Finiteness in the Minimal Models of Sullivan

By

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Abstract. Let X be a 1-connected topological space such that the vector spaces $\Pi_\ast(X) \otimes \mathbb{Q}$ and $H^\ast(X; \mathbb{Q})$ are finite dimensional. Then $H^\ast(X; \mathbb{Q})$ satisfies Poincaré duality. Set $\chi_\Pi = \sum (-1)^d \dim \Pi_\ast(X) \otimes \mathbb{Q}$ and $\chi_\ast = \sum (-1)^d \dim H^\ast(X; \mathbb{Q})$. Then $\chi_\Pi < 0$ and $\chi_\ast > 0$. Moreover the conditions:

(1) $\chi_\Pi = 0$, (2) $\chi_\ast > 0$, $H^\ast(X; \mathbb{Q})$ evenly graded, are equivalent. In this case $H^\ast(X; \mathbb{Q})$ is a polynomial algebra truncated by a Borel ideal.

Finally, if $X$ is a finite 1-connected C.W. complex, and an $r$-torus acts continuously on $X$ with only finite isotropy, then $\chi_\Pi < -r$.

1. Introduction. In this paper all vector spaces are defined over a field, $\Gamma$, of characteristic zero. We shall consider positively graded finite dimensional vector spaces $R = \bigoplus_{k \geq 0} R^k$ ($R^k$ is the subspace of elements of degree $k$) with homogeneous bases $x_1, \ldots, x_n$. The free commutative algebra over $R$ is written $F(R)$ or $F(x_1, \ldots, x_n)$. $[F(R)]^k$ denotes the subspace spanned by elements of the form $x_1 \cdots x_k$ with $\sum \deg x_i = k$. Such elements are called homogeneous of degree $k$.

Write $R = Q \oplus P$ where $Q$ (respectively $P$) is the space spanned by the elements of even (respectively odd) degree. Then $F(R) = \bigvee Q \otimes \bigwedge P$ is the tensor product of the symmetric algebra $\bigvee Q$ over $Q$ with the exterior algebra $\bigwedge P$ over $P$. We can also write $F(R) = F(x_1) \otimes \cdots \otimes F(x_n)$.

Now suppose $(A, d_A)$ is a graded commutative differential algebra (positively graded, associative, with identity $1 \in A^0$) and suppose $\tau : R \to A \otimes F(R)$ is a linear map, homogeneous of degree 1. Then $\tau$ extends to a unique derivation, $d_{\tau}$, of degree 1 in $A \otimes F(R)$ such that $d_{\tau}(a \otimes 1) = 0$. Extend $d_{\tau}$ to $A \otimes F(R)$ by writing $d_{\tau}(a \otimes z) = d_A a \otimes z$.

Definition. $(A, d_A; \tau; x_1, \ldots, x_n)$ will be called a finite tower over $A$ if

(1) $\tau(x_1) \in A$, $\tau(x_i) \in A \otimes F(x_1, \ldots, x_{i-1})$ $(i > 2)$

and

(2) $(d_{\tau} + d_A)^2 = 0$. 

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The graded differential algebra \((A \otimes F(R), d_t + d_a)\) is called the Koszul complex of the tower, and the cohomology algebra \(H(A \otimes F(R))\) is called the cohomology of the tower.

If \(A = \Gamma\) then \((\tau; x_1, \ldots, x_n)\) will be called simply a finite tower. (In this case \(\tau(x_i) = 0\).) If \(\deg x_i > 0\) (respectively \(> k\)) for all \(i\) then the tower is called connected (respectively \(k\)-connected).

Let \((F(R), d_t)\) be the Koszul complex of a finite tower. The number

\[
\chi_\Pi = \sum_k (-1)^k \dim R^k
\]

is called the homotopy Euler characteristic of the tower. If \(\dim H(F(R), d_t) < \infty\) the tower is called \(c\)-finite; in this case

\[
\chi_c = \sum_k (-1)^k \dim H^k(F(R))
\]

is called the cohomology Euler characteristic. Finally, if

\[
\tau(x_i) \in F^+(x_1, \ldots, x_{i-1}) \cdot F^+(x_1, \ldots, x_{i-1}), \quad i = 2, 3, \ldots,
\]

then the tower is called minimal.

Among the principal results of this paper is the following:

**Theorem 1.** Let \((\tau; x_1, \ldots, x_n)\) be a connected, finite, \(c\)-finite minimal tower. Then \(\chi_\Pi < 0\) and \(\chi_c > 0\). Moreover, the following conditions are equivalent:

1. \(\chi_\Pi = 0\).
2. \(\chi_c > 0\).
3. \(H(F(x_1, \ldots, x_n))\) is evenly graded.

(In fact we shall show that for each \(p\), \(\sum_{i>p} (-1)^i \dim R^i < 0\) (Corollary 2 in §6) where \(R^i\) is the span of the \(x_j\) with \(\deg x_j = i\).)

The proof of Theorem 1 is contained in the next six sections. Then, in §8, we show that under the hypotheses of Theorem 1, \(H(F(x_1, \ldots, x_n))\) satisfies Poincaré duality, and that the degree \(m\) of the top dimensional cohomology class is given by

\[
m = r - \sum_{i=1}^n (-1)^{\deg x_i} \deg x_i,
\]

where \(r\) is the number of \(x_i\) of even degree.

In §§9 and 10, we show that if \(m\) is even and \(\chi_\Pi < 0\), then the Poincaré inner product in \(\sum_i H^2(F(x_1, \ldots, x_n))\) is hyperbolic. Finally in §11, we show that if \(\chi_\Pi = 0\) then \((F(x_1, \ldots, x_n), d_t)\) is isomorphic with a Koszul complex of the form \((\sqrt{Q} \otimes \wedge P, d)\) with \(d(Q) = 0\) and \(d(P) \subset \sqrt{Q}\). In this case \(H(\sqrt{Q} \otimes \wedge P) \cong \sqrt{Q}/I\), where \(I\) is the ideal generated by \(d(P)\).
Now consider a connected topological space $X$ and let $A(X)$ be the graded commutative differential algebra of rational differential forms on the singular complex of $X$ (cf. Sullivan [5, §D]): in particular, $H(A(X)) \cong H^*(X; \mathbb{Q})$ (singular cohomology). There is a commutative connected graded differential algebra $(F(R), d)$ (over $\mathbb{Q}$) and a homomorphism $\phi: F(R) \to A(X)$ of graded differential algebras such that

1. $\phi$ induces an isomorphism of cohomology.
2. There is a homogeneous basis $\{x_\alpha\}_{\alpha \in \mathbb{R}}$ of $F(R)$, where $\mathbb{R}$ is well ordered, such that $dx_\beta$ is a polynomial in those $x_\alpha$ with $\beta < \alpha$ and $\deg x_\beta < \deg x_\alpha$. Moreover, $(F(R), d)$ is determined up to isomorphism by these conditions.

We shall call the spaces $R^k$ the pseudo dual rational homotopy spaces of $X$, and denote them by $\Pi^k_*(X)$. If $H^1(X; \mathbb{Q}) = 0$ and $H^*(X; \mathbb{Q})$ has finite type, then these spaces are finite dimensional. If, in addition, $X$ is simply connected then there are natural isomorphisms [5, §2] $\pi_k(X) \otimes \mathbb{Q} \cong \Pi^k_*(X)$.

Write $\Pi^k_*(X) = \sum_k \Pi^k_k(X)$ and $\Pi_k(X) = \sum_k \Pi_k(X)$. Then the remarks above, together with Theorem 1, yield:

**Theorem 1'.** Let $X$ be a connected topological space such that $\Pi^k_*(X)$ and $H^*(X; \mathbb{Q})$ are finite dimensional. Then

$$\sum_k (-1)^k \dim \Pi^k_*(X) < 0 \quad \text{and} \quad \sum_k (-1)^k \dim H^k(X; \mathbb{Q}) > 0.$$

Moreover, the following conditions are equivalent:

1. $\sum_k (-1)^k \dim \Pi^k_*(X) = 0$.
2. $\sum_k (-1)^k \dim H^k(X; \mathbb{Q}) > 0$.
3. $H^p(X; \mathbb{Q}) = 0$, $p$ odd.

**Corollary 1.** If $X$ is simply connected then the theorem remains true if $\Pi^k_*(X)$ is replaced by $\Pi_k(X) \otimes \mathbb{Q}$ everywhere in the statement.

Finally, we have the following application to transformation groups (see Remark 3 below):

**Theorem T.** Let a compact Lie group $G$ of rank $r$ act on a simply connected finite C.W. complex $X$ with only finite isotropy. Assume that $\Pi_*(X) \otimes \mathbb{Q}$ is finite dimensional. Then $\sum (-1)^k \dim \Pi_k(X) \otimes \mathbb{Q} \leq -r$.

**Proof.** According to Allday [1, Theorem 2.1.1, p. 177] this follows from Theorem 1'(1).

**Remarks.** 1. The special case of a finite tower over $A$ with $R$ oddly graded and $\tau(R) \subset A$ was first considered by Koszul [4] in 1950. Cartan [2] showed that the cohomology of a homogeneous space can be calculated via a Koszul complex of this form, where, in addition, $A$ is a symmetric algebra and $d_A = 0$. Cartan also obtains a special case of Theorem 1; indeed the general theorem will be established by reduction to this earlier result.
The Koszul complex of a minimal connected tower is a nilpotent minimal model as defined by Sullivan [5].

2. Historically, this paper begins with Theorem T which was conjectured by W. Y. Hsiang in 1969 or earlier. Then in 1971 Allday [1] reduced Hsiang's conjecture to Theorem 1' (\(\chi_{\Pi} \leq 0\)) in the simply connected case. The translation from Theorem 1' (1) to Theorem 1(1) was observed by Sullivan who poses it as question 5 in [5, §Q].

3. Theorem T remains valid for a much wider class of spaces, \(X\). In particular it is sufficient to assume that \(X\) is connected (but not necessarily simply connected) if we replace \(\Pi_\bullet(X) \otimes \mathbb{Q}\) by \(\Pi'_\bullet(X)\) everywhere in the statement. Precise statements and details of the proof will appear elsewhere. As a special case of this generalized Theorem T, however, we have

**Theorem H.** Let \(K \subset G\) be compact Lie groups and suppose a torus, \(T\), acts on \(G/K\) continuously, with only finite isotropy. Then \(\dim T \leq \text{rank } G - \text{rank } K\).

When \(K = (e)\) this is proved by Allday [1]. If \(G/K\) is 1-connected then Theorem H follows from the "ungeneralized" Theorem T. 

2. **Notation.** By a graded commutative differential algebra \((A, d_A)\) we mean a positively graded associative algebra \(A = \sum_{k \geq 0} A^k\) with identity \(1 \in A^0\) such that \(ab = (-1)^{rs}ba\), \(a \in A^r\), \(b \in A^s\). Here \(d_A\) denotes a derivation of degree 1 with \(d_A^2 = 0\). The cohomology algebra \(\ker d_A/\text{Im } d_A\) is written \(H(A) = \sum_k H^k(A)\). A homomorphism \(\phi: (A, d_A) \rightarrow (B, d_B)\) of graded differential algebra induces a homomorphism \(\phi^*: H(A) \rightarrow H(B)\).

The tensor product of graded algebras \(A\) and \(B\) is given the multiplication defined by \((a \otimes b)(a' \otimes b') = (-1)^{w}aa' \otimes bb'\), \(b \in B^q\), \(a' \in A^{q'}\).

The subspace of a vector space spanned by elements \(u_1, \ldots\) is denoted by \((u_1, \ldots)\). If \(U\) and \(V\) are subspaces of a vector space \(W\), \(U + V\) is the subspace spanned by \(U\) and \(V\). If \(W\) is an algebra, \(U \cdot V\) is the subspace spanned by elements of the form \(uv, u \in U, v \in V\); \(U \cdot U\) is written \(U^2\).

An evenly (respectively oddly) graded space is a space with no nonzero elements of odd (respectively even) degree.

The identity map of any set is denoted by \(\iota\).

