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INEQUALITIES FOR POLYNOMIALS ON THE UNIT INTERVAL(1)

BY

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ABSTRACT. Let $p_n(z) = \sum_{k=0}^n a_k z^k$ be a polynomial of degree at most n with real coefficients. Generalizing certain results of I. Schur related to the well-known inequalities of Chebyshev and Markov we prove that if $p_n(z)$ has at most $n-1$ distinct zeros in $(-1, 1)$, then

$$|a_n| < 2^{n-1}\left(\cos\frac{\pi}{4n}\right)^{2n} \max_{-1 \leq x \leq 1} |p_n(x)|,$$

$$\max_{-1 \leq x \leq 1} |p_n'(x)| < \left(n\cos\frac{\pi}{4n}\right)^2 \max_{-1 \leq x \leq 1} |p_n(x)|.$$

1. Introduction. Let $p_n(z) = \sum_{k=0}^n a_k z^k$ be a polynomial of degree at most n. According to a well-known result of A. Markov [4],

$$\max_{-1 \leq x \leq 1} |p_n'(x)| \leq n^2 \max_{-1 \leq x \leq 1} |p_n(x)|. \tag{1}$$

In (1) equality holds if and only if $p_n(z)$ is a constant multiple of $T_n(z)$ where

$$T_n(z) = 2^{n-1} \prod_{\nu=1}^{n} \left\{ z - \cos\left(\left(\nu - \frac{1}{2}\right)\pi/n\right)\right\}$$

is the so-called Chebyshev polynomial of the first kind of degree n.

The influence of the location of the zeros of $p_n(z)$ on the bound in Markov's inequality (1) has been studied by Schur [7], Erdös [1], Eröd [2], Rahman [5], Scheick [6] and others. It was shown by Erdös [1] that if all the zeros of $p_n(z)$ are real but lie outside $(-1, 1)$, then (1) can be replaced by

$$\max_{-1 \leq x \leq 1} |p_n'(x)| \leq \frac{1}{2} en \max_{-1 \leq x \leq 1} |p_n(x)|. \tag{2}$$

Scheick [6] obtained the same estimate under the weaker assumption that $p_n(z)$ is real for real z and does not vanish in $|z| < 1$. Schur [7] prescribed one of the zeros of $p_n(z)$ to lie at one of the end points of the interval $[-1, +1]$ and showed that then

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(3) $$\max_{-1\leqslant x\leqslant 1}|p_n'(x)| \leqslant \left(n\cos\frac{\pi}{4n}\right)^2 \max_{-1\leqslant x\leqslant 1}|p_n(x)|.$$

An analogous problem concerning Bernstein's inequality for polynomials on the unit disk was recently studied by Giroux and Rahman [3, Theorems 1, 2].

With respect to the problem considered by Schur it is natural to ask what can be said about

$$\left(\max_{-1\leqslant x\leqslant 1}|p_n'(x)|\right) \Big/ \left(\max_{-1\leqslant x\leqslant 1}|p_n(x)|\right)$$

if we simply assume that $p_n(z)$ is a real polynomial of degree n having at most $n-1$ distinct zeros in $(-1, 1)$. This question is answered in Theorem 1.

Improving upon the well-known estimate of Chebyshev

(4) $$|a_n| \leqslant 2^{n-1} \max_{-1\leqslant x\leqslant 1}|p_n(x)|$$

for the leading coefficient of a polynomial $p_n(z)$ of degree n in terms of $\max_{-1\leqslant x\leqslant 1}|p_n(x)|$, Schur [7, Theorem III*] proved that if $p_n(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial of degree n vanishing at $+1$ or -1, then

(5) $$|a_n| \leqslant 2^{n-1}\left(\cos\frac{\pi}{4n}\right)^{2n} \max_{-1\leqslant x\leqslant 1}|p_n(x)|.$$

We show that the same estimate holds (see Theorem 2 below) for all real polynomials having at most $n-1$ distinct zeros in $(-1, 1)$.

2. Statement of results.

Notation. We shall denote by $\mathcal{P}_n$ the class of all polynomials $p_n(z) = \sum_{k=0}^{n} a_k z^k$ of degree n with real coefficients.

