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CONJUGATE POINTS OF VECTOR-MATRIX DIFFERENTIAL EQUATIONS

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ABSTRACT. The system of equations

$$\sum_{k=0}^{n}(-1)^{n-k}\left(P_k(x)y^{(n-k)}(x)\right)^{(n-k)}=0 \qquad (0<x<\infty)$$

is considered where the coefficients are real, continuous, symmetric matrices, y is a vector, and $P_0(x)$ is positive definite.

It is shown that the well-known quadratic functional criterion for existence of conjugate points for this system can be further utilized to extend results of the associated scalar equation to the vector-matrix case, and in some cases the scalar results are also improved. The existence and nonexistence criteria for conjugate points of this system are stated in terms of integral conditions on the eigenvalues or norms of the coefficient matrices.

Differential vector operators generated by the system

$$\sum_{k=0}^{n}(-1)^{n-k}\left(P_k(x)y^{(n-k)}(x)\right)^{(n-k)}=\lambda y \qquad (0<x<\infty), \tag{1}$$

are considered where the coefficients are real, continuous, $m\times m$, symmetric matrices, $y(x)$ is an m-dimensional vector, and $P_0(x)$ is positive definite.

The purpose of this paper is to extend some of the results concerning the oscillation and nonoscillation of the scalar differential equation associated with (1) or, equivalently, the existence and nonexistence of conjugate points $\eta(a)$ for all $a>0$.

V. V. Martynov [11] studies the spectral properties of the even order, two-term, vector operator and extends many of the theorems obtained in the scalar case. A result of this paper improves a condition for nonoscillation given by Martynov. Theorems of Ahlbrandt [1] and Kaufman and Sternberg [7] are also improved. Other papers connected with this topic are those of Etgen [2], [3], Howard [6], Kreith [8], Noussair and Swanson [14], and Tomastik [17], [18] where primary attention is given to the oscillation of a second order matrix differential equation, which is equivalent to the oscilla-

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tion of the associated vector-matrix differential equation. Under certain conditions some of these results are extended and improved here. The books of W. T. Reid [15] and Gelfand and Fomin [4] are suggested for elaboration on the history of the problem and the associated calculus of variations considerations.

1. **Introduction.** Let L be generated by the differential expression

$$l(y) = \sum_{k=0}^{n} (-1)^{n-k}\left(P_k y^{(n-k)}\right)^{(n-k)} \tag{2}$$

in the Hilbert space of real vector-valued functions $y(x) = (y_1(x), \ldots, y_m(x))^*$ with inner product

$$[a, b] = \int_{-\infty}^{\infty} (a, b)\, dx = \int_{-\infty}^{\infty} \sum_{k=1}^{m} a_k(x) b_k(x)\, dx.$$

We denote any selfadjoint extension of L by $\tilde{L}$.

If there is a number $b > a$ such that $l(y) = 0$ has a nontrivial solution satisfying

$$y^{(i)}(a) = 0 = y^{(i)}(b) \qquad (0 \leqslant i \leqslant n-1),$$

then b is called a *conjugate point of* a and the least such b is denoted by $\eta(a)$. The system $l(y) = 0$ is said to be *oscillatory* if any positive number a has an associated conjugate point $\eta(a)$. Otherwise, $l(y) = 0$ is said to be *nonoscillatory*.

I. M. Glazman [5, pp. 34, 43, 95] has shown that the set of points in the spectrum of $\tilde{L}$ lying to the left of λ_0 will be finite or infinite depending on whether system (1) with the $\lambda = \lambda_0$ is nonoscillatory or oscillatory, respectively.

Let $\mathfrak{D}_N(L)$ denote the set of all vector-valued functions, $y(x)$, that have compact support in (N, ∞), a continuous $n-1$ derivative, and a piecewise continuous derivative of order n.

By fixing $\lambda = 0$, we have the following equivalent condition to the nonoscillation of (1) due to Glazman [5, pp. 15, 34, 43]. The theorem also follows as a corollary to Theorem 5.1 of Reid [15, p. 337].