Let \(R = (x_1, \ldots, x_n)\) be as in the introduction, and suppose \((A, d_A; \tau; x_1, \ldots, x_n)\) is a tower over \(A\) with Koszul complex \((A \otimes F(R), d)\). Then for each \(m\), \((A, d_A; \tau; x_1, \ldots, x_m)\) is a tower over \(A\) with Koszul complex the subdifferential algebra \((A \otimes F(x_1, \ldots, x_m), d)\). Write this \((B, d_B)\).

Then \(A \otimes F(R) = B \otimes F(x_{m+1}, \ldots, x_n)\) and so we may regard \(\tau\) as a linear map \(\tau: (x_{m+1}, \ldots, x_n) \rightarrow B \otimes F(x_{m+1}, \ldots, x_n)\). Clearly \((B, d_B; \tau; x_{m+1}, \ldots, x_n)\) is a tower over \(B\) whose Koszul complex coincides
with the Koszul complex \((A \otimes F(R), d)\).

Next, let \((\tau; x_1, \ldots, x_n)\) be a finite tower and denote by \((B, d_B)\) the subdifferential algebra \(F(x_1, \ldots, x_m)\) of \((F(R), d)\). Then, as above, \((F(R), d)\) is also the Koszul complex of the tower \((B, d_B; \tau; x_{m+1}, \ldots, x_n)\) over \(B\). The projection \(\rho: B \to \Gamma\) satisfies \(\rho \circ d_B = 0\). Hence by Lemma 1, below, it determines a tower \((\tilde{\tau}; x_{m+1}, \ldots, x_n)\) with

\[
\tilde{\tau}(x_i) = (\rho \otimes i)(\tau x_i) \in F(x_{m+1}, \ldots, x_n), \quad i = m + 1, \ldots, n.
\]

The maps

\[
F(x_1, \ldots, x_m) \to F(x_1, \ldots, x_n)
\]

and

\[
\rho \otimes i: F(x_1, \ldots, x_n) \to F(x_{m+1}, \ldots, x_n)
\]

are homomorphisms of graded differential algebras. They will be called, respectively, a base inclusion and a fibre projection.

Finally, suppose \((A, d_A; \tau; x_1, \ldots, x_n)\) is a tower over \(A\). Let \(\omega \in S_n\) be some permutation such that for each \(i\), \(\tau(x_{\omega(i)}) \in A \otimes F(x_{\omega(1)}, \ldots, x_{\omega(i-1)})\). Then \((A, d_A; \tau; x_{\omega(1)}, x_{\omega(2)}, \ldots, x_{\omega(n)})\) is again a tower over \(A\); it is called a rearrangement of the original tower, and has the same Koszul complex.

Observe that the following properties of a tower \((\tau; x_1, \ldots, x_n)\): c-finiteness, \(k\)-connectivity, minimality depend only on the Koszul complex, and so hold for any rearrangement. (In particular, the tower is minimal if and only if \(\tau(R) \subseteq F^+(R) \cdot F^+(R)\).) If \((\tau; x_1, \ldots, x_n)\) is a minimal connected tower then there is a permutation, \(\omega\), such that \(\deg x_{\omega(1)} < \deg x_{\omega(2)} < \ldots\), and \((\tau; x_{\omega(1)}, \ldots, x_{\omega(n)})\) is again a tower.

**Lemma 1.** Suppose \((A, d_A; \tau; x_1, \ldots, x_n)\) is a tower, and let \(\phi: (A, d_A) \to (B, d_B)\) be a homomorphism of graded commutative differential algebras. Define \(\sigma: R \to B \otimes F(R)\) by \(\sigma(x_i) = (\phi \otimes i)(\tau x_i)\).

Then \((B, d_B; \sigma; x_1, \ldots, x_n)\) is a tower over \(B\) and \(\phi \otimes i: A \otimes F(R) \to B \otimes F(R)\) is a homomorphism of graded differential algebras. Moreover if \(\phi^*: H(A) \to H(B)\) is an isomorphism then \((\phi \otimes i)^*\) is an isomorphism.

**Proof.** Clearly \(\sigma(x_i) \in B \otimes F(x_1, \ldots, x_{i-1})\). Moreover

\[
d_\sigma \circ (\phi \otimes i)(1 \otimes x_i) = \sigma(x_i) = (\phi \otimes i) \circ d_\tau(1 \otimes x_i)
\]

and

\[
d_\sigma \circ (\phi \otimes i)(a \otimes 1) = d_\sigma(\phi a \otimes 1) = 0 = (\phi \otimes i) \circ d_\tau(a \otimes 1).
\]

Since \(d_\sigma \circ (\phi \otimes i) - (\phi \otimes i) \circ d_\tau\) is a \((\phi \otimes i)\)-derivation these equations imply that it is zero:

\[
d_\sigma \circ (\phi \otimes i) = (\phi \otimes i) \circ d_\tau.
\]

Hence also \((d_\sigma + d_B) \circ (\phi \otimes i) = (\phi \otimes i) \circ (d_\sigma + d_B)\).
Now we obtain
\[(d_\alpha + d_B)^2(1 \otimes x_i) = (d_\alpha + d_B)^2(\phi \otimes \iota)(1 \otimes x_i)\]
\[= (\phi \otimes \iota)(d_\tau + d_A)^2(1 \otimes x_i) = 0.\]

Since \((d_\alpha + d_B)^2(b \otimes 1) = d_B^2(b) \otimes 1 = 0\), it follows that \((d_\alpha + d_B)^2 = 0\).

Thus \((B, d_B; \sigma; x_1, \ldots, x_\mu)\) is a tower and \(\phi \otimes \iota\) is a homomorphism of graded differential algebras.

Finally, suppose \(\phi^*\) is an isomorphism. We shall show (by induction on \(m\)) that the restrictions
\[(\phi \otimes \iota)_m: A \otimes F(x_1, \ldots, x_m) \rightarrow B \otimes F(x_1, \ldots, x_m)\]
induce isomorphisms of cohomology.

Suppose first that \(m = 1\). Filter \(A \otimes F(x_1)\) and \(B \otimes F(x_1)\) by the subspaces
\[L^p = \sum_{j=0}^p A \otimes F^j(x_1)\quad \text{and}\quad \hat{L}^p = \sum_{j=0}^p B \otimes F^j(x_1), \quad p = 0, 1, \ldots.\]

Then \(\phi \otimes \iota\) is filtration preserving, and so it induces a homomorphism \(\alpha_1: (E_1, d_1) \rightarrow (\hat{E}_1, \hat{d}_1)\) of spectral sequences. In particular, \(\alpha_1\) is given by
\[\alpha_1 = \phi^* \otimes \iota: H(A) \otimes F(x_1) \rightarrow H(B) \otimes F(x_1).\]

Thus each \(\alpha_i\) \((1 < i < \infty)\) is an isomorphism. Since \(E^{p,q}_i = 0 = \hat{E}^{p,q}_i\) for \(p > 0\) we have \(E_{\infty}^{p,q} = \text{ind lim } E^{p,q}_i\) \((i \text{ large})\). It follows that \(\alpha_{\infty}\) is an isomorphism. Hence \((\phi \otimes \iota)^*\) induces an isomorphism in the bigraded algebra determined by the filtrations in \(H(A \otimes F(x_1))\) and \(H(B \otimes F(x_1))\). This implies that \((\phi \otimes \iota)^*\) is an isomorphism.

Finally, assume by induction that \((\phi \otimes \iota)^*_m - 1\) is an isomorphism. Write
\[(\phi \otimes \iota)_{m-1} = \psi_1: A \otimes F(x_1, \ldots, x_{m-1}) = A', B \otimes F(x_1, \ldots, x_{m-1}) = B'.\]

Apply the argument above to
\[\phi_m \otimes \iota = \psi \otimes \iota: A' \otimes F(x_m) \rightarrow B' \otimes F(x_m)\]
to obtain that \((\phi_m \otimes \iota)^*\) is an isomorphism. Q.E.D.

**Example.** Let \((A, d_A; \tau; x_1)\) be a tower with \(\deg x_1\) odd. Its Koszul complex is given by \((A \otimes \wedge x_1, d)\), where
\[d(a \otimes x_1 + b \otimes 1) = d_\alpha a \otimes x_1 + ((-1)^{\deg a}a \cdot \tau(x_1) + d_\beta b) \otimes 1.\]

In particular \(\tau(x_1)\) is a cocycle representing a class \(\alpha \in H(A)\).

A short exact sequence \(0 \rightarrow A \rightarrow A \otimes \wedge x_1 \rightarrow \Psi A \rightarrow 0\) is given by \(\phi a = a \otimes 1, \Psi(a \otimes x_1 + b \otimes 1) = a\). The ensuing long exact (Gysin) sequence in cohomology has connecting homomorphism \(\partial: H(A) \rightarrow H(A)\) given by \(\partial \beta = \alpha \cdot \beta\). This sequence yields the short exact sequence
0 → Coker δ → H(A ⊗ x₁) → Ker δ → 0.

(Cf. [3, Chapter III] for details.)

3. Pure towers. Let (σ; x₁, ..., xₙ) be a finite tower. As in §1 write R = (x₁, ..., xₙ) = Q ⊕ P where Q is evenly graded and P is oddly graded. The tower will be called pure if σ(P) ⊂ V Q and σ(Q) = 0. Koszul complexes of pure towers were studied by Koszul [4] and H. Cartan [2]; we recall here some of their results.

Let (\vee Q ⊗ xₙ P, d) be the Koszul complex of a pure tower (σ; x₁, ..., xₙ). Then d: \vee Q ⊗ xₙ P → \vee Q ⊗ xₙ⁻¹ P, and thus the gradation \vee Q ⊗ xₙ P = \sum_k \vee Q ⊗ xₙ⁻¹ P leads to a gradation of H(\vee Q ⊗ xₙ P), written H(\vee Q ⊗ xₙ P) = \sum_k H_k(\vee Q ⊗ xₙ P). Let \vee Q ⊗ P be the ideal in \vee Q generated by σ(P); then the inclusion i: \vee Q ⊗ P → H(\vee Q ⊗ xₙ P) induces an isomorphism i*: \vee Q / \vee Q ⊗ P → H_0(\vee Q ⊗ xₙ P).

If P₁ ⊂ P is any graded subspace then \vee Q ⊗ xₙ P₁ is a subdifferential algebra of \vee Q ⊗ xₙ P.

**Lemma 2.** If H_k(\vee Q ⊗ xₙ P₁) = 0 then H_k(\vee Q ⊗ xₙ P) ≠ 0.

**Proof.** By considering a sequence of spaces P₁ ⊂ P₂ ⊂ ... ⊂ P_m = P we can reduce to the case P = P₁ ⊂ (x). Set (A, dₐ) = (\vee Q ⊗ xₙ P₁, d); then \vee Q ⊗ xₙ P = A ⊗ x is the Koszul complex of the tower (A, dₐ; σ; x).

Now apply the example of §2 to obtain a Gysin sequence in which the connecting homomorphism δ: H(A) → H(A) is multiplication by the class α ∈ H(σ(x)). Since σ(x) ∈ \vee Q it follows that for some p > 0, α ∈ Hₚ(\vee Q ⊗ xₙ P₁). This implies that δ restricts to linear maps δₖ: Hₖ(\vee Q ⊗ xₙ P₁) → Hₖ(\vee Q ⊗ xₙ P₁) of positive degree. In particular, since Hₖ(\vee Q ⊗ xₙ P₁) ≠ 0, then Coker δₖ ≠ 0.

Finally note that the inclusion Coker δ → H(A ⊗ xₙ x) of the example in §2 is the direct sum of inclusions Coker δ → H(\vee Q ⊗ xₙ P). Thus since Coker δₖ ≠ 0 we have Hₖ(\vee Q ⊗ xₙ P) ≠ 0. Q.E.D.

The following is due to Cartan [2]. A detailed proof is given in [3, Chapter 2].

**Theorem 2.** Let (\vee Q ⊗ xₙ P, d) be the Koszul complex of a connected pure tower such that dim H(\vee Q ⊗ xₙ P) < ∞. Then H(\vee Q ⊗ xₙ P) has nonnegative Euler characteristic χ. Moreover dim P − dim Q is the nonnegative integer k with the property

\[ H_k(\vee Q ⊗ xₙ P) ≠ 0, \quad H_{k+p}(\vee Q ⊗ xₙ P) = 0, \quad p > 1. \]

Finally, the following conditions are equivalent:

(i) dim P = dim Q.