THEOREM 1. *Inequality* (3) *holds for all polynomials* $p_n(z)$ *in* $\mathcal{P}_n$ *which have at most* $n-1$ *distinct zeros in* $(-1, 1)$. *Equality is attained if and only if* $p_n(z)$ *is a constant multiple of*

$$T_n\left(\pm\left(\cos\frac{\pi}{4n}\right)^2 z + \left(\sin\frac{\pi}{4n}\right)^2\right).$$

In particular (3) holds for all polynomials $p_n(z)$ in $\mathcal{P}_n$ which vanish at $+1$ or -1. Here the restriction that $p_n(z)$ has real coefficients can be easily dropped. In fact, if $p_n(z) = \sum_{k=0}^{n} a_k z^k$ is an arbitrary polynomial of degree n vanishing at $+1$ or -1 and the maximum of $|p_n'(x)|$ in $[-1, 1]$ is attained at $x_0 \in [-1, 1]$ where $p_n'(x_0) = |p_n'(x_0)|e^{i\gamma}$, then $A_n(z) = \sum_{k=0}^{n} \mathcal{Re}(a_k e^{-i\gamma}) z^k$ is a polynomial in $\mathcal{P}_n$ vanishing at $+1$ or -1 with

$$\max_{-1\leqslant x\leqslant 1}|A_n'(x)| = \max_{-1\leqslant x\leqslant 1}|p_n'(x)|, \quad \max_{-1\leqslant x\leqslant 1}|A_n(x)| \leqslant \max_{-1\leqslant x\leqslant 1}|p_n(x)|.$$

Since by Theorem 1,

$$\max_{-1\leqslant x\leqslant 1}|A_n'(x)| \leqslant \left(n\cos\frac{\pi}{4n}\right)^2 \max_{-1\leqslant x\leqslant 1}|A_n(x)|,$$

we see that (3) holds for all polynomials $p_n(z)$ of degree n vanishing at $+1$ or -1. We thus get an alternative proof of Schur's result in its full generality.

Note that if in Theorem 1, $p_n(z)$ is allowed to have complex coefficients, then nothing better than Markov's result can hold.

THEOREM 2. *Inequality* (5) *holds for all polynomials* $p_n(z)$ *in* $\mathcal{P}_n$ *which have at most* $n-1$ *distinct zeros in* $(-1, 1)$. *Equality is attained if and only if* $p_n(z)$ *is a constant multiple of* $T_n(\pm(\cos(\pi/4n))^2 z + (\sin(\pi/4n))^2)$.

Here again the coefficients of $p_n(z)$ cannot be allowed to be complex. Nevertheless, Schur's result that (5) holds for *all* polynomials $p_n(z) = \sum_{k=0}^{n} a_k z^k$ vanishing at $+1$ or -1 can be easily deduced.

As an immediate consequence of Theorem 2, we obtain

COROLLARY. *All the zeros of a monic polynomial* $p_n(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ *in* $\mathcal{P}_n$ *with*

$$\max_{-1<x<1} |p_n(x)| < 2^{1-n}\left(\cos \frac{\pi}{4n}\right)^{-2n}$$

are distinct and lie in $(-1, 1)$.

3. Lemmas.

Notation. We shall denote by $\mathcal{T}_n$ the class of all real trigonometric polynomials

$$t(\theta) = a_0 + \sum_{\nu=1}^{n} (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta)$$

with $a_n^2 + b_n^2 = 4^{1-n}$, and having a double zero at $\theta = 0$, i.e.

$$\sum_{\nu=0}^{n} a_\nu = 0 = \sum_{\nu=1}^{n} \nu b_\nu.$$

Theorem 2 will be deduced from the following two lemmas.

