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MULTIPLIER CRITERIA OF MARCINKIEWICZ TYPE FOR JACOBI EXPANSIONS

BY

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ABSTRACT. It is shown how an integral representation for the product of Jacobi polynomials can be used to derive a certain integral Lipschitz type condition for the Cesàro kernel for Jacobi expansions. This result is then used to give criteria of Marcinkiewicz type for a sequence to be multiplier of type (p,p), $1 < p < \infty$, for Jacobi expansions.

1. **Introduction.** Fix $\alpha \geqslant \beta \geqslant -\frac{1}{2}$ and let $L^p = L^p_{(\alpha,\beta)}$, $1 \leqslant p < \infty$, denote the space of measurable functions $f(\theta)$ on $[0,\pi]$ for which

$$\|f\|_p = \left(\int_0^\pi |f(\theta)|^p \, d\mu(\theta)\right)^{1/p} < \infty,$$

where

$$d\mu(\theta) = d\mu^{(\alpha,\beta)}(\theta) = \left(\sin\frac{\theta}{2}\right)^{2\alpha+1}\left(\cos\frac{\theta}{2}\right)^{2\beta+1} d\theta.$$

Also let

$$R_k(x) = R_k^{(\alpha,\beta)}(x) = P_k^{(\alpha,\beta)}(x)/P_k^{(\alpha,\beta)}(1),$$

where $P_k^{(\alpha,\beta)}(x)$ is the Jacobi polynomial of order (α,β) [16]. Each $f \in L^p$ has the Jacobi expansion

$$f(\theta) \sim \sum_{k=0}^{\infty} f^\wedge(k) h_k R_k(\cos\theta),$$

where

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$$f^\wedge(k) = \int_0^\pi f(\theta) R_k(\cos\theta)\, d\mu(\theta),$$

$$\text{(1.1)} \qquad h_k = h_k^{(\alpha,\beta)} = \left\{ \int_0^\pi (R_k(\cos\theta))^2\, d\mu(\theta) \right\}^{-1}$$

$$= \frac{(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)\Gamma(k+\alpha+1)}{\Gamma(k+\beta+1)\Gamma(k+1)\Gamma(\alpha+1)\Gamma(\alpha+1)}.$$

A sequence $\eta = \{\eta_k\}_{k=0}^\infty$ is called a multiplier of type (p,q), notation $\eta \in M_p^q$, if for each $f \in L^p$ there exists a function $f^\eta \in L^q$ with

$$f^\eta(\theta) \sim \sum_{k=0}^\infty \eta_k f^\wedge(k) h_k R_k(\cos\theta)$$

and $\|f^\eta\|_q \leqslant C_{p,q,\eta} \|f\|_p$. Here, as elsewhere, C is a positive constant, not always the same at different occasions, which depends only on the parameters indicated by subscripts, except that the dependence on the parameters α, β, or γ will not be indicated. Also, we will let $\int \cdots d\mu(\theta)$ denote integration over the interval $[0,\pi]$.

Our aim is to give sufficient multiplier criteria of Marcinkiewicz type for a sequence to belong to M_p^p, $1 < p < \infty$. For this purpose we consider the fractional difference operator Δ^λ defined by [4]

$$\Delta^\lambda \eta_k = \sum_{j=k}^\infty A_{j-k}^{-\lambda-1} \eta_j, \qquad A_k^\lambda = \binom{k+\lambda}{k} = \frac{\Gamma(k+\lambda+1)}{\Gamma(k+1)\Gamma(\lambda+1)},$$

whenever the series converges, and for $\gamma \geqslant 0$ we define the space $\mathrm{wbv}_{\gamma+1}$ of sequences of weak bounded variation of order $\gamma + 1$ by

$$\mathrm{wbv}_{\gamma+1} = \left\{ \eta \in l^\infty : \|\eta\|_{\gamma+1,w} = \|\eta\|_\infty + \sup_{m \geqslant 1} \sum_{2^{m-1}}^{2^m-1} A_k^\gamma |\Delta^{\gamma+1}\eta_k| < \infty \right\},$$

where l^∞ denotes the space of bounded sequences $\eta = \{\eta_k\}_{k=0}^\infty$ with the sup norm $\|\eta\|_\infty$. The space $\mathrm{bv}_{\gamma+1}$ of sequences of bounded variation of order $\gamma + 1$ is defined as above except that k runs from 0 to ∞, see [17, p. 20]. Our main result can then be stated as follows.