THEOREM 1.1. *The system* (1) *is nonoscillatory if, and only if for some* $N > 0$ *the functional*

$$\int_{N}^{\infty} \sum_{k=0}^{n} \left(P_k(x) y^{(n-k)}(x), y^{(n-k)}(x)\right) dx$$

is nonnegative for all $y \in \mathfrak{D}_N(L)$.

The functional above is the inner product $[L(y), y]$.

We say that $A \geqslant B$ if and only if $A - B$ is positive semidefinite. The following comparison theorem is a direct corollary of Theorem 1.1.

COROLLARY 1.1. *Let $l(y)$ denote the differential expression* (2) *and $l_2(y)$ denote the differential expression* (2) *with each P_k replaced by Q_k. Suppose $P_k(x) \geqslant Q_k(x)$ for each k and all x, then $l(y)$ is nonoscillatory if $l_2(y)$ is nonoscillatory.*

By $\|\xi\|$, we mean the Euclidean vector norm $\|\xi\| = (\xi_1^2 + \cdots + \xi_m^2)^{1/2}$.

LEMMA 1.1. *If the vector $\xi(x)$ has a continuous derivative on $[a, b]$ and vanishes at a and b, then*

$$(\alpha+1)^2 \int_a^b x^\alpha \|\xi(x)\|^2\,dx \leqslant 4\int_a^b x^{\alpha+2}\|\xi'(x)\|^2\,dx \tag{3}$$

for any constant α.

PROOF. We may assume that $\alpha \neq -1$ since (3) is obvious when $\alpha = -1$. By using the Cauchy-Schwarz inequality, we have that

$$\begin{aligned}
\int_a^b x^\alpha\|\xi(x)\|^2\,dx &= \int_a^b (\xi(x), \xi(x))\left(\int_a^x t^\alpha\,dt\right)' dx \\
&= \frac{-2}{\alpha+1}\int_a^b (\xi(x), \xi'(x))x^{\alpha+1}\,dx \\
&\leqslant \frac{2}{|\alpha+1|}\int_a^b x^{\alpha/2}\|\xi(x)\|\cdot x^{\alpha/2+1}\|\xi'(x)\|\,dx \\
&\leqslant \frac{2}{|\alpha+1|}\left[\int_a^b x^\alpha\|\xi(x)\|^2\,dx\right]^{1/2}\left[\int_a^b x^{\alpha+2}\|\xi'(x)\|^2\,dx\right]^{1/2}.
\end{aligned}$$

Inequality (3) now follows by squaring both sides of the above inequality.

LEMMA 1.2. *Define*

$$J(n) = \frac{2^{n+1}}{1\cdot 3\cdot \cdots \cdot (2n-1)} + \sum_{i=1}^{n-1}\binom{n}{i}\frac{2^n}{(1\cdot 3\cdot\cdots\cdot(2i-1))(1\cdot 3\cdot \cdots \cdot (2(n-i)-1))}$$

for $n > 1$ and $J(1) = 4$. For all positive integers n, $J(n) = 2^{4n-1}n!/(2n)!$.

PROOF. A simplification shows that for $n > 1$

$$J(n) = \left[2^{2n}n!/(2n)!\right]\sum_{i=0}^{n}\binom{2n}{2i}.$$

Since

$$0 = (1-1)^{2n} = \sum_{i=0}^{2n} (-1)^i \binom{2n}{i} = \sum_{i=0}^{n} \binom{2n}{2i} - \sum_{i=1}^{n} \binom{2n}{2i-1},$$

then

$$2^{2n-1} = \tfrac{1}{2}(1+1)^{2n} = \frac{1}{2}\left[\sum_{i=0}^{n}\binom{2n}{2i} + \sum_{i=1}^{n}\binom{2n}{2i-1}\right] = \sum_{i=0}^{n}\binom{2n}{2i}.$$

Hence, $J(n) = 2^{4n-1}n!/(2n)!$. We remark that $J(n+1) = 8\,J(n)/(2n+1)$.