LEMMA 1. *Let* $t(\theta)$ *be a trigonometric polynomial in the class* $\mathcal{T}_n$ *with* $\max_{-\pi<\theta<\pi}|t(\theta)| = M$. *If* $|t(\theta)|$ *is equal to* M *at* $2n-1$ *different points in* $[-\pi, \pi)$, *then*

$$t(\theta) = \pm 2^{1-n}\left(\cos \frac{\pi}{4n}\right)^{-2n} T_n\left(-\left(\cos \frac{\pi}{4n}\right)^2 \cos\theta + \left(\sin \frac{\pi}{4n}\right)^2\right),$$

where, as usual, $T_n(z)$ *is the Chebyshev polynomial of the first kind of degree* n.

PROOF. We show first that under the assumptions of the lemma, $t(\theta)$ is a cosine polynomial. Since $t(0) = t'(0) = 0$, we see that $t(\theta)$ has exactly $2n$ critical points in $[-\pi, \pi)$, which we may list as

$$-\pi \leqslant \varphi_1 < \varphi_2 < \cdots < \varphi_{2n} < \pi,$$

where for some k $(1 \leqslant k \leqslant 2n)$ $\varphi_k = 0$. Further, in each of the subintervals $[-\pi, 0)$ and $(0, \pi)$ the signs of $t(\theta)$ at consecutive critical points are alternating (provided the subinterval in question contains at least two critical points). If φ_j and φ_{j+1} $(j \neq k \neq j+1)$ are two consecutive critical points of $t(\theta)$ such that

$$\operatorname{sgn} t(\varphi_j) = -\operatorname{sgn} t(\varphi_{j+1}),$$

and

$$|t(\varphi_j)| = |t(\varphi_{j+1})| = \max_{-\pi \leqslant \theta < \pi} |t(\theta)|,$$

then for every ε $(0 < \varepsilon < 1)$ the graph of $(1-\varepsilon)t(-\theta)$ crosses the graph of $t(\theta)$ in $(\varphi_j, \varphi_{j+1})$. Hence, whatever k $(1 \leqslant k \leqslant 2n)$ may be, $s(\varepsilon, \theta) = t(\theta) - (1-\varepsilon)t(-\theta)$ has at least $2n-3$ zeros in

$$E = \{\theta \colon \varphi_1 \leqslant \theta \leqslant \varphi_{2n}\} \cap \{\theta \colon |\theta| \geqslant \delta\}$$

where δ is a suitably small positive number not depending on ε. As E is a closed set the number of zeros of $s(\varepsilon, \theta)$ in E cannot decrease when $\varepsilon \to 0$. Hence $s(\theta) = t(\theta) - t(-\theta)$ has at least $2n-3$ zeros in E. If $\varphi_1 = -\pi$, then taking the periodicity of $t(\theta)$ into account we see that one of the zeros of $s(\varepsilon, \theta)$ lying in E tends to $-\pi$ as $\varepsilon \to 0$, where it becomes a zero of multiplicity at least two. If $\varphi_1 > -\pi$, then $s(\theta)$ has at least a simple zero at $-\pi$, since $t(-\pi) = t(\pi)$. Besides, in any case $s(\theta)$has a zero of multiplicity at least three at $\theta = 0$. Hence $s(\theta)$ has at least $2n+1$ zeros in $[-\pi, \pi)$ if a multiple zero is counted as many times as its multiplicity. Since $s(\theta)$ is of degree at most n this is possible only if $s(\theta) \equiv 0$, i.e. $t(\theta)$ is a cosine polynomial.

A similar discussion shows that taking into account the multiplicity of the zero at $\theta = 0$ each of the two polynomials

$$t(\theta) \pm 2^{1-n}\left(\cos \frac{\pi}{4n}\right)^{-2n} T_n\left(-\left(\cos \frac{\pi}{4n}\right)^2 \cos\theta + \left(\sin \frac{\pi}{4n}\right)^2\right)$$

has at least $2n-1$ zeros in $[-\pi, \pi)$. But clearly, one of these two polynomials is of degree at most $n-1$, and hence must be identically zero. This completes the proof of Lemma 1.