THEOREM 1. *If* $\alpha \geqslant \beta \geqslant -\frac{1}{2}$, $\gamma > \alpha + \frac{1}{2}$, *and* $\eta \in \mathrm{wbv}_{\gamma+1}$, *i.e.*

$$\|\eta\|_\infty + \sup_m \sum_{2^{m-1}}^{2^m-1} A_k^\gamma |\Delta^{\gamma+1}\eta_k| < \infty,$$

then $\eta \in M_p^p$, $1 < p < \infty$.

Recently, Connnett and Schwartz [6], [7] proved essentially

THEOREM A. *Let* $\alpha \geqslant \beta > -\frac{1}{2}$ *and* $\gamma = [\alpha + 1]$, *where* $[\alpha + 1]$ *is the integer part of* $\alpha + 1$. *If*

$$\|\eta\|_\infty + \sup_m \left(\sum_{2^{m-1}}^{2^m - 1} k^{-1} |A_k^{\gamma+1} \Delta^{\gamma+1} \eta_k|^2 \right)^{1/2} < \infty,$$

then $\eta \in M_p^p$, $1 < p < \infty$.

By Schwarz's inequality it follows that Theorem 1 implies Theorem A when $[\alpha + 1] > \alpha + \frac{1}{2}$. Let us mention that if $\gamma > \alpha + \frac{1}{2}$ then

$$\|\eta\|_\infty + \sum_{k=0}^{\infty} A_k^\gamma |\Delta^{\gamma+1} \eta_k| < \infty$$

is a sufficient condition for $\eta \in M_1^1$, see [17, pp. 21, 88]. For M_p^q-multipliers, $1 \leqslant p < q \leqslant \infty$, see [2]. Also see Butzer, Nessel, Trebels [17, ref. 33].

For the proof of Theorem 1 we need two useful lemmas concerning the space $\mathrm{wbv}_{\gamma+1}$, which will be proved in §§2 and 3.

LEMMA 1. *If* $0 \leqslant \gamma' < \gamma$ *then* $\mathrm{wbv}_{\gamma+1} \subset \mathrm{wbv}_{\gamma'+1}$, *where the inclusion sign means that the identity map is continuous.*

LEMMA 2. *Let* $G(t) \in C^\infty[0, \infty)$ *be monotone decreasing with* $G(t) = 1$ *if* $t \leqslant 1$ *and* $G(t) = 0$ *if* $t \geqslant 2$. *If* $\gamma \geqslant 0$ *and* $u > 0$, *then*

$$\|\{\eta_k G(k/u)\}\|_{\gamma+1,w} \leqslant C \|\eta\|_{\gamma+1,w}$$

with C independent of u.

In particular, it should be noted that, since

$$R_k^{(-1/2,-1/2)}(\cos\theta) = \cos k\theta,$$

the case $\alpha = \beta = -\frac{1}{2}$ of Theorem 1 follows directly from Lemma 1 and the classical Marcinkiewicz multiplier theorem [14], [18, p. 232]. In §4 we shall use these lemmas and the observations in Coifman and Weiss [5, pp. 74, 75] to reduce the proof of Theorem 1 to showing that a certain kernel satisfies a uniform Hörmander condition and to show that this condition follows from the following proposition, which is the crucial part of this paper.

PROPOSITION 1. *Let* $\alpha \geqslant \beta \geqslant -\frac{1}{2}$ *and* $\alpha + \frac{1}{2} < \gamma \leqslant \alpha + \frac{3}{2}$. *Then the Cesàro kernel*

$$\sigma_n^\gamma(\theta, \varphi) = \sum_{k=0}^{n} \frac{A_{n-k}^\gamma}{A_n^\gamma} h_k R_k(\cos\theta) R_k(\cos\varphi) \tag{1.2}$$

satisfies the inequalities

$$\int |\sigma_n^\gamma(\theta,\varphi) - \sigma_n^\gamma(\theta,\varphi_0)|\,d\mu(\theta) \leqslant Cn|\varphi - \varphi_0|, \tag{1.3}$$

$$\int |\sigma_n^\gamma(\theta,\varphi)|(n|\theta - \varphi|)^\varepsilon\,d\mu(\theta) \leqslant C_\varepsilon, \qquad 0 < \varepsilon < \gamma - \alpha - \tfrac{1}{2}. \tag{1.4}$$