2. **Nonoscillation.** The theorem which follows is an extension of a theorem by the author [9] to the vector-matrix system (1). Theorem 6 of Martynov [11] follows as a corollary.

We let $\|A\|$ denote the norm of the matrix A induced by the Euclidean vector norm, $\|A\| = \sup_{\|\xi\|=1}\|A\xi\|$, and let $\|A\|_E$ denote the Euclidean matrix norm, $\|A\|_E = [\sum_{i,j=1}^{m}|a_{ij}|^2]^{1/2}$. Let $P_k^0(x) = P_k(x)$ and for $i \geq 1$ define

$$P_k^i(x) = \int_x^\infty P_k^{i-1}(t)\,dt$$

when the integral exists.

THEOREM 2.1. *Suppose $P_0(t) \equiv I$, the identity matrix, and for $k = 1, \ldots, n$ and $i = 0, 1, \ldots, k-1$, $-\infty < \int_a^\infty P_k^i(t)\,dt < \infty$. If there are numbers δ_k such that $\sum_{k=1}^n \delta_k J(k) = 1$ and $x^k\|P_k^k(x)\| \leq \delta_k$ for all $x \geq a$, then the system $l(y) = 0$ is nonoscillatory.*

PROOF. Note that

$$\int (A'(t)x(t), y(t))\,dt$$
$$= (A(t)x(t), y(t)) - \int (A(t)x(t), y'(t))\,dt - \int (A(t)x'(t), y(t))\,dt.$$

Hence, for any $y \in \mathfrak{D}_N(L)$ and each k

$$-\int_N^\infty (P_k(t)y^{(n-k)}(t), y^{(n-k)}(t))\,dt$$
$$= -\sum_{i=0}^{k}\binom{k}{i}\int_N^\infty (P_k^k(t)y^{(n-k+i)}(t), y^{(n-i)}(t))\,dt$$
$$\leq \sum_{i=0}^{k}\binom{k}{i}\int_N^\infty \frac{\|y^{(n-k+i)}\|}{t^{k-i}}\cdot\frac{\|y^{(n-i)}\|}{t^i}\,t^k\|P_k^k\|\,dt$$

$$\leqslant \delta_k \sum_{i=0}^{k} \binom{k}{i} \left[\int_N^\infty t^{-2(k-i)} \| y^{(n-k+i)} \|^2 \, dt \right]^{1/2} \left[\int_N^\infty t^{-2i} \| y^{(n-i)} \|^2 \, dt \right]^{1/2}$$

$$= \delta_k \left\{ 2 \left[\int_N^\infty t^{-2k} \| y^{(n-k)} \|^2 \, dt \right]^{1/2} \left[\int_N^\infty \| y^{(n)} \|^2 \, dt \right]^{1/2} \right.$$

$$+ \sum_{i=1}^{k-1} \binom{k}{i} \left[\int_N^\infty t^{-2(k-i)} \| y^{(n-k+i)} \|^2 \, dt \right]^{1/2}$$

$$\left. \cdot \left[\int_N^\infty t^{-2i} \| y^{(n-i)} \|^2 \, dt \right]^{1/2} \right\}$$

$$\leqslant \delta_k J(k) \int_N^\infty \| y^{(n)} \|^2 \, dt$$

by the Cauchy-Schwarz inequality and Lemma 1.1. Summing both sides over k yields

$$- \sum_{k=1}^{n} \int_N^\infty \left(P_k(t) y^{(n-k)}(t), y^{(n-k)}(t) \right) dt \leqslant \int_N^\infty \| y^{(n)}(t) \|^2 \, dt.$$

The conclusion follows by Theorem 1.1.

Corollary 2.1. *If $P_0(t) \equiv I$ and for $k = 1, \ldots, n$ and some $c > 0$*

$$\int_c^\infty x^{2k-1} \| P_k(x) \| \, dx < \infty$$

then the system $l(y) = 0$ is nonoscillatory.