LEMMA 2. *Let $t(\theta)$ be a trigonometric polynomial in the class $\mathfrak{T}_n$ with* $\max_{-\pi \leqslant \theta < \pi} |t(\theta)| = M$. *If $|t(\theta)|$ is equal to M at less than $2n-1$ different points in $[-\pi, \pi)$, then $t(\theta)$ cannot be of smallest supremum norm in $\mathfrak{T}_n$.*

PROOF. We may assume that $|t(\theta)|$ attains its maximum at exactly $2n-2$ points in $[-\pi, \pi)$ with alternating signs in the subintervals $[-\pi, 0)$ and $(0, \pi)$, for otherwise we can add a trigonometric polynomial of degree less than n such that the resulting trigonometric polynomial still belongs to $\mathfrak{T}_n$, but has smaller supremum norm.

Since $t'(\theta)$ is a real trigonometric polynomial it has an even number of zeros in $[-\pi, \pi)$. Hence either $\xi = 0$ is a zero of $t(\theta)$ of multiplicity three, or else there is one (and only one) critical point η of $t(\theta)$ other than 0 with $|t(\eta)| < M$. It is easily seen that ξ and η must be consecutive critical points if $t(\theta)$ is to be a trigonometric polynomial of smallest supremum norm in $\mathfrak{T}_n$. In any case, we may assume without loss of generality, that we have two consecutive critical points $\xi = 0$ and η with $\xi \leqslant \eta$ and $0 = |t(\xi)| \leqslant |t(\eta)| < M$. If a multiple zero is counted as many times as its multiplicity then we see that $t(\theta)$ has a total number of $2n$ zeros θ_ν $(1 \leqslant \nu \leqslant 2n)$ in $[-\pi, \pi)$, which may be arranged as

$$-\pi \leqslant \theta_1 \leqslant \theta_2 \leqslant \cdots \leqslant \theta_{2n} < \pi.$$

Putting $\theta_0 = \theta_{2n} - 2\pi$ and $\theta_{2n+1} = \theta_1 + 2\pi$, we have for some k $(2 \leqslant k \leqslant 2n)$,

$$\theta_{k-2} < \theta_{k-1} = \theta_k = \xi = 0 \leqslant \eta \leqslant \theta_{k+1}.$$

As

$$\begin{aligned} |t(\theta)| &= 2^n \prod_{\nu=1}^{2n} \left|\sin \frac{\theta - \theta_\nu}{2}\right| = 2^{-n} \prod_{\nu=1}^{2n} \left|e^{i(\theta-\theta_\nu)/2} - e^{-i(\theta-\theta_\nu)/2}\right| \\ &= 2^{-n} \prod_{\nu=1}^{2n} \left|e^{i\theta} - e^{i\theta_\nu}\right|, \end{aligned} \tag{6}$$

it is sufficient to show that we can decrease the maximum modulus of $F(z) = \prod_{\nu=1}^{2n}(z - e^{i\theta_\nu})$ on the unit circle by moving some of the θ_ν's on the real axis keeping $\theta_{k-1} = \theta_k$. For this purpose we consider

$$F(\alpha, z) = \frac{D(\alpha, z)}{D(0, z)} F(z),$$

where

$$D(\alpha, z) = (z - e^{-i\alpha})^2 (z - e^{i(\theta_{k+1} + 2\alpha)}).$$

On discussing the behaviour of

$$|D(\alpha, e^{i\theta})| = 8\left\{\sin\left(\frac{\theta + \alpha}{2}\right)\right\}^2 \left|\sin\left(\frac{\theta - \theta_{k+1} - 2\alpha}{2}\right)\right|$$

we see that, indeed, for small positive α

$$\max_{|z|=1} |F(\alpha, z)| < \max_{|z|=1} |F(z)|.$$

Through the relationship (6) there corresponds to $F(\alpha, z)$ a trigonometric polynomial $t(\alpha, \theta)$ which is simply a translation of an element in $\mathfrak{T}_n$ and has smaller supremum norm than $t(\theta)$.