(ii) χ > 0.
(iii) $H(\sqrt{Q} \otimes \wedge P)$ is evenly graded.
(iv) $H(\sqrt{Q} \otimes \wedge P) = H_0(\sqrt{Q} \otimes \wedge P)$.

Remark. Dim $H(\sqrt{Q} \otimes \wedge P) < \infty$ if and only if dim $\sqrt{Q}/\sqrt{Q} \circ P < \infty$. In fact note that ker $d$ is a $\sqrt{Q}$-submodule of the finitely generated $\sqrt{Q}$-module $\sqrt{Q} \otimes \wedge P$. Because $\sqrt{Q}$ is noetherian ker $d$ is finitely generated. Thus $H(\sqrt{Q} \otimes \wedge P)$ is a finitely generated $\sqrt{Q}$ module.

This implies (clearly) that $H(\sqrt{Q} \otimes \wedge P)$ is a finitely generated module over $H_0(\sqrt{Q} \otimes \wedge P)$. Thus dim $H(\sqrt{Q} \otimes \wedge P) < \infty$ if and only if dim $H_0(\sqrt{Q} \otimes \wedge P) < \infty$; i.e., if and only if dim $\sqrt{Q}/\sqrt{Q} \circ P < \infty$.

4. The $S$-spectral sequence. As in §1 let $R = (x_1, \ldots, x_n)$. Assume deg $x_i > 0$, $i = 1, \ldots, n$. Let $S$ be a subspace spanned by some of the $x_i$ and let $T$ be the subspace spanned by the remaining $x_j$. (Then $R = T \oplus S$.)

Now suppose $(A, d_A; \tau; x_1, \ldots, x_n)$ is a tower over $A$. Then $A \otimes F(R) = A \otimes F(T) \otimes F(S)$ and so a bigradation of $A \otimes F(R)$ is given by

$$[A \otimes F(R)]^{p+q} = [A \otimes F(T) \otimes F^q(S)]^{p+q}.$$

Write $(A \otimes F(R), d_A + d_T) = (C, d_\sigma)$; $C = \sum_{p,q} C^{p,q}$.

Clearly $C^{p+r,q+s} \subseteq C^{p+r,q+s}$ and so $C$ is filtered by the ideals $I^p = \sum_{q \geq p} C^p$. (Note $C^p = 0$ if $p < 0$.)

Now let $\sigma: R \to A \otimes F(T)$ be the unique linear map such that $\sigma(x) = 0$, $x \in T$ and $\sigma(x) - \tau(x) \in A \otimes F(T) \otimes F^+(S)$, $x \in S$. Extend $\sigma$ to a derivation $d_\sigma$ in $C$ such that $d_\sigma(A) = 0$. Clearly $d_\sigma^2 = 0$.

Lemma 3. (i) $d_\sigma$ is homogeneous of bidegree $(0, 1)$.
(ii) $d_\sigma - d_\sigma: I^p \to I^{p+1}$.

Proof. Clear.

The lemma shows that the $I^p$ filter the graded differential algebra $(C, d_\sigma)$, and that the first term of the resulting spectral sequence (of graded differential algebras) is given by

$$(E_0, d_0) = (C, d_\sigma).$$

Moreover, because the elements in $F^q(S)$ have degree at least $q$, it follows that $C^{p,q} = 0$ unless $0 \leq -2q < p$. This implies that the spectral sequence converges to $H(C, d_\sigma)$. This spectral sequence will be called the $S$-spectral sequence.

In particular, if $P$ denotes the subspace of $R$ of elements of odd degree then the $P$-spectral sequence will be called the odd spectral sequence.

5. The odd spectral sequence of a tower. Let $R = (x_1, \ldots, x_n)$ and suppose $(\tau; x_1, \ldots, x_n)$ is a connected finite tower. As usual write $F(R) = \sqrt{Q} \otimes \wedge P$. 
Let \( \sigma: R \to \bigvee Q \) be the linear map defined by \( \sigma(Q) = 0 \) and \( \sigma(x) - \tau(x) \in \bigvee Q \otimes \wedge^+ P, x \in P \). Then \((\sigma; x_1, \ldots, x_n)\) is a pure tower, called the associated pure tower for \((\tau; x_1, \ldots, x_n)\).

Observe as well that \( \tau(Q) \subset F(R)^{odd} \subset \bigvee Q \otimes \wedge^+ P \). It follows that
\[
(5) \quad d_\ell - d_\ell: \bigvee Q \otimes \wedge P \to \bigvee Q \otimes \wedge^+ P.
\]

If \((E, d)\) is the odd spectral sequence for the original tower then
\[
(6) \quad (E_0, d_0) \cong (\bigvee Q \otimes \wedge P, d_0)
\]
(cf. formula (4), §4). This isomorphism restricts to isomorphisms \( E_\ell^{r,q} \cong (\bigvee Q \otimes \wedge^{-q} P)^{r+q} \). Thus there is an algebra isomorphism
\[
(7) \quad E_1 \cong H(\bigvee Q \otimes \wedge P, d_0)
\]
which restricts to isomorphisms
\[
(8) \quad E_1^{r,q} \cong H_{r+q}^{q+q}(\bigvee Q \otimes \wedge P, d_0).
\]

Now we show that \( d_1 = 0 \), so that \( E_2 \cong E_1 \). In fact by (5), \((d_\ell - d_\ell)(Q) \subset \bigvee Q \otimes \wedge^+ P\) while
\[
\begin{align*}
(d_\ell - d_\ell)(P) & \subset (\bigvee Q \otimes \wedge^+ P) \cap \left( \sum_{r \in \mathbb{Z}^+} (\bigvee Q \otimes \wedge P)^r \right) \\
& \subset \bigvee Q \otimes \wedge^+ P.
\end{align*}
\]
Thus
\[
(9) \quad (d_\ell - d_\ell): (\bigvee Q \otimes \wedge^{-q} P)^{r+q} \to \sum_{j \geq 2} (\bigvee Q \otimes \wedge^{-q+j} P)^{r+q+1}.
\]
It follows that \( d_\ell - d_\ell: I^p \to I^{p+2} \) (the \( I^p \) are the ideals defining the spectral sequence) and hence the differential \( d_1 = 0 \).

Similarly, it follows at once from the definition that
\[
I^p \cap F(R)^r = \sum_{k > p-r} (\bigvee Q \otimes \wedge^k P)^r.
\]
Thus if \( J = \bigvee Q \otimes \wedge^+ P \) and \( J_k = J \cdots \cdot J \) (\( k \) factors), then \( I^p \cap F(R)^r = \sum_{k > p-r} J_k \cap F(R)^r \).

Now suppose \( F(R') = \bigvee Q' \otimes \wedge P' \) is the Koszul complex of a second finite tower and assume \( \phi: F(R) \to F(R') \) is a homomorphism of graded differential algebras. Since \( J \) and \( J' \) are the ideals generated by elements of odd degree, \( \phi(J_k) \subset J'_k \) for \( k = 1, 2, \ldots \).

Now the formula above for \( I^p \) shows that \( \phi \) is filtration preserving. Hence it induces a homomorphism of spectral sequences. In particular, if \( \phi \) is an isomorphism then \( \phi^{-1} \) is also filtration preserving and so \( \phi \) and \( \phi^{-1} \) induce inverse isomorphisms of the odd spectral sequences.
6. Proof of Theorem 1. Recall from §2 that a tower $(\tau; x_1, \ldots, x_n)$ determines towers $(\tilde{\tau}; x_p, \ldots, x_n)$. Since $\tilde{\tau}(x_p) = 0$, $x_p$ is a cocycle in $(F(x_p, \ldots, x_n))$. Let $[x_p] \in H(F(x_p, \ldots, x_n), d_r)$ be the class represented by $x_p$.

**Proposition 1.** Let $(\tau; x_1, \ldots, x_n)$ be a connected, finite, minimal tower. Write $R = (x_1, \ldots, x_n)$, $F(R) = \bigvee Q \otimes \bigwedge P$. Suppose $(E_i, d_i)$ denotes the odd spectral sequence and $(\sigma; x_1, \ldots, x_n)$ is the associated pure tower. Then the following are equivalent:

1. The tower is c-finite: $\dim H(\bigvee Q \otimes \bigwedge P, d_i) < \infty$.
2. For each $p$ the class $[x_p] \in H(F(x_p, \ldots, x_n), d_i)$ is nilpotent: $[x_p]^k = 0$, some $k$.
3. $\dim H(\bigvee Q \otimes \bigwedge P, d_i) < \infty$.
4. $\dim E_1 < \infty$.

**Proof.** (1) $\Rightarrow$ (2). This is deferred until §7 (Lemma 5).

(2) $\Rightarrow$ (3). Denote by $Q_p$ the subspace of $Q$ spanned by the $x_i$ of even degree with $i < p$, and set $Q_0 = 0$. We show first by induction on $p$ that the elements of $Q_p$ represent nilpotent classes in $H(\bigvee Q \otimes \bigwedge P, d_i)$.

This is certainly true for $p = 0$. Suppose it is true for $p - 1$. If $x_p$ has odd degree then $Q_p = Q_{p-1}$ and our claim is true for $p$. If $x_p$ has even degree our hypothesis shows that for some $u, v_i \in \bigvee Q \otimes \bigwedge P$ and some $k > 1$:

$$x_p^k = d_i u - \sum_{i=1}^{p-1} x_i \cdot v_i.$$

Hence $d_i u - x_p^k \in Q_{p-1} \cdot \bigvee Q + \bigvee Q \otimes \bigwedge^+ P$. Thus formula (5) yields

$$d_i u - x_p^k \in Q_{p-1} \cdot \bigvee Q + \bigvee Q \otimes \bigwedge^+ P.$$

Now write $u = \sum u_i$, $u_i \in \bigvee Q \otimes \bigwedge^i P$. Since $d_i: \bigvee Q \otimes \bigwedge^i P \to \bigvee Q \otimes \bigwedge^{i-1} P$, it follows from (9) that

$$d_i u_i - x_p^k \in Q_{p-1} \cdot \bigvee Q.$$

Since the elements of $\bigvee Q$ are $d_i$-cocycles and the elements of $Q_{p-1}$ represent nilpotent classes in $H(\bigvee Q \otimes \bigwedge P, d_i)$, it follows that the elements of $Q_{p-1} \cdot \bigvee Q$ represent nilpotent classes. Hence the equation above implies that $x_p$ represents a nilpotent class. The induction is now closed.

We have now shown that the elements in $Q$ represent nilpotent classes in $H(\bigvee Q \otimes \bigwedge P, d_i)$. This implies that $\bigvee Q$ has finite dimensional image in $H(\bigvee Q \otimes \bigwedge P, d_i)$. The remark in §3 now implies that

$$\dim H(\bigvee Q \otimes \bigwedge P, d_i) < \infty.$$

(3) $\Rightarrow$ (4). Apply formula (7).

(4) $\Rightarrow$ (1). Recall that the spectral sequence converges to
H(\sqrt Q \otimes \wedge P, d_n). \quad \text{Q.E.D.}

**Corollary.** If \((\tau; x_1, \ldots, x_n)\) is a connected, finite, c-finite, minimal tower, then for each \(p\) the tower \((\tau; x_p, \ldots, x_n)\) is also c-finite.

**Theorem 1.** Let \((\tau; x_1, \ldots, x_n)\) be a connected, finite, c-finite, minimal tower. Then \(\chi_\Pi < 0\) and \(\chi_c > 0\). Moreover, the following conditions are equivalent.

1. \(\chi_\Pi = 0\).
2. \(\chi_c > 0\).
3. \(H(F(x_1, \ldots, x_n), d_t)\) is evenly graded.

**Proof.** We adopt the notation of Proposition 1. Then according to Proposition 1, \(H(\sqrt Q \otimes \wedge P, d_n)\) has finite dimension. Denote its Euler characteristic by \(\chi\). Since \(H(\sqrt Q \otimes \wedge P, d_n) \cong E_1\) and since \((E_i, d_i)\) converges to \(H(\sqrt Q \otimes \wedge P, d_n)\) it follows that \(\chi = \chi_c\).