Proof. By the hypothesis, for any arbitrarily small $\varepsilon > 0$ we know that $\int_t^\infty \|P_k(x)\| \, dx \leqslant \varepsilon t^{1-2k}$ for large t. This implies that $\|P_k^1(t)\| \leqslant \varepsilon t^{1-2k}$ which implies that

$$\int_t^\infty \| P_k^1(x) \| \, dx \leqslant \varepsilon \cdot t^{2-2k} / (2k-2).$$

Continuing this procedure we can show that $\|P_k^k(t)\| \leqslant \delta_k t^{-k}$ by choosing ε correctly. Now, the conclusion follows from Theorem 2.1.

The next corollary of Theorem 2.1 is Theorem 6 of Martynov [11]. For each symmetric matrix Q we define $Q^- = \frac{1}{2}(Q - |Q|) \leqslant 0$ where $|Q|$ is the arithmetic square root of Q^2 that has the same rank as Q^2 and is positive semidefinite.

Corollary 2.2. *If*

$$\limsup_{x \to \infty} x^{2n-1} \left\| \int_x^\infty Q^-(t) \, dt \right\| < \frac{\alpha_n^2}{2n-1}$$

where $\alpha_n = (1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1))/2^n$ then the system

$$(-1)^n y^{(2n)} + Qy = 0$$

is nonoscillatory.

PROOF. There is some number $a > 0$ such that $x \geqslant a$ implies that

$$\left\| \int_x^\infty Q^-(t)\, dt \right\| < \frac{\alpha_n^2}{2n-1} x^{1-2n}.$$

If we let $P_n^0(t) = Q^-(t)$ then

$$\left\| \int_x^\infty P_n^1(t)\, dt \right\| \leqslant \int_x^\infty \|P_n^1(t)\|\, dt$$

$$= \int_x^\infty \left\| \int_t^\infty Q^-(s)\, ds \right\| dt < \frac{\alpha_n^2}{(2n-1)(2n-2)} x^{2-2n}.$$

Continuing this procedure it follows that

$$\left\| \int_x^\infty P_n^{n-1}(t)\, dt \right\| < \frac{\alpha_n^2}{(2n-1)(2n-2)\cdot\cdots\cdot(n)} x^{-n}$$

or

$$x^n \|P_n^n(x)\| < 1/J(n).$$

By Theorem 2.1 the system $(-1)^n y^{(2n)} + Q^- y = 0$ is nonoscillatory which, according to Corollary 1.1, completes the proof.

Since $\|A\| \leqslant \|A\|_E$ for any matrix A, then it follows that Theorem 2.1 and its corollaries also are true when we replace the induced norm with the Euclidean norm.

THEOREM 2.2. *Suppose $P_0(t) \geqslant Ct^\delta I$ for constants C and δ. If the integral $\int_0^\infty t^{2k-\delta-1} P_k(t)\, dt$ exists for each $k \geqslant 1$, then the system $l(y) = 0$ is nonoscillatory.*

PROOF. For any $N > 0$ and $y \in \mathfrak{D}_N(L)$ the following holds for each $k \geqslant 1$:

$$-\int_N^\infty (P_k(t)y(t), y(t))\, dt$$

$$= (1 - 2k + \delta) \int_N^\infty t^{\delta-2k} \left(\int_t^\infty x^{2k-\delta-1} P_k(x)\, dx\, y(t), y(t) \right) dt$$

$$- 2 \int_N^\infty t^{\delta+1-2k} \left(\int_t^\infty x^{2k-\delta-1} P_k(x)\, dx\, y(t), y'(t) \right) dt$$

$$\leqslant \int_N^\infty \left\| \int_t^\infty x^{2k-\delta-1} P_k(x)\, dx \right\|_E$$

$$\cdot \left(|1 - 2k + \delta| t^{\delta-2k} \|y(t)\|^2 + 2 \frac{\|y(t)\|}{t^{k-\delta/2}} \frac{\|y'(t)\|}{t^{k-1-\delta/2}} \right) dt$$