4. Proofs of the theorems. We will prove Theorem 2 first since we shall need it for the proof of Theorem 1.

PROOF OF THEOREM 2. We will prove the equivalent fact that if $p_n(z)$ is a monic polynomial in $\mathcal{P}_n$ having at most $n-1$ zeros in $(-1, 1)$ and $M = \max_{-1<x<1}|p_n(x)|$, then

$$M \geqslant 2^{1-n}\left(\cos \frac{\pi}{4n}\right)^{-2n} \tag{7}$$

where equality is possible if and only if $p_n(z) = 2^{1-n}(\cos(\pi/4n))^{-2n}P_*(z)$ or $p_n(z) = (-1)^n 2^{1-n}(\cos(\pi/4n))^{-2n}P_*(-z)$, where

$$P_*(z) = T_n\left(\left(\cos \frac{\pi}{4n}\right)^2 z + \left(\sin \frac{\pi}{4n}\right)^2\right).$$

Since $2 \geqslant (\cos(\pi/4n))^{-2n}$ $(n \geqslant 1)$, Chebyshev's inequality (4) shows that (7) holds for all monic polynomials of degree less than n. If $p_n(z)$ has a real zero outside $[-1, 1]$ or pairs of complex conjugate zeros, $\max_{-1<x<1}|p_n(x)|$ can be decreased by moving these zeros appropriately and keeping them outside the unit interval. So, we may suppose that $p_n(z)$ is a polynomial of degree n vanishing at one of the end points of the unit interval, or having a double zero in $(-1, 1)$. Then the trigonometric polynomial $p_n(\cos \theta)$ is also of degree n and has at least one double zero in $[-\pi, \pi)$. For a suitable choice of α the trigonometric polynomial $t(\theta) = p_n(\cos(\theta - \alpha))$ belongs to $\mathcal{T}_n$. Lemmas 1 and 2 show that

$$\pm 2^{1-n}\left(\cos \frac{\pi}{4n}\right)^{-2n} P_*(-\cos \theta)$$

are the only elements of smallest supremum norm in $\mathcal{T}_n$. Hence (7) holds, with equality if and only if

$$p_n(z) = 2^{1-n}\left(\cos \frac{\pi}{4n}\right)^{-2n} P_*(z) \quad \text{or}$$

$$p_n(z) = (-1)^n 2^{1-n}\left(\cos \frac{\pi}{4n}\right)^{-2n} P_*(-z).$$

With this Theorem 2 is proved.

PROOF OF THEOREM 1. Without loss of generality we may restrict ourselves to polynomials whose absolute value does not exceed 1 on the unit interval. Now let $\mathcal{C}$ denote the (sub-) class consisting of all polynomials $p_n(z)$ in $\mathcal{P}_n$ which have at most $n-1$ distinct zeros in $(-1, 1)$ and which satisfy $|p_n(x)| \leqslant 1$ for $-1 \leqslant x \leqslant 1$. Then

$$P_*(\pm z) = T_n\left(\pm\left(\cos \frac{\pi}{4n}\right)^2 z + \left(\sin \frac{\pi}{4n}\right)^2\right) \in \mathcal{C}.$$

A straightforward calculation shows that

$$\max_{-1\leqslant x\leqslant 1}|P_*'(x)| = (n\cos(\pi/4n))^2 = P_*'(1).$$

In view of this and the fact that for a polynomial $p_n(z)$ in $\mathfrak{P}_n$ for which $\max_{-1\leqslant x\leqslant 1}|p_n(x)| \leqslant 1$ we have [7, p. 275]

$$\max_{-1\leqslant x\leqslant 1}|p_n'(x)| \leqslant \frac{n^2}{2} < P_*'(1) \quad (n > 2),$$

whenever $\max_{-1\leqslant x\leqslant 1}|p_n'(x)|$ is attained in $(-1, 1)$, it is enough to show (in order to establish Theorem 1) that $|p_n'(1)| \leqslant (n\cos(\pi/4n))^2 = P_*'(1)$ for all $p_n(z) \in \mathcal{C}$ with equality if and only if $p_n(z) = \pm P_*(z)$.