Moreover, since \(H(\sqrt Q \otimes \wedge P, d_n)\) has finite dimension we can apply Theorem 2, §3 to obtain \(\chi_\Pi = \dim Q - \dim P < 0\) and \(\chi_c = \chi > 0\).

The equivalence of conditions (i), (ii) and (iii) in Theorem 2 implies that conditions (1) and (2) are equivalent, and hold if and only if

\[ H(\sqrt Q \otimes \wedge P, d_n) \]

is evenly graded. But in this case \(E_1\) is evenly graded and so the odd spectral sequence collapses at the \(E_1\)-term. In particular, \(H(\sqrt Q \otimes \wedge P, d_n)\) is evenly graded. Thus (1) \(\iff\) (2) \(\implies\) (3). But clearly (3) \(\implies\) (2). \(\text{Q.E.D.}\)

**Corollary 1.** Let \((E_i, d_i)\) be the odd spectral sequence of a connected, finite, c-finite minimal tower. Then for \(i > 1\):

\[ E_{i,q}^p = 0, \quad q < \chi_\Pi. \]

**Proof.** By formula (8), \(E_{i,q}^p = H_{i,q}^{p+q}(\sqrt Q \otimes \wedge P, d_n)\). If \(q < \chi_\Pi\) then \(H_{-q}(\sqrt Q \otimes \wedge P, d_n) = 0\) by formula (3), Theorem 2. Thus \(E_{i,q}^p = 0, \quad q < \chi_\Pi\) and so \(E_{i,q}^p = 0, \quad q < \chi_\Pi\). \(\text{Q.E.D.}\)

**Corollary 2.** Let \((\tau; x_1, \ldots, x_n)\) be a connected, finite, c-finite minimal tower. Then for each \(p\),

\[ \sum_{i=p}^{n} (-1)^{\text{deg } x_i} < 0. \]

In particular, if \(R = (x_1, \ldots, x_n)\) then for each \(p\),

\[ \sum_{i \geq p} (-1)^i \dim R^i < 0. \]

**Proof.** In view of the corollary to Proposition 1 we may apply Theorem 1 to the tower \((\tau; x_p, \ldots, x_n)\) to obtain (10). Next note (cf. §2) that we may
r rearrange the $x_i$ so that $\deg x_\omega(1) \leq \deg x_\omega(2) \leq \ldots$. Now (11) is a special case of (10) (with $x_i$ replaced by $x_\omega(1)$). Q.E.D.

**Corollary 3.** Let $X$ be a connected topological space such that $H^*(X; \mathbb{Q})$ and $\Pi_\omega^*(X)$ are finite dimensional. Then for each $p$, $\sum_{i \geq p} (-1)^i \dim \Pi_i^*(X) < 0$. If $X$ is simply connected then for each $p$, $\sum_{i \geq p} (-1)^i \dim \Pi_i^*(X) \otimes \mathbb{Q} < 0$.

**Proof.** This follows from Corollary 2 in the same way Theorem 1′ followed from Theorem 1 (cf. §1). Q.E.D.

**Corollary 4.** The odd spectral sequence for a connected, finite, $c$-finite, minimal tower with $\chi_{\Pi} = 0$ collapses at the $E_i$-term.

Proposition 2 below and its proof are due to C. Allday (private communication). It is a special case of his conjecture ** in [1]; the general case remains open.

Let $A = \sum_{k \geq 0} A^k$ be a graded vector space of finite type. Its Poincaré series is the formal series $f_A(t) = \sum_k \dim A^k t^k$. Following Hsiang set

$$\rho_0(A) = \inf \{ \alpha \in \mathbb{R} | (1 - t)\alpha f_A(t) \to 0 \text{ as } t \to 1 \}.$$

If $g = \sum a_k t^k$ and $h = \sum b_k t^k$ are two formal series with integer coefficients we write $g < h$ to mean $a_k < b_k$, each $k$.

**Proposition 2.** Suppose $(F(R), d_\tau)$ is the Koszul complex of a connected, finite, minimal tower $(\tau; x_1, \ldots, x_n)$ with homotopy Euler characteristic $\chi_\Pi$. Assume $H(F(R))$ is finitely generated as an algebra over $\Gamma$. Then $\chi_\Pi < \rho_0(H(F(R)))$.

**Remark.** As will appear in the proof, $\rho_0(H(F(R)))$ is the Krull dimension of the commutative algebra $\sum_k H^{2k}(F(R))$.

**Proof.** Denote the commutative subalgebra $\sum_k H^{2k}(F(R))$ by $A$. Using the argument of [7, p. 201] we construct a sequence $z_0, \ldots, z_i$ of homogeneous elements in $A^+$ as follows: assume $z_0, \ldots, z_i$ are constructed with $z_0 = 0$, and generate an ideal $Z_i$ with isolated prime ideals $J_1, \ldots, J_k$ (cf. [6, p. 211]). These are necessarily graded, and hence in $A^+$. Thus either $k = 1$ and $J_1 = A^+$ or there is a homogeneous element $z_{i+1}$ in $A^+$ such that $z_{i+1} \notin \bigcup J_j^\infty$.

The sequence $Z_1, Z_2, \ldots$ terminates at some $Z_l$ because $A$ is noetherian; in particular, $A^+$ is the unique prime ideal for $Z_l$ and so $A/Z_l$ has finite dimension.

Choose a sequence $K_l \supset \cdots \supset K_1 \supset K_0$ with $K_i$ an isolated prime ideal for $Z_l$; then $z_i \notin K_{i-1}$, $i > 1$. Thus an easy induction on $l - i$ shows that the obvious homomorphism $\vee(z_{i+1}, \ldots, z_l) \to A/Z_l$ is injective. In particular we have an inclusion $\bigvee(z_1, \ldots, z_l) \to A$. On the other hand, if $F$ is a (finite dimensional) graded space such that $F \oplus Z_l = A$, then the obvious map $\bigvee(z_1, \ldots, z_l) \otimes F \to A$ is surjective. It follows that (denoting $\bigvee(z_1, \ldots, z_l)$...
by B) that \( f_B \leq f_A \leq f_B f_F \), whence \( \rho_0(A) = \rho_0(B) = l \).

Moreover, if \( S \subset H^{\text{odd}}(F(R)) \) is a finite dimensional graded subspace, which, together with \( A \) generates \( H(F(R)) \), then \( f_A \leq f_{H(F(R))} \leq f_A \cdot f_\wedge S \). Hence \( \rho_0(H(F(R))) = \rho_0(A) = l \).

On the other hand, let \( y_1, \ldots, y_l \) be cocycles representing \( z_1, \ldots, z_l \) and let \( U = (u_1, \ldots, u_l) \) be a graded space with deg \( u_i = \deg z_j - 1 \). Define Koszul complexes \( (F(R) \otimes \wedge U, d) \) and \( (H(F(R)) \otimes \wedge U, d') \) by

\[
d(\Phi \otimes 1) = d_\Phi \otimes 1, \quad d(1 \otimes u_i) = y_i \otimes 1, \quad d'(1 \otimes u_i) = z_i \otimes 1.
\]

According to [4] there is a spectral sequence converging to

\[
H(F(R) \otimes \wedge U, d)
\]

with \( E_2 \)-term \( H(H(F(R)) \otimes \wedge U, d') \). (See [3, Chapter III] for details.) (This spectral sequence, introduced by Koszul, is a special case of the Eilenberg-Moore spectral sequence.) Since \( A/Z_l \) has finite dimension, the argument in the remark of §3 shows that, so does \( H(H(F(R)) \otimes \wedge U) \).

It follows that \( H(F(R) \otimes \wedge U) \) has finite dimension. Its homotopy Euler characteristic is given by \( \chi_\Pi - l \) (\( \chi_\Pi \) the homotopy Euler characteristic of \( F(R) \)). Now apply Theorem 1 to get \( \chi_\Pi - l < 0 \); i.e. \( \chi_\Pi < l = \rho_0(H(F(R))) \).

Q.E.D.

**Corollary.** Let \( X \) be a connected topological space with \( \dim \Pi_\sigma^*(X) < \infty \) and \( H^*(X; \mathbb{Q}) \) finitely generated. Set \( \chi_\Pi(X) = \Sigma(-1)^k \dim \Pi_\sigma^k(X) \) and \( \rho_0(X) = \rho_0(H^*(X; \mathbb{Q})) \). Then \( \chi_\Pi(X) < \rho_0(X) \).

**7. Two lemmas.**

**Lemma 4.** Suppose \((\tau; x_1, \ldots, x_n)\) is a connected, finite, minimal tower. Then there is a tower \((\sigma; x_1, x_2, \ldots, x_n, y_1, \ldots, y_n)\) with \( \deg y_i = \deg x_i - 1 \) and with the following properties for each \( i \) (\( 1 \leq i \leq n \)):

(i) \( \sigma(x_i) = \tau(x_i) \).

(ii) \( \sigma(y_i) \in (x_1, \ldots, x_{i-1}) \cdot F(x_1, \ldots, x_{i-1}, y_1, \ldots, y_{i-1}) \).

(iii) \( H(F^+(x_1, \ldots, x_i, y_1, \ldots, y_i), d_\sigma) = 0 \). (Note that (ii) implies that \( F(x_1, \ldots, x_i, y_1, \ldots, y_i) \) is stable under \( d_\sigma \)).

(iv) If \( i < n \) then for some \( w \in (x_1, \ldots, x_i) \cdot F(x_1, \ldots, x_i, y_1, \ldots, y_i) \),

\[
d_\sigma(x_{i+1} - w) = 0.
\]

**Proof.** We use induction on \( p \) to define elements

\[
\sigma(y_p) \in F(x_1, \ldots, x_p, y_1, \ldots, y_p)
\]

so that conditions (i)–(iv) hold for \( i \leq p \).

If \( p = 1 \) set \( \sigma(y_1) = x_1 \). Since \( \tau(x_1) = 0 \) it follows that \((\sigma; x_1, y_1)\) and \((\sigma; x_1, \ldots, x_n, y_1)\) are towers. Condition (ii) is obvious, while (iii) asserts that \( H(F^+(x_1, y_1)) = 0 \); this is a simple and classical calculation. (If \( \deg x_1 \) is odd
it is essential that $\Gamma$ have characteristic 0!)

Finally, since the original tower was minimal, for some $a \in F^+(x_1)$, $\tau(x_2) = ax_1$. Set $w = (-1)^{deg} a y_1$. Then in $F(x_1, x_2, y_1)$, $d_o(x_2 - w) = ax_1 - ax_1 = 0$.

Suppose now that $\sigma(y_j)$ is constructed for $j < p$ so that (i)-(iv) hold for $j < p$. Then $(\sigma; x_1, \ldots, x_p, y_1, \ldots, y_{p-1})$ is a tower, and by (iv) there is an element $w \in (x_1, \ldots, x_{p-1}) \cdot F(x_1, \ldots, x_{p-1}, y_1, \ldots, y_{p-1})$ such that $d_o(x_p - w) = 0$. Set $\sigma(y_p) = x_p - w$. Then $d_o^2(y_p) = d_o(x_p - w) = 0$ and so $(\sigma; x_1, \ldots, x_p, y_1, \ldots, y_p)$ is a tower. Hence so is $(\tau; x_1, \ldots, x_n, y_1, \ldots, y_p)$.

Moreover (ii) (for $i = p$) is immediate from the definition. To check (iii) write $(F(x_1, \ldots, x_{p-1}, y_1, \ldots, y_{p-1}), d_o) = (A, d_A)$. Then

$$(F(x_1, \ldots, x_p, y_1, \ldots, y_p), d_o)$$

is the Koszul complex of the tower $(A, d_A; \sigma, x_p, y_p)$ over $(A, d_A)$.

Let $p: A \to \Gamma$ and $\rho \otimes \iota: A \otimes F(x_p, y_p) \to F(x_p, y_p)$ be the projections. By (i) and (ii) (for $i = p$) \((\rho \otimes \iota)(\sigma x_p) = 0\) and \((\rho \otimes \iota)(\sigma y_p) = y_p\). Thus if we define \((\tilde{\sigma}; x_p, y_p)\) by \(\tilde{\sigma}(y_p) = x_p\), \(\tilde{\sigma}(x_p) = y_p\), then \((\rho \otimes \iota) \circ d_o = d_o \circ (\rho \otimes \iota)\).