Our proof of (1.3) in §5 is of special interest since it is quite different from and much simpler computationally than that which Connett and Schwartz used in [6], [7] to prove (with a slightly different notation) that if $\alpha \geqslant \beta > -\frac{1}{2}$, then the Poisson kernel

$$W_r(\theta,\varphi) = \sum_{k=0}^{\infty} (1-r)^k h_k R_k(\cos\theta) R_k(\cos\varphi) \tag{1.5}$$

satisfies the inequality

$$\int |W_r(\theta,\varphi) - W_r(\theta,\varphi_0)|\,d\mu(\theta) \leqslant \frac{C}{r}|\varphi - \varphi_0|. \tag{1.6}$$

Recall that for the ultraspherical case $\alpha = \beta > -\frac{1}{2}$ in [6] they used the Muckenhoupt and Stein [15] bounds for the kernel (1.5) and an integral representation of Watson to derive bounds for a partial derivative of (1.5), and then used these bounds, the Mean Value Theorem, and Watson's integral to prove (1.6); while for the case $\alpha > \beta > -\frac{1}{2}$ the main computations in [7] were directed at the derivation of bounds for (1.5) and a partial derivative of it. Our method for proving (1.3) uses both of the integral representations for the product of Jacobi polynomials in Koornwinder [13] and Gasper [9], [10] in such a way that, instead of having to derive and use bounds for a partial derivative of a two-variable kernel, we only need to consider the derivative of a one-variable kernel. In §6 we shall show that this method also yields a very short proof of (1.6) for $\alpha \geqslant \beta \geqslant -\frac{1}{2}$, and hence an extension of Theorem A to $\alpha \geqslant \beta \geqslant -\frac{1}{2}$.

2. **Proof of Lemma 1.** First note that for $\eta \in l^\infty$ we have

$$\Delta^a(\Delta^b \eta_k) = \Delta^{a+b}\eta_k \tag{2.1}$$

if $a > -1$, $b \geqslant 0$, $a + b > 0$; see [4, Lemma 1]. If $\gamma' \geqslant 0$, $\gamma - 1 < \gamma' < \gamma$, and we set $d = \gamma - \gamma'$, then $0 < d < 1$ and, by (2.1),

$$\begin{aligned}\sum_{k=2^{m-1}}^{2^m-1} A_k^{\gamma'} |\Delta^{\gamma'+1}\eta_k| &\leqslant \sum_{2^{m-1}}^{2^m-1} A_k^{\gamma-d} \sum_{j=k}^{\infty} A_{j-k}^{d-1} |\Delta^{\gamma+1}\eta_j| \\ &= \sum_{k=2^{m-1}}^{2^m-1} \left(\sum_{j=k}^{2^m-1} + \sum_{j=2^m}^{\infty} \right) = \mathrm{I}_1 + \mathrm{I}_2 .\end{aligned}$$

A change in the order of summation in I_1 gives

$$\mathrm{I}_1 = \sum_{j=2^{m-1}}^{2^m-1} |\Delta^{\gamma+1}\eta_j| \sum_{k=2^{m-1}}^{j} A_k^{\gamma-d} A_{j-k}^{d-1} \leqslant \|\eta\|_{\gamma+1,w}$$

uniformly in m. Also

$$\begin{aligned}
\mathrm{I}_2 &= \sum_{2^{m-1}}^{2^m-1} A_k^{\gamma-d} \sum_{i=m+1}^{\infty} \sum_{j=2^{i-1}}^{2^i-1} A_{j-k}^{d-1} |\Delta^{\gamma+1}\eta_j| \\
&\leqslant C\|\eta\|_{\gamma+1,w} \sum_{i=m+1}^{\infty} (2^{i-1})^{-\gamma} \sum_{k=2^{m-1}}^{2^m-1} A_k^{\gamma-d} A_{2^m-1-k}^{d-1} \\
&\leqslant C\|\eta\|_{\gamma+1,w} (2^m)^{\gamma} \sum_{i=m}^{\infty} (2^i)^{-\gamma} \leqslant C\|\eta\|_{\gamma+1,w}
\end{aligned}$$

uniformly in m. Hence Lemma 1 follows if $\gamma' \geqslant 0$ is such that $\gamma - 1 < \gamma' < \gamma$. If $\gamma - 1 \geqslant 0$ then one need but iterate the previous case finitely often, and thus Lemma 1 is proved.

