^*}(\lambda)$$

and that the boundary condition (2.28) is equivalent to (2.18) with this value of γ, cf. Fulton [3, pp. 43–46]. Despite the arbitrariness of his boundary condition, Kodaira does not make any statement to the effect that 'all' symmetric boundary conditions have been accounted for. This conclusion seems to require the abstract theory of deficiency indices, which is also absent from Kodaira's paper.

Remark 2.3. M. H. Stone was the first to give a parametrization of the boundary conditions at a limit-circle endpoint together with a proof that all symmetric possibilities are accounted for, cf. Stone [14, p. 475, Theorem 10.17]. Stone's boundary conditions correspond to putting

$$(2.32) \qquad w(x) = \frac{\exp(i\theta/2)\phi_i(x) - \exp(-i\theta/2)\overline{\phi_i(x)}}{2i}$$

(15) Kodaira's nontriviality condition [8, p. 924, equation (1.6)] guarantees that $(Sw)_1(\infty)$ and $(Sw)_2(\infty)$ are not both zero in (2.30).

with $\theta \in [0, 2\pi)$ in (2.28). The connection between our parameter γ and Stone's θ-parameter is therefore obtainable from (2.30), cf. Fulton [3, pp. 92–96].

REMARK 2.4. The representation (2.19) for the Green's function differs from the similar representation given by Dunford and Schwartz [2, p. 1329, Theorem 16] in that both numerator and denominator are known to be entire in λ. The solution of (1.2) occuring in the Dunford-Schwartz representation which corresponds to our solution $\chi_\lambda(x)$ is defined merely by specifying that it satisfy (for $\operatorname{Im}\lambda \neq 0$) the boundary condition at ∞, and its λ-dependence can therefore be rather arbitrary. (The existence of a solution of (1.2) satisfying a given boundary condition at ∞, which is infinitely differentiable in λ for $\lambda \in (-\infty, \infty)$, follows from Dunford and Schwartz [2, p. 1472, Lemma 42] (which applies with $\lambda_0 = \infty$ when the limit-circle case occurs at both endpoints), but it does not seem that this lemma establishes analyticity in λ.) Because of this arbitrariness in λ, the computation of the residue of the Resolvent Operator as in (2.21) and the corresponding contour integration proof of the expansion theorem cannot be carried out. Since, however, the spectral theory and expansion theorem for problems of the type (1.2), (1.3), (2.18) is accomplished in the book of Dunford and Schwartz by appealing to the Compactness of the Resolvent Operator (cf. [2, p. 1331, Theorem 2 ($n = 2$)]), there is actually no need to know the meromorphic character of either the Green's function or the Resolvent Operator(16). Compare also Stone [14, p. 484, Theorem 10.19] where the Hilbert-Schmidt theory is also used to obtain the expansion theorem.

REMARK 2.5. For problems which are singular and limit-circle at both endpoints the theory can be carried out without essential change, yielding the expansion formula in the form (2.23). One has only to make use of 'end conditions' of the type (2.5) at both endpoints. This contrasts with Titchmarsh's discussion in [15, pp. 42–43] where the basic interval is broken into two half-open intervals and two $m(\lambda)$-functions must be introduced to define the Green's function, cf. [15, p. 42, equation 2.18.4]. For problems singular at both endpoints, Kodaira, however, introduces a simplification by reducing his 'characteristic matrix' to the so-called 'normal form' [8, pp. 934–937], and his form of the expansion formula [8, p. 937, equation (4.10)] may be compared to (2.23). For problems limit-circle at both endpoints Kodaira's 'normal form'

(16) The meromorphic character of the limit-circle $m(\lambda)$ functions, the associated Green's function, and the Resolvent Operator may be deduced by specializing the Dunford-Schwartz version of the general case of the Titchmarsh-Kodaira theory (cf. [2, p. 1364, Theorem 18]) to the case with one regular and one limit-circle endpoint which we are considering. As far as characterizing the class of all limit-circle $m(\lambda)$-functions, however, the theory of the general case [2, Chapter 13, §5] is not helpful; here only the existence of a meromorphic $m(\lambda)$-function can be concluded once a selfadjoint extension is chosen.

is nevertheless not uniquely defined and actually depends on the choice of fundamental system used in the reduction. This arises from the fact that Kodaira uses a boundary condition of the type (2.28) to define one of the solutions of (1.2) to be used in the reduction to normal form, cf. [8, p. 934]. A boundary condition does not suffice to guarantee any kind of smoothness with respect to λ and even when the analyticity in λ is imposed as an additional condition there actually remains some leeway in the choice of his fundamental system. This difficulty can be avoided by employing a fundamental system defined via (2.4) by 'end conditions' of the type (2.5) in the reduction to normal form. The end conditions (2.5) fix the λ-dependence of the solutions so defined, guaranteeing in particular the analyticity in λ, and evidently serve the same purpose at a singular endpoint as the initial conditions (1.4) do at a regular endpoint. Our approach, of course, using two solutions defined by end conditions at either endpoint actually obviates the need for Kodaira's 'normal form' altogether.

A CONCLUDING REMARK. In conclusion it can be said that there is a rather close kinship between Sturm-Liouville eigenvalue problems on a finite closed interval and singular limit-circle problems. For this reason it would seem to be desirable to abandon the simultaneous treatment of the limit-circle case and the 'discrete' limit-point case as it appears in Chapter 2 of Titchmarsh's book in favour of a separate treatment of the regular case and the limit-circle case on the one hand, and the limit-point case on the other hand. Our approach to the theory of the limit-circle case evidently lends support to the following observation made by H. Weyl in his original 1910 paper [18, p. 230]:

> "Da sich Gleichungen (vom Grenzkreistypus) in jeder Hinsicht wie Gleichungen ohne Singularitäten verhalten, hat man danach den Grenzkreisfall als den regulären aufzufassen."

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(**Added in Proof.** Since this paper was submitted a revised version of reference [11] with improvements by K. Jörgens and J. Weidmann has been published: K. Jörgens and F. Rellich, *Eigenwerttheorie gewöhnlicher Differentialgleichungen*, Springer-Verlag, Berlin and New York, 1976. While the previous manuscripts of Rellich [11] and Jörgens [7] did not contain results comparable to Theorem 1 above, the new Jörgens-Rellich manuscript does contain the integral equation equivalent of our 'terminal value problem' (2.4), (2.5), cf. Chapter III, §6. The proof of the uniqueness of solutions and their analyticity in λ is accomplished, however, by direct consideration of a Taylor expansion in λ, and is thus independent of ours, the estimates (2.11) and (2.12) of Theorem 1 not being exploited. Since the new Jörgens-Rellich manuscript again makes use of Rellich's 'Initial Numbers' to give a parametrization of the boundary conditions at a limit-circle endpoint we wish to comment on a point of pedagogics: The conclusion that 'all' (separated) symmetric boundary conditions are accounted for is most easily made merely by interpreting the Rellich Initial Numbers as 'boundary values for τ' in the sense of Dunford and Schwartz, cf. footnote (10) above, or Fulton [3, pp. 96–98]. The argument on this point in the Jörgens-Rellich manuscript [pp. 134–35 and pp. 138–141], as well as the previous arguments of Rellich [11, pp. 60–63] and Jörgens [7, pp. 9.9–9.11], is, on the other hand more complicated, and incidentally makes no appeal to the general results of J. von Neumann's Deficiency Index Theory.)

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DEPARTMENT OF MATHEMATICS, PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PENNSYLVANIA 16802