By our induction hypothesis (iii) (for $i = p - 1$), $p^*$ is an isomorphism. Hence (cf. Lemma 1, §2) \((\rho \otimes \iota)^*\) is an isomorphism. Thus

$$H(F^+(x_1, \ldots, x_p, y_1, \ldots, y_p), d_o) \cong H(F^+(x_p, y_p), d_o) = 0.$$  

It remains to prove (iv). Since $\tau(x_{p+1})$ is a cocycle in $F(x_1, \ldots, x_p)$, it is a cocycle in $F(x_1, \ldots, x_p, y_1, \ldots, y_p)$. By (iii) (for $i = p$) we can write $\tau(x_{p+1}) = d_o(w)$ for some $w \in F(x_1, \ldots, x_p, y_1, \ldots, y_p)$. In view of (i) this gives

$$(12) d_o(x_{p+1} - w) = 0.$$  

Write $w = u + v$, $u \in (x_1, \ldots, x_p) \cdot F(x_1, \ldots, x_p, y_1, \ldots, y_p)$, $v \in F(y_1, \ldots, y_p)$.

We prove (iv) by showing that $v = 0$. If $v \neq 0$ then for some $q$,

$$v = \sum_{k=0}^{m} y_q b_k,$$

where $b_k \in F(y_1, \ldots, y_{q-1})$, $b_m \neq 0$ and $m \geq 1$. Suppose this is the case.

Let $I = (x_1, \ldots, x_p) \cdot F(x_1, \ldots, x_p, y_1, \ldots, y_p)$ and set $J = (x_1, \ldots, x_{q-1}) \cdot F(x_1, \ldots, x_p, y_1, \ldots, y_p)$. (If $q = 1$ set $J = 0$.) Then we have the short exact sequence

$$0 \to I \cdot J \to F(x_1, \ldots, x_p, y_1, \ldots, y_p)$$

$$\to \prod \Gamma \oplus (x_{q'}, \ldots, x_q) \otimes F(y_1, \ldots, y_p) \to 0.$$  

It follows from (ii) that $\sigma(y_j) \in I$ ($i < p$). The minimality of $(\tau; x_1, \ldots, x_n)$ implies that $\sigma(x_i) = \tau(x_i) \in I \cdot I$ ($i < p + 1$). Hence $d_o(I) \subset I \cdot I$ and $d_o(x_{p+1}) \subset I \cdot I$. Thus applying $\Pi$ to equation (12) we find that
\[ \Pi d_\sigma v = \Pi d_\sigma (x_{p+1}) - \Pi d_\sigma u = 0. \]

Moreover it follows from (ii) that \( d_\sigma y_i \in J(i < q) \) and \( d_\sigma y_q - x_q \in J \). Hence

\[ \Pi d_\sigma v = \sum_{k=1}^m k \Pi (x_q y_q^{k-1} b_k) = \sum_{k=1}^m k x_q y_q^{k-1} b_k \neq 0. \]

This contradiction shows that \( v = 0 \). The induction is now closed and the proof is complete. Q.E.D.

**Corollary.** For each \( p \) \( (\sigma; x_1, \ldots, x_{p-1}, y_1, \ldots, y_{p-1}, x_p, \ldots, x_n) \) is a tower, and the induced fibre projection

\[ \Pi: (F(x_1, \ldots, x_n, y_1, \ldots, y_{p-1}), d_\sigma) \to (F(x_p, \ldots, x_n), d_\tau) \]

induces an isomorphism in cohomology.

**Lemma 5.** Let \( (\tau; x_1, \ldots, x_n) \) be a connected, finite, \( c \)-finite minimal tower. Let \( [x_p] \in H(F(x_p, \ldots, x_n)) \) be the class represented by \( x_p \). Then for each \( p \) \((1 < p < n)\), \( [x_p] \) is nilpotent.

**Proof.** Let \( (\sigma; x_1, \ldots, x_n, y_1, \ldots, y_n) \) be the tower of Lemma 4, and fix \( p \). Part (iv) of Lemma 4 implies that for some \( w \) of the form

\[ w = \sum_{i=1}^{p-1} x_i u_i, \quad u_i \in F(x_1, \ldots, x_{i-1}, y_1, \ldots, y_{i-1}), \]

we have \( d_\sigma (x_p - w) = 0 \) in \( F(x_1, \ldots, x_n, y_1, \ldots, y_{p-1}) \).

Thus if \( \Pi \) is the projection in the corollary to Lemma 4, \( \Pi(x_p - w) = x_p \).

Let \( \alpha \in H(F(x_1, \ldots, x_n, y_1, \ldots, y_{p-1}), d_\sigma) \) be the cohomology class represented by \( x_p - w \). Then \( \Pi^* \alpha = [x_p] \). Since \( \Pi^* \) is an isomorphism we need only prove that \( \alpha^k = 0 \), for some \( k \).

Denote by \((X, d_X)\) the Koszul complex of \((\tau; x_1, \ldots, x_n)\). Since \( H(X) \) is finite dimensional there is an integer \( k > 2 \) such that \( H^j(X) = 0, j > k \). Let \( C \subset X^{k-1} \) be a subspace such that \( X^{k-1} = C \oplus (\ker d_X)^{k-1} \). Define a graded, \( d_X \)-stable ideal \( I \subset X \) by \( I^j = 0 \) \((j < k - 1)\), \( I^k = C, I^{j} = X^j \) \((j > k)\).

Let \( A = X/I \) and let \( d_A \) be the derivation induced by \( d_X \) in \( A \). Then the projection \( \rho: X \to A \) is a homomorphism of graded differential algebras, and \( \rho^* \) is an isomorphism.

On the other hand, \((F(x_1, \ldots, x_n, y_1, \ldots, y_{p-1}), d_\sigma)\) is the Koszul complex of the tower \((X, d_X; \sigma; y_1, \ldots, y_{p-1})\) over \((X, d_X)\). Thus by Lemma 1, §2 there is a tower \((A, d_A; \lambda; y_1, \ldots, y_{p-1})\) such that

\[ (\rho \otimes \iota): (X \otimes F(y_1, \ldots, y_{p-1}), d_\sigma) \to (A \otimes F(y_1, \ldots, y_{p-1}), d_A + d_\lambda) \]

is a homomorphism of graded differential algebras. Moreover \((\rho \otimes \iota)^* \) is an isomorphism.
But $x_p - w \in X^+ \otimes F(y_1, \ldots, y_{p-1})$ and so $(\rho \otimes \iota)(x_p - w) \in A^+ \otimes F(y_1, \ldots, y_{p-1})$. Since $A^j = 0$, $j > k$ this implies that

$$(\rho \otimes \iota)(x_p - w)^k = \left[ (\rho \otimes \iota)(x_p - w) \right]^k = 0.$$ 

Hence $(\rho \otimes \iota)^* (\alpha^k) = 0$. Since $(\rho \otimes \iota)^*$ is an isomorphism, $\alpha^k = 0$. Q.E.D.

8. **Poincaré duality.** A finite dimensional graded algebra $A = \sum_{p=0}^n A^p$ is said to have formal dimension $n$ if $A^n \neq 0$ (and $A^p = 0$, $p > n$). A Poincaré duality algebra (P.d.a.) is a finite dimensional graded commutative algebra $A = \sum_{p=0}^n A^p$ such that $\dim A^n = 1$ and such that multiplication defines nondegenerate bilinear maps $A^p \times A^{n-p} \to A^n (\cong \Gamma)$, $p = 0, 1, \ldots$. If $e$ is a nonzero element in $A^n$ then the scalar product $\langle \cdot, \cdot \rangle$ in $A$ given by

$$\langle \alpha, \beta \rangle = 0, \quad \deg \alpha + \deg \beta \neq n, \quad \langle \alpha, \beta \rangle e = \alpha \cdot \beta, \quad \deg \alpha + \deg \beta = n$$

induces isomorphisms $A^{n-p} \cong (A^p)^*$. These are called, respectively, the Poincaré scalar product and the Poincaré isomorphism.

The tensor product of two graded commutative algebras is a P.d.a. if and only if each factor is a P.d.a. If $(A, d_A)$ is a graded differential algebra such that $A$ and $H(A)$ both have formal dimension $n$, and if $A$ is a P.d.a., then so is $H(A)$.

In this section we establish

**Theorem 3.** Let $(\tau; x_1, \ldots, x_n)$ be a connected, finite, c-finite minimal tower with odd spectral sequence $(E_i, d_i)$. Then

(i) $H(F(x_1, \ldots, x_n), d_t)$ and each $E_i$ ($i \geq 1$) have the same formal dimension $m$, given by

$$m = r - \sum_{i=1}^n (-1)^{\deg x_i} \deg x_i,$$

where $r$ is the number of $x_i$ of even degree.

(ii) $H(F(x_1, \ldots, x_n), d_t)$ and each $E_i$ are P.d.a.'s.

(iii) For $i \geq 1$ $E_i^{*+n} = 0$, $q < \chi_i$ and $E_i^{*q} \neq 0$, $q = \chi_i - 1$.

Exactly as in §1 (Theorem 1 \Rightarrow Theorem 1'), Theorem 3 yields

**Theorem 3'.** Let $X$ be a connected topological space such that $H^*(X; \mathbb{Q})$ and $\Pi_{\psi}^*(X)$ are finite dimensional. Then $H^*(X; \mathbb{Q})$ is a P.d.a. of formal dimension $m$ given by

$$m = \sum_i \dim \Pi_{\psi}^i(X) - \sum_k (-1)^k k \dim \Pi_{\psi}^k(X).$$

If $X$ is simply connected the theorem remains true if $\Pi_{\psi}^*(X)$ is replaced by $\Pi_{\psi}^*(X) \otimes \mathbb{Q}$ everywhere in the statement.

**Lemma 6.** Theorem 3 is correct for pure towers.
PROOF. Suppose \( \tau; x_1, \ldots, x_n \) is a pure tower satisfying the hypotheses of the theorem. Write \( R = (x_1, \ldots, x_n) \); \( F(R) = \bigvee Q \otimes \wedge P \). Let \( z_1, \ldots, z_r \) be a homogeneous basis of \( Q \) and choose an integer \( k > 2 \) so that \( z_i^k = d_i w_i, \ i = 1, \ldots, r \).

Let \( U = (u_1, \ldots, u_r) \) be a graded vector space with \( \deg u_i = k \deg z_i - 1 \). Define a graded differential algebra \( (\bigvee Q \otimes \wedge P \otimes \wedge U, d) \) by

\[
d\Phi = d_i \Phi, \quad \Phi \in \bigvee Q \otimes \wedge P \quad \text{and} \quad du_i = z_i^k, \quad i = 1, \ldots, r.
\]

Then an isomorphism \( \phi: (\bigvee Q \otimes \wedge P \otimes \wedge U, d_r \otimes \iota) \to \bigwedge (\bigvee Q \otimes \wedge P \otimes \wedge U, d) \) is given by

\[
\phi \Phi = \Phi, \quad \Phi \in \bigvee Q \otimes \wedge P, \quad \phi u_i = u_i - w_i, \quad i = 1, \ldots, r.
\]

This yields an isomorphism of graded algebras

(13) \( H(\bigvee Q \otimes \wedge P, d_r) \otimes \wedge U \cong H(\bigvee Q \otimes \wedge P \otimes \wedge U, d) \).

On the other hand, let \( A_i \) be the truncated polynomial algebra \( \bigvee (z_i)/z_i^k \) and set \( A = A_1 \otimes \cdots \otimes A_r \). The projection \( \Pi: \bigvee Q \to A \) determines a graded differential algebra \( (A \otimes \wedge P, \tilde{d}) \).

Moreover, \( \Pi \) extends to the homomorphism \( \Pi: (\bigvee Q \otimes \wedge P \otimes \wedge U, d) \to (A \otimes \wedge P, \tilde{d}) \) of graded differential algebras given by \( \Pi(y) = y, \ y \in P \) and \( \Pi(u) = 0, \ u \in U \). Since the restriction of \( \Pi \) to \( \bigvee Q \otimes \wedge U \) induces an isomorphism \( H(\bigvee Q \otimes \wedge U) \to \bigwedge A \), Lemma 1, §2 shows that \( \Pi \) induces an isomorphism of graded algebras

(14) \( \Pi^*: H(\bigvee Q \otimes \wedge P \otimes \wedge U, d) \cong H(A \otimes \wedge P, \tilde{d}) \).