by the Cauchy-Schwarz inequality. For an arbitrarily small $\varepsilon > 0$ we choose N so large that for $t \geqslant N$

$$\left\| \int_t^\infty x^{2k-\delta-1} P_k(x)\, dx \right\|_E < \varepsilon.$$

Consequently, by summing over $k \geqslant 1$ and using the Cauchy-Schwarz inequality we obtain the inequality

$$-\sum_{k=1}^{n} \int_N^\infty (P_k(t)y(t), y(t))\, dt \leqslant \varepsilon \sum_{k=1}^{n} \int_N^\infty |1 - 2k + \delta| t^{\delta-2k} \|y(t)\|^2\, dt$$

$$+ 2\varepsilon \sum_{k=1}^{n} \left[\int_N^\infty t^{\delta-2k} \|y(t)\|^2\, dt \right]^{1/2} \left[\int_N^\infty t^{\delta+2-2k} \|y'(t)\|^2\, dt \right]^{1/2}.$$

By using Lemma 1.1 repeatedly, we have that for some constant B

$$-\sum_{k=1}^{n} \int_N^\infty (P_k(t)y(t), y(t))\, dt \leqslant \varepsilon B \int_N^\infty C\, t^{\delta} \|y^{(n)}(t)\|^2\, dt$$

$$< \int_N^\infty \left(P_0(t) y^{(n)}(t), y^{(n)}(t)\right) dt$$

if we choose $\varepsilon < 1/B$. By Theorem 1.1, the proof is complete.

Theorem 2.2 improves a theorem of Kaufman and Sternberg [7], and Theorem 6.1 of Ahlbrandt [1], which is concerned only with the scalar differential equation, is improved and extended.

Finally, since $|a_{ij}| \leqslant \|A\|_E$ for any matrix A, the next result follows easily from Theorem 2.2.

Corollary 2.3. *Suppose $P_0(t) \geqslant Ct^{\delta} I$ for constants C and δ. If*

$$\lim_{x\to\infty} \left\| \int_0^x t^{2k-\delta-1} P_k(t)\, dt \right\| < \infty$$

for each $k > 1$, then the system $l(y) = 0$ is nonoscillatory.

When the limit of Corollary 2.3 does not exist, it is still possible to establish nonoscillation criteria as evidenced by the next theorem. For convenience in stating the theorem, we consider only the case where $P_0(t) \equiv I$.

Let $B(k) = 2^{4k+1}(2k-1)(k!)^2/((2k)!)^2$.

Theorem 2.3. *If there are constants ρ_k and N such that $\sum_{k=1}^{n} \rho_k B(k) \leqslant 1$ and*

$$\limsup_{t\to\infty} \left\| \int_N^t x^{2k-1} P_k(x)\, dx \right\| \leqslant \rho_k \qquad (k = 1, \ldots, n)$$

then the system $l(y) = 0$ is nonoscillatory.

PROOF. For any $y \in \mathfrak{D}_N(L)$ and each $k \geq 1$,

$$-\int_N^\infty (P_k(t)y^{(n-k)}(t), y^{(n-k)}(t))\,dt$$

$$= -\int^\infty \left(\left(\int_N^t x^{2k-1}P_k(x)\,dx\right)' y^{(n-k)}(t), y^{(n-k)}(t)/t^{2k-1}\right) dt$$

$$\leq \int_N^\infty \left[2\frac{\|y^{(n-k)}(t)\|}{t^k}\frac{\|y^{n-(k-1)}(t)\|}{t^{k-1}} + (2k-1)\frac{\|y^{(n-k)}(t)\|^2}{t^{2k}}\right]\left\|\int_N^t x^{2k-1}P_k(x)\,dx\right\| dt$$

by integrating by parts and using the Schwarz inequality. Proceeding in a manner which is similar to the proof of Theorem 2.1 we can establish that

$$-\int_N^\infty (P_k(t)y^{(n-k)}(t), y^{(n-k)}(t))\,dt \leq \rho_k B(k)\int_N^\infty \|y^{(n)}\|^2\,dt.$$

Summing over k and using Theorem 1.1 completes the proof.