Let $Q_*(z)$ be a polynomial in $\mathcal{C}$ for which

(8) $$|Q_*'(1)| = \sup_{p_n(z)\in\mathcal{C}} |p_n'(1)|.$$

Since $(n-1)^2 < (n\cos(\pi/4n))^2 = P_*'(1)$, we see by A. Markov's theorem that $Q_*(z)$ is of degree n. Suppose

(9) $$|Q_*'(1)| > |P_*'(1)|.$$

Denote by $\xi_1, \xi_2, \ldots, \xi_k$ the zeros (multiple zeros appearing as many times as their multiplicity) of $Q_*(z)$ lying in $(-1, 1)$. We distinguish three cases:

Case (i). If $k \leqslant n-2$, then for suitable choice of the real quantity σ

$$Q(z) = Q_*(z) + \sigma(z-1)^2 \prod_{j=1}^{k} (z-\xi_j)$$

is a polynomial of degree n with $Q'(1) = Q_*'(1)$, and

$$\mu = \max_{-1\leqslant x\leqslant 1}|Q(x)| < 1,$$

so that $\mu^{-1}Q(z)$ belongs to $\mathcal{C}$, but $|\mu^{-1}Q'(1)| > |Q_*'(1)|$. This contradicts (8).

Case (ii). If $k = n-1$, we denote by ξ_n the (real) zero of $Q_*(z)$ lying outside $(-1, 1)$. Note that if ξ_n were $\leqslant -1$ then the graph of $P_*'(1)Q_*(x)/Q_*'(1)$ would cross that of $P_*(x)$ at least n times on $[-1, 1)$. Hence the polynomial

$$S(x) = P_*(x) - \frac{P_*'(1)}{Q_*'(1)} Q_*(x),$$

which is clearly $\not\equiv 0$ would have all its zeros in $[-1, 1)$ which is a contradiction since $S'(1) = 0$. On the other hand, the same reasoning can be used to show that in the case $\xi_n \geqslant 1$ the largest critical point η of $Q_*(x)$ cannot be larger than 1. But if $\eta \leqslant 1 \leqslant \xi_n$, then for sufficiently small $\varepsilon > 0$ the polynomial $Q_*(z+\varepsilon)$ still belongs to $\mathcal{C}$, and $|Q_*'(1+\varepsilon)| > |Q_*'(1)|$, which contradicts (8).

Case (iii). If $k = n$, then all the zeros of $Q_*(z)$ lie in $(-1, 1)$, and at least one of them is of multiplicity at least two. Denote by p and q the coefficients of z^n in $P_*(z)$ and $Q_*(z)$ respectively. Without loss of generality we may assume $q > 0$. By Theorem 2 we have $p > q$. Since all the zeros of $Q_*(z)$ lie in $(-1, 1)$, $Q_*'(x)$ is monotone increasing for $x \geqslant 1$. We must have $Q_*(1) = 1$, because otherwise for appropriate $\varepsilon > 0$ the polynomial $Q_*(z + \varepsilon)$ would contradict the extremal property of $Q_*(z)$. Consequently, if

$$U(z) = P_*(z) - Q_*(z),$$

then

$$U(1) = 0,\ U'(1) < 0,$$

and $\operatorname{sgn} U(x) = \operatorname{sgn}(p - q) = +1$ for $x \to \infty$, so that $U(x)$ has a zero on $(1, \infty)$. Furthermore, comparing the graphs of $P_*(x)$ and $(1 - \varepsilon)Q_*(x)$ for $\varepsilon > 0$ and letting ε tend to zero, we see that $U(x)$ has at least (and hence exactly) $n - 1$ zeros in $(-1, 1]$. Therefore, $U(x) \neq 0$ for $x \leqslant -1$. It follows that

$$\operatorname{sgn} U(x) = \operatorname{sgn}(-1)^n(p - q) = (-1)^n \quad \text{for } x \leqslant -1,$$

but

$$\operatorname{sgn} U(-1) = -\operatorname{sgn} Q_*(-1) = (-1)^{n+1}\operatorname{sgn} q = (-1)^{n+1},$$

which is a contradiction.

Hence in any case $|Q_*'(1)| = |P_*'(1)|$. Investigating the above three cases under this hypothesis, we obtain using similar reasonings that $Q_*(x) = \pm P_*(x)$. This completes the proof of Theorem 1.

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