Now \( A \) and \( \wedge P \) are obviously P.d.a.'s of formal dimensions \( a \) and \( d \) given by

\[
a = \sum_{i=1}^{r} (k - 1) \deg z_i \quad \text{and} \quad d = \sum_{i=1}^{s} \deg y_i,
\]

where \( y_1, \ldots, y_s \) is any homogeneous basis of \( P \). Hence \( A \otimes \wedge P \) is a P.d.a. of formal dimension \( a + d \), and \( (A \otimes \wedge P)^{a+d} = A^{a} \otimes \wedge^{s} P \).

Since \( \mathrm{Im} \tilde{d} \subset \sum_{j<s} A \otimes \wedge^j P \) it follows that the elements in \( A^{a} \otimes \wedge^{s} P \) are not coboundaries; hence \( H(A \otimes \wedge P) \) is a P.d.a. of formal dimension \( a + d \). Now the isomorphisms (13) and (14) show that \( H(\bigvee Q \otimes \wedge P, d_r) \) is a P.d.a. of formal dimension

\[
m = a + d - \sum_{i=1}^{r} \deg u_i.
\]

A simple calculation shows now that \( m \) is given by the formula of Theorem 3(i).
This proves parts (i) and (ii) of Theorem 3 for pure towers. (The odd spectral sequence collapses in this case!) Part (iii) follows at once from formula (3) of Theorem 2. Q.E.D.

**Lemma 7.** Let \((\tau; x_1, \ldots, x_n)\) be a tower satisfying the hypotheses of Theorem 3. Then \(H(F(x_1, \ldots, x_n), d_\tau)\) has formal dimension \(m\), where \(m\) is given by Theorem 3(i).

**Proof.** By induction on \(n\). For \(n = 1\), \(\tau = 0\), \(x_1\) has odd degree and the lemma is trivial. Assume it holds for \(n - 1\) and distinguish two cases:

**Case 1.** \(\deg x_1\) is odd. Write \(F(x_1, \ldots, x_n) = \bigwedge x_1 \otimes F(x_2, \ldots, x_n)\) and filter by the ideals \(I^p = \sum_{j > p} (\bigwedge x_1)^j \otimes F(x_2, \ldots, x_n)\). The resulting spectral sequence \(\tilde{E}_i\) satisfies (if \(\deg x_1 > 1\))

\[
\tilde{E}_2^{p,q} = (\bigwedge x_1)^p \otimes H^q(F(x_2, \ldots, x_n), d_\tau).
\]

According to the corollary to Proposition 1, §6, the tower \((\tilde{\tau}; x_2, \ldots, x_n)\) also satisfies the hypotheses of Theorem 3. Thus by the induction hypothesis \(H(F(x_2, \ldots, x_n))\) has formal dimension

\[
l = r - \sum_{i=2}^{n} (-1)^{\deg x_i} \deg x_i = m - \deg x_1.
\]

The formal dimension of \(\bigwedge x_1\) is simply \(\deg x_1 = m - l\). Hence \(\tilde{E}_2^{p,q} = 0\) if \(p > m - l\) or \(q > l\), while \(\tilde{E}_2^{-l,l} \neq 0\). It follows that \(\tilde{E}_2^{p,q} = 0\) if \(p > m - l\) or \(q > l\) and \(\tilde{E}_2^{-l,l} \neq 0\). Hence \(\tilde{E}_\infty\) and so \(H(F(x_1, \ldots, x_n), d_\tau)\) have formal dimension \(m\). The case \(\deg x_1 = 1\) is left to the reader.

**Case 2.** \(\deg x_1\) is even. Choose \(k\) so that \(x_1^k = d_\tau w\) and let \(U = (u)\) be a 1-dimensional graded space with \(\deg u = k \deg x_1 - 1\). Let \(A\) be the truncated polynomial algebra \(\sqrt{(x_1)}/x_1^k\).

The projection \(\sqrt{(x_1)} \to A\) defines a tower \((A, 0; \rho, x_2, \ldots, x_n)\). Moreover a slight modification of the proof of Lemma 6 yields an isomorphism of graded algebras:

\[
H(F(x_1, \ldots, x_n), d_\tau) \otimes \sqrt{(u)} \cong H(A \otimes F(x_2, \ldots, x_n), d_\rho).
\]

Now filter \(A \otimes F(x_2, \ldots, x_n)\) and repeat the argument of Case 1 (with \(A\) replacing \(\sqrt{(x_1)}\)) to complete the proof. Q.E.D.

**Proof of Theorem 3.** (i) Let \((\sigma; x_1, \ldots, x_n)\) be the associated pure tower; according to Proposition 1, §6 it is c-finite. Hence Lemma 6 applies and shows that \(H(F(x_1, \ldots, x_n), d_\sigma)\) is a P.d.a. of formal dimension \(m\). This is therefore true of \(E_1\) as well (cf. §5).

On the other hand by Lemma 7, \(H(F(x_1, \ldots, x_n), d_\tau)\) also has formal dimension \(m\). Since for each \(i\)

\[
\text{formal dim } E_1 \geq \text{formal dim } E_i \geq \text{formal dim } H(F(x_1, \ldots, x_n), d_\tau),
\]

190  STEPHEN HALPERIN
it follows that all the $E_i$ have formal dimension $m$.

(ii) By Lemma 6, $E_1$ is a P.d.a. Since $E_i = H(E_{i-1})$ and since $E_i$ and $E_{i-1}$ have the same formal dimension, an inductive argument shows that each $E_i$ is a P.d.a. Hence $E_\omega$ is a P.d.a. and so $H(F(x_1, \ldots, x_n), d_j)$ is a P.d.a.

(iii) The statement $E^+\equiv 0$, $q < \chi_\Pi$, is Corollary 1 to Theorem 1, §6. Let $q = \chi_\Pi$. Then $E^+_F = H_\chi(\sqrt{Q} \otimes \wedge P, d_0)$ is a nonzero ideal in $E_1$ (cf. Theorem 2, §3). Since $E_1$ is a P.d.a. of formal dimension $m$ (cf. Lemma 6) we have $E^{(m)}_F \subset E^+\equiv$. It follows that if $i > 1$, $0 \neq E^{(m)}_i \subset E^+\equiv$. Q.E.D.

9. Hyperbolic towers. Suppose $A$ is a P.d.a. of even formal dimension $2m$. Then the Poincaré scalar product restricts to a symmetric inner product in the subspace $\Sigma_j A^{2j}$. An inner product space $(X, \langle , \rangle)$ is called hyperbolic if there is a subspace $Y$ such that $\langle y_1, y_2 \rangle = 0$ ($y_i \in Y$) (then $Y$ is called isotropic) and such that $\dim X = 2 \dim Y$. If $\Sigma_j A^{2j}$ is hyperbolic we say $A$ is a hyperbolic P.d.a. Note that this is independent of the choice of basis vector in $A^{2m}$. In this section we prove

**Theorem 4.** Let $(\tau; x_1, \ldots, x_n)$ be a connected, finite, c-finite, minimal tower such that $H(F(x_1, \ldots, x_n), d_j)$ has formal dimension $2m$.

Assume $\chi_\Pi < 0$. Then $H(F(x_1, \ldots, x_n), d_j)$ is a hyperbolic P.d.a. In particular (if $\Gamma \subset \mathbb{R}$) the inner product space $\Sigma_j H^{2j}(F(x_1, \ldots, x_n), d_j)$ has zero signature.

**Corollary.** Let $M$ be a simply connected, compact oriented $4k$-manifold such that $\pi_*(M; \mathbb{Q})$ is finite dimensional. Assume that $H^j(M; \mathbb{Q}) \neq 0$ for some odd $j$. Then $\text{sign}(M) = 0$.

**Proof.** It follows from Theorem 1' that $\chi_\Pi < 0$. Now apply Theorem 4.

**Proof of Theorem 4.** In the next section we show (Proposition 3) that the theorem holds for pure towers. Hence it holds for the $E_i$-term of the odd spectral sequence. On the other hand since $E_i$ and $H(E_i)$ have the same formal dimension $2m$ it follows that there is an isometry $\Sigma_j E^{(2j)}_i = \Sigma_j H^{2j}(E_i) \oplus X$ where $X$ is a hyperbolic inner product space and $\oplus$ means orthogonal direct sum. If $\Sigma_j E^{(2j)}_i$ is hyperbolic this implies that $\Sigma_j H^{2j}(E_i)$ is hyperbolic.

Thus an induction argument shows that $\Sigma_j E^{(2j)}_\omega$ is hyperbolic; the same must then be true for $\Sigma_j H^{2j}(F(x_1, \ldots, x_n), d_j)$. Q.E.D.

**Remark.** Theorem 4 shows that the only “interesting” inner products arise when $\dim P = \dim Q$. In this case (cf. Theorem 5, §11) the Koszul complex is the Koszul complex of a pure tower, totally determined by a linear map $\sigma$: $P \rightarrow \sqrt{Q}$.

It would be interesting and useful to have an explicit means of calculating invariants of the inner product (e.g. signature) directly from $\sigma$. 
10. The pure case. In this section we prove

PROPOSITION 3. Let \((\sqrt{Q} \otimes \wedge P, d_p)\) be the Koszul complex of a connected finite, c-finite, pure tower. Suppose \(H(\sqrt{Q} \otimes \wedge P)\) has formal dimension 2m, and assume \(\dim P > \dim Q\). Then \(\Sigma_j H^{2j}(\sqrt{Q} \otimes \wedge P)\) is hyperbolic.

LEMMA 8. There is a basis \(u_1, \ldots, u_s\) of \(P\) (not necessarily homogeneous) with the following properties: Let \(I_i \subset \sqrt{Q}\) be the ideal generated by \(\sigma(u_1), \ldots, \sigma(u_i)\). Let \(I_0 = 0\). Then
(i) \(\sigma(u_i) \in \sqrt{Q}/I_{i-1}\) is not a zero divisor, \(1 < i < r\), where \(r = \dim Q\).
(ii) \(\dim \sqrt{Q}/I_r < \infty\).

PROOF. We construct \(u_k\) (\(1 < k < r\)) by induction on \(k\) and extend to any basis of \(P\). If \(k = 1\), let \(u_1\) be any nonzero element of \(P\). Now suppose (for some \(k < r\)) \(u_1, \ldots, u_k\) are constructed, and that (i) holds for \(i < k\).

By the Noether decomposition theorem \(I_k\) is the finite irredundant intersection of primary ideals in \(\sqrt{Q}\); denote the associated prime ideals by \(J_1, \ldots, J_l\) (cf. [6, Chapter 4]). Let \(d(J_i)\) be the transcendence degree of \(\sqrt{Q}/J_i\).

Suppose \(J_i\) is not contained in any \(J_j\). Then according to [7, p. 394, Appendix 6], \(J_i\) has height \(k\). Hence by [7, Theorem 20, Chapter 7], \(d(J_i) = r - k\). Thus Macaulay's theorem [7, Theorem 26, Chapter 7] applies and asserts that \(I_k\) is unmixed; i.e., \(d(J_i) = r - k, i = 1, \ldots, l\). We now distinguish two cases:

Case 1. For some element \(u_{k+1}\) of \(P\), \(\sigma(u_{k+1}) \in \sqrt{Q}/I_k\) is not a zero divisor. In this case we have constructed a sequence \(u_1, \ldots, u_{k+1}\) satisfying (i); repeating the argument above yields ideals \(J\) with \(d(J) = r - k - 1\), and so \(r > k + 1\).

Case 2. Every \(u \in P\) yields a zero divisor \(\sigma(u)\) in \(\sqrt{Q}/I_k\). Choose an infinite sequence \(w_1, w_2, \ldots\) of elements in \(P\) such that any subsequence of length \(s\) is a basis (possible because \(\text{char} \Gamma = 0\) and so \(\Gamma\) is infinite). Each \(\sigma(w_i)\) is a zero divisor in \(\sqrt{Q}/I_k\). Hence by [6, Theorem 11, Chapter 4] \(\sigma(w_i) \in \bigcup J_j\). By renumbering the \(J_j\) we can arrange that infinitely many \(\sigma(w_i) \in J_1\).