3. **Oscillation.** In this section we shall let μA and νA denote the smallest and largest eigenvalue of A, respectively.

THEOREM 3.1. *Suppose $P_n(t)$ is negative semidefinite and*

$$\lim_{x\to\infty} \mu\int_1^x t^\gamma P_n(t)\,dt = -\infty$$

for some number γ. If

$$\int_1^\infty t^{\gamma-2(n-k)}\|P_k(t)\|\,dt < \infty \qquad (k = 0, 1, \ldots, n-1),$$

then the system $l(y) = 0$ is oscillatory.

PROOF. According to Theorem 1.1, it will suffice to exhibit a vector function $y(t) \in \mathfrak{D}_N(L)$ for each $N > 0$ such that

$$\sum_{k=0}^n \int_N^\infty (P_k(t)y^{(n-k)}(t), y^{(n-k)}(t))\,dt < 0.$$

Let $\alpha(t)$ be the $2n-1$ degree polynomial satisfying $\alpha^{(i)}(0) = \alpha^{(i)}(1) = \alpha(1) = 0$ $(1 \leq i \leq n-1)$ and $\alpha(0) = 1$. For each $N > 0$ we define

$$\beta(t) = t^{\gamma/2}\alpha((2N-t)/N), \quad t \in [N, 2N),$$
$$\beta(t) = t^{\gamma/2}, \quad t \in [2N, M),$$
$$\beta(t) = t^{\gamma/2}\alpha((t-M)/M), \quad t \in [M, 2M),$$

and $\beta(t) \equiv 0$ otherwise, where M will be taken to be greater than $2N$.

By noting that

$$\alpha(t) = C\int_1^t \left[x(x-1)^{n-1}\right] dx$$

where

$$C^{-1} = \int_0^1 \left[t(t-1)\right]^{n-1} dt,$$

it is easy to show that $\alpha(t) \geqslant 0$ and $\alpha'(t) \leqslant 0$ on [0, 1]. Hence, $0 \leqslant \alpha(t) \leqslant 1$.

Let ξ_M be a unit eigenvector corresponding to the eigenvalue $\mu\int_{2N}^M t^\gamma P_n(t)\, dt$, and let $y(t) = \beta(t) \cdot \xi_M$.

Since $\| \beta^{(n-k)}(t)\|^2 \leqslant Bt^{\gamma-2(n-k)}/n$ for some constant B which is independent of M, it follows that

$$\sum_{k=0}^{n} \int_N^\infty \left(P_k(t) y^{(n-k)}(t), y^{(n-k)}(t)\right) dt$$

$$\leqslant B\sum_{k=0}^{n-1} \int_N^\infty t^{\gamma-2(n-k)}\|P_k(t)\|\, dt + \mu\int_{2N}^M t^\gamma P_n(t)\, dt$$

by using the Cauchy-Schwarz inequality. Consequently, we can choose M so large that

$$\sum_{k=0}^{n} \int_N^\infty \left(P_k(t) y^{(n-k)}(t), y^{(n-k)}(t)\right) dt < 0.$$

By Theorem 1.1, this completes the proof.

The condition $P_n(t) \leqslant 0$ of Theorem 3.1 could be replaced by the weaker requirement that $\nu\int_1^x t^\gamma P_n(t)\, dt$ be bounded above for all x. It would be interesting to know if this condition could be relaxed completely as in the scalar case (see Lewis [10]). Also, with sign restrictions on $P_0(t)$ and $P_1(t)$ and $n = 1$ Tomastik [18] established an oscillation criterion involving conditions on eigenvalues of integrals of both $P_0(t)$ and $P_1(t)$.

Since $P_n(t)$ is negative semidefinite in Theorem 3.1, then

$$\left(\mu\int_1^x t^\gamma P_n(t)\, dt\right)^2 = \nu\left(\int_1^x t^\gamma P_n(t)\, dt\right)^2$$

or

$$\left|\mu\int_1^x t^\gamma P_n(t)\, dt\right| = \left\|\int_1^x t^\gamma P_n(t)\, dt\right\|$$

(see Noble [13, p. 429]) which yields the following corollary.

COROLLARY 3.1. *If* $P_n(t) \leqslant 0$ *and for some* γ

$$\lim_{x\to\infty}\left\|\int_1^x t^\gamma P_n(t)\,dt\right\| = \infty,$$

then the system $l(y) = 0$ *is oscillatory provided*

$$\int_1^\infty t^{\gamma-2(n-k)}\|P_k(t)\|\,dt < \infty \qquad (k = 0, 1, \ldots, n-1).$$

Since $\|A\| \leqslant \|A\|_E \leqslant m^{1/2}\|A\|$ for any m by m matrix A, we could replace the induced norms of Theorem 3.1 and Corollary 3.1 with Euclidean norms.

Let $\operatorname{tr}(A)$ denote the trace of the matrix A, which is the sum of the diagonal elements of A and, also, the sum of the eigenvalues of A. When $P_n(t) \leqslant 0$ then it is easy to see that $\lim_{x\to\infty}\mu\int_1^x t^\gamma P_n(t)\,dt = -\infty$ if and only if

$$\lim_{x\to\infty}\operatorname{tr}\int_1^x t^\gamma P_n(t)\,dt = -\infty.$$

Consequently, the system $l(y) = 0$ is also oscillatory when the minimum eigenvalue is replaced by the trace in the hypothesis of Theorem 3.1.

When $P_0(t) \equiv I$ and $P_k(t) \equiv 0$ for $k = 1, \ldots, n-1$, we have the following result as a corollary of Theorem 3.1.

COROLLARY 3.2. *If* $P(t) \leqslant 0$ *and for some* $\gamma < 2n - 1$

$$\lim_{x\to\infty}\operatorname{tr}\left(\int_1^x t^\gamma P(t)\,dt\right) = -\infty \tag{4}$$

then the system

$$(-1)^n y^{(2n)} + P(t)y = 0 \tag{5}$$

is oscillatory.

If we let $\gamma = 2n - 1$, there are choices of $P(t)$ for which (4) is satisfied and (5) is nonoscillatory (see [5, p. 96]).

When $n = 1$, Etgen [2, Theorem 3] has shown that the condition $P \leqslant 0$ is not needed in Corollary 3.2. Etgen's results apply to the more general quasilinear matrix differential equation

$$(R(x)V')' - P(x, V, V')V = 0 \tag{6}$$

where $P(x, V, V')$ is continuous and symmetric. The quadratic functional criterion for the oscillation of $l(y) = 0$ which was established in Theorem 1.1 has been extended to equation (6) by Swanson [16]. Therefore, the sufficient conditions for oscillation of $l(y) = 0$ that are established here can also be applied to equation (6).

The next theorem relates the oscillation of $l(y) = 0$ to the oscillation of the scalar differential equation

$$\sum_{k=0}^{n}(-1)^{n-k}\left((P_k\xi, \xi)\phi^{(n-k)}\right)^{(n-k)} = 0 \tag{7}$$

where ξ is a constant unit vector. It follows as an immediate corollary of Theorem 1.1.

THEOREM 3.2. *If there is some constant unit vector ξ such that* (7) *is oscillatory, then the system $l(y) = 0$ is oscillatory.*

This theorem yields results for $l(y) = 0$ like those of Swanson [16] and Noussair and Swanson [14], which apply to (6).

The next corollary follows from Theorem 3.2 by letting

$$\xi^* = m^{-1/2}(1, 1, \ldots, 1).$$

Also, note that $P(t) = (p_{ij}(t))$.

COROLLARY 3.3. *System* (5) *is oscillatory provided the scalar equation*

$$(-1)^n \phi^{(2n)} + m^{-1} \sum_{i,j=1}^{m} p_{ij}(t)\phi = 0$$

is oscillatory.

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