It follows that \(J_1\) contains \(\sigma(P)\) and so \(\sqrt{Q} \cdot \sigma(P) \subset J_1\). Thus
\[
\dim \sqrt{Q}/J_1 < \dim \sqrt{Q}/\sqrt{Q} \cdot \sigma(P) < \dim H(\sqrt{Q} \otimes \wedge P) < \infty.
\]
It follows that \(d(J_1) = 0\) and so \(k = r\).

Thus \(u_1, \ldots, u_s\) are constructed. Moreover, \(d(J_j) = 0, j = 1, \ldots, l\), and so \(\dim \sqrt{Q}/J_j < \infty, j = 1, \ldots, l\). This implies that \(\dim \sqrt{Q}/I_r < \infty\). Q.E.D.

Now define differential algebras \((A_{p,q}, d)\), \(p < q \leq s\), by \((A_{p,q}, d) = (\sqrt{Q}/I_p, 0)\) and
\[ A_{p,t} = (\bigvee Q/I_p) \otimes (u_{p+1}, \ldots, u_t), \]

\[ d(\Phi \otimes u_{a_0} \wedge \cdots \wedge u_{a_q}) = \sum_{j=0}^{q} (-1)^j \Phi \cdot \sigma(u_{a_j}) \otimes u_{a_0} \wedge \cdots \wedge \hat{u}_{a_j} \wedge \cdots \wedge u_{a_q}, \]

\[ 1 \leq p \leq r. \]

Note that these are not graded differential algebras. Lemma 8 has the following corollary.

**Corollary.** There is an isomorphism of algebras

\[ H(\bigvee Q \otimes \wedge P, d_g) \cong H(\bigvee Q/I_r \otimes (u_{r+1}, \ldots, u_t)) \]

which restricts to isomorphisms

\[ H_k(\bigvee Q \otimes \wedge P) \cong H_k(\bigvee Q/I_r \otimes (u_{r+1}, \ldots, u_t)). \]

**Proof.** Extend the projections \( \Pi : \bigvee Q/I_p \to \bigvee Q/I_{p+1} \) to homomorphisms \( \Pi : \bigvee Q/I_p \otimes (u_{p+1}) \to \bigvee Q/I_{p+1} \) by setting \( \Pi(u_{p+1}) = 0 \). It follows directly from Lemma 8(i) that \( \Pi^* \) is an isomorphism from

\[ H(\bigvee Q/I_p \otimes (u_{p+1})) \]

onto \( \bigvee Q/I_{p+1} \).

Write \( \Pi_k = \Pi \otimes i : A_{p,p+k} \to A_{p+1,p+k} \). Assume by induction on \( k \) that \( \Pi_k^* \) is an isomorphism. Write \( \Pi_{k+1}^* \) in the form

\[ \Pi_{k+1}^* = \Pi_k \otimes i : A_{p,p+k} \otimes (u_{p+k+1}) \to A_{p+1,p+k} \otimes (u_{p+k+1}). \]

Both sides have a Gysin sequence (cf. the example in §2) and so the 5-lemma implies that \( \Pi_{k+1}^* \) is an isomorphism.

In this way we obtain a sequence of isomorphisms

\[ H(\bigvee Q/I_p \otimes (u_{p+1}, \ldots, u_t)) \cong H(\bigvee Q/I_{p+1} \otimes (u_{p+2}, \ldots, u_t)), \]

\[ 0 \leq p < r. \]

Composing them gives the desired isomorphism. Q.E.D.

**Lemma 9.** There is an ideal \( I \subset \bigvee^+ Q \) and a basis (not necessarily homogeneous) \( u_1, \ldots, u_s \) of \( P \) with the following properties: Let \( \rho(u_i) \in \bigvee Q/I \) be the image of \( \sigma(u_i) \) under the projection \( \bigvee Q \to \bigvee Q/I \). Then

(i) \( \bigvee Q/I \) has finite dimension and the elements in \( \bigvee^+ Q/I \) are nilpotent.

(ii) There is an isomorphism of algebras,

\[ \Psi : H(\bigvee Q \otimes \wedge P, d_g) \cong H(\bigvee Q/I \otimes (u_{r+1}, \ldots, u_s), d_p) \]

which restricts to isomorphisms.
\[ H_k \left( \sqrt{Q} \otimes \bigwedge P \right) \cong H_k \left( \sqrt{Q/I} \otimes \bigwedge (u_{r+1}, \ldots, u_s) \right). \]

Proof. Let \( u_1, \ldots, u_s \) be the basis of Lemma 8, and write \( B = \sqrt{Q/I} \).

Multiplication by \( \sigma(u_i) \) is a linear transformation \( \phi_i \) of the finite dimensional commutative algebra \( B \); in particular \( \phi_i \) commutes with multiplication by elements of \( B \). Let \( d = \text{dim } B \).

Then ideals \( K_{r+1}, \ldots, K_s, L_{r+1}, \ldots, L_s \) are defined by the equations:
\[
K_{r+1} = \phi_{r+1}^d(B), \quad \phi_{r+1}^d(L_{r+1}) = 0, \quad B = K_{r+1} \oplus L_{r+1},
\]
and for \( i > r + 1, \)
\[
K_i = \phi_i^d(L_{i-1}), \quad \phi_i^d(L_i) = 0, \quad L_{i-1} = K_i \oplus L_i.
\]
\( \phi_i \) restricts to an automorphism of \( K_i \) while each \( \phi_i \) is nilpotent in \( L_s \).

Moreover \( B = K_{r+1} \oplus \cdots \oplus K_s \oplus L_s \).

Now let \( I \) be the inverse image of \( K_{r+1} \oplus \cdots \oplus K_s \) under the canonical projection \( \sqrt{Q} \to B \). For \( z \in Q \) we know that some \( z^k \in \sqrt{Q} \cdot \sigma(P) \). It follows that if \( \tilde{z} \) is the image of \( z \) in \( B \) then multiplication by \( \tilde{z} \) is nilpotent in \( L_s \). Hence \( \tilde{z}^j \in K_{r+1} \oplus \cdots \oplus K_s \) for some \( j \) and so \( z^j \in I \). Thus the elements of \( \sqrt{+Q} \) determine nilpotent elements in \( \sqrt{Q/I} \). This implies (clearly) that \( I \subset \sqrt{+Q} \), and (i) is proved.

To prove (ii) we need only show that projection
\[
\sqrt{Q/I} \otimes \bigwedge (u_{r+1}, \ldots, u_s) \to \sqrt{Q/I} \otimes \bigwedge (u_{r+1}, \ldots, u_s)
\]
induces an isomorphism in cohomology. Since \( K_i, L_i \) are ideals, we have
\[
H \left( B \otimes \bigwedge (u_{r+1}, \ldots, u_s) \right) = \sum_{i=r+1}^s H \left( K_i \otimes \bigwedge (u_{r+1}, \ldots, u_s) \right)
\]
\[
\oplus H \left( L_i \otimes \bigwedge (u_{r+1}, \ldots, u_s) \right).
\]

Now \( \phi_i \) is multiplication by a coboundary, hence it induces zero in cohomology. On the other hand each \( K_i, L_i \) is stable under \( \phi_i \) and \( \phi_i \) is an isomorphism in \( K_i \). Hence \( \phi_i \) induces an isomorphism in
\[
H \left( K_i \otimes \bigwedge (u_{r+1}, \ldots, u_s) \right).
\]
This implies that \( H \left( K_i \otimes \bigwedge (u_{r+1}, \ldots, u_s) \right) = 0, \ i = r + 1, \ldots, s \), and so (ii) follows from formula (15). Q.E.D.

Denote \( \sqrt{Q/I} \) by \( A \), \( \sqrt{+Q/I} \) by \( A^+ \). (But note that \( A \) is not graded!) Let \( K \subset A \) be the subspace of elements \( x \) such that \( x \cdot A^+ = 0 \). Finally, denote \((u_{r+1}, \ldots, u_s)\) by \( U \). Then clearly
\[
K \otimes u_{r+1} \wedge \cdots \wedge u_s \subset (A \otimes \bigwedge^+ U) \cap \ker d_p \subset H_{s-r} \left( A \otimes \bigwedge^i U \right).
\]
(The last inclusion is an inclusion because \( \text{Im } d_p \subset \Sigma_{j<s} A \otimes \bigwedge^j U \).)

Corollary. The space \( K \) satisfies \( \text{dim } K = 1. \) If \( 0 \neq a \in K \) then the class \([a \otimes u_{r+1} \wedge \cdots \wedge u_s] \) corresponds under the isomorphism \( \Psi \) of Lemma 9 to
an element of $H^{2m}(\vee Q \otimes \wedge P)$. (Recall that $2m$ is the formal dimension of $H(\vee Q \otimes \wedge P)$.)

**Proof.** It follows from Lemma 9(i) that $K \neq 0$. Let $a \in K$. Then $(a \otimes u_{r+1} \wedge \cdots \wedge u_s) \cdot (A^{+} \otimes \wedge U + A \otimes \wedge^{+} U) = 0$. If $a \in H(\vee Q \otimes \wedge P)$ is defined by $\Psi(a) = [a \otimes u_{r+1} \wedge \cdots \wedge u_s]$, then this equation implies that $a \cdot H^{+}(\vee Q \otimes \wedge P) = 0$. But this condition characterizes $H^{2m}(\vee Q \otimes \wedge P)$, and $\dim H^{2m}(\vee Q \otimes \wedge P) = 1$ by Lemma 6, §8. Q.E.D.

**Lemma 10.** There is a subspace $X \subset \Sigma_{j}H^{2j}(\vee Q \otimes \wedge P)$ with the following properties:

(i) $2 \dim A = \dim(\Sigma_{j}H^{2j}(\vee Q \otimes \wedge P))$.

(ii) $X \cdot X \cap H^{2m}(\vee Q \otimes \wedge P) = 0$.

**Proof.** Clearly $\Sigma_{j}H^{2j}(\vee Q \otimes \wedge P) = \Sigma_{j}H_{2j}(\vee Q \otimes \wedge P)$. Thus we need only find a subspace $Z \subset \Sigma_{j}H_{2j}(A \otimes \wedge U)$ such that $2 \dim Z = \dim(\Sigma_{j}H_{2j}(A \otimes \wedge U))$ and $Z \cdot Z \cap K \otimes \wedge^{s-r} U = 0$ (cf. Lemma 9 and its corollary).

Furthermore, since $H^{2m}(\vee Q \otimes \wedge P) \subset H_{s-r}(\vee Q \otimes \wedge P)$, it follows that $s - r = 2k$.

Choose a subspace $N \subset A$ so that $A = N \oplus K = N \oplus (a)$ (a, a basis vector for $K$). Define a bilinear function $\langle , , \rangle: A \times A \to \Gamma$ by $a_{1}a_{2} = \langle a_{1}, a_{2} \rangle a \in N$. Since the elements of $A^{+}$ are nilpotent, and since $a$ is a basis for $K$, it follows easily that this is a nondegenerate inner product in $A$.

Next assign $\wedge U$ the standard Poincaré scalar product determined by the basis vector $u_{r+1} \wedge \cdots \wedge u_{s}$ in $\wedge^{s-r} U$:

$$\Phi \wedge \Psi - \langle \Phi, \Psi \rangle u_{r+1} \wedge \cdots \wedge u_{s} \in \sum_{j<s-r} \wedge^{j} U.$$ 

These two scalar products define a scalar product $\langle , , \rangle$ in $A \otimes \wedge U$, for which $\langle A \otimes \wedge^{j} U, A \otimes \wedge^{l} U \rangle = 0$ unless $j + l = s - r$. In particular, $\langle \Im d_{p}, 1 \rangle = 0$.

A simple calculation shows as well that

$$\langle \Phi, \Psi \rangle = \langle \Phi \cdot \Psi, 1 \rangle, \quad \Phi, \Psi \in A \otimes \wedge U,$$

whence

$$\langle d_{p} \Phi, \Psi \rangle + (-1)^{p} \langle \Phi, d_{p} \Psi \rangle = 0,$$

$$\Phi \in A \otimes \wedge^{p} U, \Psi \in A \otimes \wedge U.$$ 

Thus the scalar product of two cocycles depends only on their respective cohomology classes, and so a scalar product is induced in $H(A \otimes \wedge U)$. It satisfies
Moreover $\langle H_j(A \otimes \bigwedge U), H_l(A \otimes \bigwedge U) \rangle = 0$ if $j + l \neq s - r$; since $s - r = 2k$ the spaces $\Sigma_j A \otimes \bigwedge^{2j} U$ and $\Sigma_j H_{2j}(A \otimes \bigwedge U)$ are inner product spaces.

Now choose subspaces $C_j \subset A \otimes \bigwedge^j U$ such that $C_j \oplus d_p(A \otimes \bigwedge^{j+1} U) = \ker d_p \cap (A \otimes \bigwedge^j U)$. Then the restriction of $\langle , \rangle$ to $\Sigma_j C_{2j}$ is nondegenerate and the inner product spaces $\Sigma_j C_{2j}$ and $\Sigma_j H_{2j}(A \otimes \bigwedge U)$ are isometric. Write

$$\sum_j A \otimes \bigwedge^{2j} U = \sum_j C_{2j} \oplus \left( \sum_j C_{2j} \right)'.$$

The left-hand side is obviously hyperbolic. Moreover, $\Sigma_j d_p(A \otimes \bigwedge^{2j+1} U)$ is an isotropic subspace of $\left( \sum_j C_{2j} \right)'$ and

$$2 \dim \sum_j d_p(A \otimes \bigwedge^{2j+1} U) = \dim \left( \sum_j C_{2j} \right)' .$$

Hence $\left( \sum_j C_{2j} \right)'$ is hyperbolic.

It follows that $\Sigma_j C_{2j}$ is hyperbolic; hence so is $\Sigma_j H_{2j}(A \otimes \bigwedge U)$. Choose an isotropic subspace $Z \subset \Sigma_j H_{2j}(A \otimes \bigwedge U)$ such that $2 \dim Z = \dim (\Sigma_j H_{2j}(A \otimes \bigwedge U))$. Formula (18) implies that $Z \cdot Z \cap (K \otimes u_{r+1} \wedge \cdots \wedge u_0) = 0$. Q.E.D.

**Proof of Proposition 3.** Let $X$ be the subspace of Lemma 10. Then there is a basis, $\alpha_1, \ldots, \alpha_N$ of $X$ with the following property: There are linearly independent elements $\beta_1, \ldots, \beta_N$ in $\Sigma_j H^{2j}(\bigvee Q \otimes \bigwedge P)$ such that

(i) $\beta_i$ is homogeneous.

(ii) $\beta_i - \beta_i \in \Sigma_j H_j(\bigvee Q \otimes \bigwedge P)$ ($|\beta_i| = \deg \beta_i$).

Now let $\langle , \rangle$ denote the Poincaré scalar product in $H(\bigvee Q \otimes \bigwedge P)$. Then $\langle \beta_i, \beta_j \rangle = 0$ if $|\beta_i| + |\beta_j| < 2m$. On the other hand, if $|\beta_i| + |\beta_j| = 2m$ then (ii) implies that $\langle \beta_i, \beta_j \rangle \epsilon = \beta_i \cdot \beta_j = \alpha_i \cdot \alpha_j$. Since

$$X \cdot X \cap H^{2m}(\bigvee Q \otimes \bigwedge P) = 0$$

this equation implies $\alpha_i \alpha_j = 0$; i.e. $\langle \beta_i, \beta_j \rangle = 0$ if $|\beta_i| + |\beta_j| = 2m$. Thus the $\beta_j$ span an isotropic space $Y \subset \Sigma_j H^{2j}(\bigvee Q \otimes \bigwedge P)$. Since

$$\dim Y = \dim X = \frac{1}{2} \sum_j \dim H^{2j}(\bigvee Q \otimes \bigwedge P),$$

the inner product space $\Sigma_j H^{2j}(\bigvee Q \otimes \bigwedge P)$ is hyperbolic. Q.E.D.

11. **The case that $X_{\Pi} = 0$.** The object of this section is to establish

**Theorem 5.** Let $(\tau; x_1, \ldots, x_n)$ be a connected, finite, e-finite, minimal tower. Assume $X_{\Pi} = 0$. Then the Koszul complex of the tower and the Koszul complex of the associated pure tower are isomorphic as graded differential
algebras: $(\bigvee Q \otimes \land P, d_i) \cong (\bigvee Q \otimes \land P, d_i)$.

Throughout the section $(\tau; x_1, \ldots, x_n)$ denotes a fixed tower satisfying the hypotheses of the theorem; $R = (x_1, \ldots, x_n); F(R) = \bigvee Q \otimes \land P; (\sigma; x_1, \ldots, x_n)$ is the associated pure tower. To establish the theorem we may assume without loss of generality that

$$\text{(19) } \deg x_1 < \deg x_2 < \ldots .$$

We assume this throughout the section.

**Lemma 11.** Suppose $x_i$ has even degree. Then for some

$$u_i \in F^+(x_1, \ldots, x_{i-1}) \cdot F^+(x_1, \ldots, x_{i-1}), \quad d_i(x_i + u_i) = 0.$$

**Proof.** By Corollary 4, §6, the odd spectral sequence collapses at the $E_1$-term. Moreover $d_0(x_i) = 0$. Thus $x_i$ represents an element in $E_1^{p,0} (p = \deg x_i)$. It follows that $d_i(x_i + u_i) = 0$ for some $u_i \in \sum_{j>0} F(R)^{i+p-j}$. But $F(R)^{i+p-j} \subset \bigvee Q \otimes \land^j P$. Since $u_i$ has even degree $p$ this gives

$$u_i \in \sum_{j>2} \bigvee Q \otimes \land^j P \subset F^+(R) \cdot F^+(R).$$

It follows now from (19) that $u_i \in F^+(x_1, \ldots, x_{i-1}) \cdot F^+(x_1, \ldots, x_{i-1})$.

Q.E.D.

Now define an automorphism $\phi$ of the graded algebra $F(R)$ by setting

$$\phi x_i = x_i \quad (x_i \in P) \quad \text{and} \quad \phi x_i = x_i + u_i \quad (x_i \in Q).$$

Then $\phi$ restricts to automorphisms of each $F(x_1, \ldots, x_i)$. Hence a tower $(\rho; x_1, \ldots, x_n)$ is defined by

$$\rho(x_i) = \phi^{-1}d\phi(x_i), \quad i = 1, \ldots, n.$$

Lemma 11 yields

$$\text{(20) } \rho(x) = 0, \quad x \in Q.$$

Clearly $\phi: (F(R), d_i) \rightarrow (F(R), d_i)$ is an isomorphism of graded differential algebras. It follows (cf. §2) that $(\rho; x_1, \ldots, x_n)$ is a connected, finite, $c$-finite, minimal tower, with zero homotopy Euler characteristic. In view of (20) this tower can be rearranged (cf. §2) in the form $(\rho; z_1, \ldots, z_m, y_1, \ldots, y_m)$, where the $z_i$ are a basis of $Q$ and the $y_i$ a basis of $P$. ($z_1, \ldots, y_m$ is a permutation of $x_1, \ldots, x_n$.)

Let $(\lambda; z_1, \ldots, z_m, y_1, \ldots, y_m)$ be the associated pure tower. Proposition 1, applied to $(\rho; x_1, \ldots, x_n)$ shows that $H(\bigvee Q \otimes \land P, d_\lambda)$ has finite dimension. Hence Theorem 2 implies that $H_+(\bigvee Q \otimes \land P, d_\lambda) = 0$. Let $P_i$ be the subspace of $P$ spanned by $y_1, \ldots, y_i$. Lemma 2, §3 implies now that for each $i, H_+(\bigvee Q \otimes \land P, d_\lambda) = 0$.

**Lemma 12.** For each $i$ the inclusion $\theta: \bigvee Q \rightarrow \bigvee Q \otimes \land P_i$ induces a surjective homomorphism $\theta^*: \bigvee Q \rightarrow H(\bigvee Q \otimes \land P_i, d_\rho)$. 

Finiteness in the Minimal Models of Sullivan 197
Proof. Since \( H_+(\bigwedge Q \otimes \bigwedge P, d_\lambda) = 0, \) \( H(\bigwedge Q \otimes \bigwedge P, d_\lambda) \) is evenly graded. Thus if \((F_k, d_k)\) is the odd spectral sequence for \((\rho; z_1, \ldots, z_m, y_1, \ldots, y_m)\), \( F_1 = H(\bigwedge Q \otimes \bigwedge P, d_\lambda) \) is evenly graded. Hence \( F_1 = F_\infty \).

Now filter \( \bigwedge Q \) by the ideals \( I^p = \sum_{j \geq p} (\bigwedge Q)^j \). The corresponding spectral sequence is given \( \tilde{F}_k = \bigwedge Q, \tilde{d}_k = 0 \).

Observe that \( \theta \) is filtration preserving and so induces a homomorphism \( \theta : F_k \to F_k \) of spectral sequences. In particular, \( \theta \): \( \bigwedge Q \to H(\bigwedge Q \otimes \bigwedge P, d_\lambda) \) is surjective, since \( H_+(\bigwedge Q \otimes \bigwedge P, d_\lambda) = 0 \). But \( F_1 = F_\infty, \tilde{F}_1 = \tilde{F}_\infty \); thus \( \theta_\infty = \theta_1 \) and so \( \theta_\infty \) is surjective. This implies at once that \( \theta^* \) is surjective.

Q.E.D.

Proof of Theorem 5. We continue the notation developed above. Since \((\rho; z_1, \ldots, z_m, y_1, \ldots, y_m)\) is a tower it follows that
\[
(\bigwedge Q \otimes \bigwedge P_{i-1}), \quad i = 1, 2, \ldots, m.
\]

In view of Lemma 12 above we can write
\[
(\bigwedge Q \otimes \bigwedge P_{i-1}), \quad i = 1, 2, \ldots, m.
\]

Define an automorphism \( \Psi \) of the graded algebra \( \bigwedge Q \otimes \bigwedge P \) by setting
\[
\Psi(z_i) = z_i, \quad \Psi(y_i) = y_i - w_i, \quad i = 1, \ldots, m.
\]

Then define \( \gamma : \bigwedge Q \otimes \bigwedge P \) by
\[
(21) \quad \gamma(z_i) = v_i \quad \text{and} \quad \gamma(z_i) = 0.
\]

It follows from the definition that \( \Psi d_\psi = d_\rho \Psi \).

In particular, \( \text{Im} \gamma \subset F^+(R) \cdot F^+(R) \). In view of (21) this implies that \( (\gamma; x_1, \ldots, x_n) \) is a pure, minimal tower. Thus it coincides with the associated pure tower.

Now consider the isomorphism \( \phi \circ \Psi : (F(R), d_\rho) \to \mathcal{K}(\rho, (F(R), d_\rho)) \) of Koszul complexes. According to §5 it induces an isomorphism of the odd spectral sequences. The isomorphism of the \( E_0 \)-terms can be written \( \alpha : (F(R), d_\rho) \to \mathcal{K}(F(R), d_\rho) \). Thus \( \phi \circ \Psi \circ \alpha^{-1} : (F(R), d_\rho) \to \mathcal{K}(F(R), d_\rho) \) is the desired isomorphism. Q.E.D.

Corollary 1. \( H^*(F(R), d_\rho) \cong \bigwedge Q / \bigwedge Q \cdot \sigma(P) \).

Proof. Apply Theorem 2.

Corollary 2. Suppose the bases \( y_i \) of \( P \) and \( z_i \) of \( Q \) satisfy \( \deg y_i = g_i, \deg z_i = k_i \). Then
\[
\sum_p \dim H^p (F(R), d_\rho) t^p = \prod_{i=1}^m (1 - t^i)^{-1} \prod_{i=1}^m (1 - t^{2i})^{-1}.
\]

Moreover the Euler characteristic, \( \chi_c \) (equals the dimension of the cohomology) is given by formula \( \chi_c = ((g_1 + 1) \ldots (g_m + 1))/(k_1 \ldots k_m) \).

Proof. See [4] or [3, Chapter 2].
Finiteness in the minimal models of Sullivan

References