3. **Proof of Lemma 2.** We first consider integer γ. Then Leibniz' formula gives

$$\Delta^{\gamma+1}\eta_k G\left(\frac{k}{u}\right) = \sum_{j=0}^{\gamma+1} \binom{\gamma+1}{j} (\Delta^j \eta_k) \Delta^{\gamma+1-j} G\left(\frac{j+k}{u}\right). \tag{3.1}$$

By [17, p. 26] and the hypotheses on $G(u)$,

$$\sum_{k=0}^{\infty} A_k^j \left|\Delta^{j+1} G\left(\frac{k}{u}\right)\right| \leqslant \int_0^{\infty} t^j |G^{(j+1)}(t)|\, dt \leqslant C_j$$

for $j = 0, 1, \ldots$. Hence

$$\left|k^j \Delta^j G\left(\frac{k}{u}\right)\right| = \left|k^j \sum_{i=k}^{\infty} \Delta^{j+1} G\left(\frac{i}{u}\right)\right| \leqslant C_j \sum_{i=0}^{\infty} A_i^j \left|\Delta^{j+1} G\left(\frac{i}{u}\right)\right| \leqslant C_j$$

and so, using the above and Lemma 1, we have

$$\begin{aligned}
\sum_{k=2^{m-1}}^{2^m-1} A_k^{\gamma} \left|\Delta^{\gamma+1}\eta_k G\left(\frac{k}{u}\right)\right| &\leqslant \sum_{2^{m-1}}^{2^m-1} |\eta_k| A_k^{\gamma} \left|\Delta^{\gamma+1} G\left(\frac{k}{u}\right)\right| \\
&\quad + \sum_{j=1}^{\gamma+1} \binom{\gamma+1}{j} \frac{\Gamma(j)}{\Gamma(\gamma+1)} \sum_{k=2^{m-1}}^{2^m-1} A_k^{j-1} |\Delta^j \eta_k| \\
&\quad \cdot \left\{(\gamma+k)\cdots(j+k)\left|\Delta^{\gamma+1-j} G\left(\frac{j+k}{u}\right)\right|\right\} \\
&\leqslant C\left\{\|\eta\|_{\infty} + \sum_{j=1}^{\gamma+1} \|\eta\|_{j,w}\right\} \leqslant C\|\eta\|_{\gamma+1,w}
\end{aligned}$$

uniformly in m and $u > 0$. Therefore the lemma is proved for integer γ.

For positive noninteger values of γ we set $a = [\gamma]$. Then, by (2.1) and (3.1),

$$
\begin{aligned}
\Delta^{\gamma+1}\left(\eta_k G\left(\frac{k}{u}\right)\right) &= \Delta\Delta^{\gamma-a-1}\sum_{j=0}^{a+1}\binom{a+1}{j}(\Delta^j\eta_k)\Delta^{a+1-j}G\left(\frac{j+k}{u}\right)\\
&= \Delta\sum_{i=k}^{a+1} A_{i-k}^{a-\gamma} G\left(\frac{i+a+1}{u}\right)\Delta^{a+1}\eta_i\\
&\quad +\Delta\sum_{j=0}^{a}\binom{a+1}{j}\sum_{i=k}^{\infty} A_{i-k}^{a-\gamma}(\Delta^j\eta_i)\Delta^{a+1-j}G\left(\frac{i+j}{u}\right)\\
&= \Sigma_1 + \Sigma_2 .
\end{aligned}
$$

By (2.1),

$$
\Delta^{a+1}\eta_i = \sum_{j=i}^{\infty} A_{j-i}^{\gamma-a-1}\Delta^{\gamma+1}\eta_j,
$$

and hence, after a change in order of summation,

$$
\begin{aligned}
\Sigma_1 &= \Delta\sum_{j=k}^{\infty}\Delta^{\gamma+1}\eta_j\sum_{i=k}^{j} A_{i-k}^{a-\gamma}A_{j-i}^{\gamma-a-1}G\left(\frac{i+a+1}{u}\right)\\
&= \lambda_k^{(k)}\Delta^{\gamma+1}\eta_k + \sum_{j=k+1}^{\infty}(\lambda_k^{(j)} - \lambda_{k+1}^{(j)})\Delta^{\gamma+1}\eta_j,
\end{aligned}
$$

where

$$
\lambda_k^{(j)} = \begin{cases} \displaystyle\sum_{i=k}^{j} A_{i-k}^{a-\gamma}A_{j-i}^{\gamma-a-1}G\left(\frac{i+a+1}{u}\right), & k \leqslant j,\\ 0, & k > j. \end{cases}
$$

It is not hard to check that $\{\lambda_k^{(j)}\}_{k=0}^{\infty}$ is monotone decreasing and bounded by 1. In particular

$$
\sum_{k=0}^{\infty} |\Delta\lambda_k^{(j)}| \leqslant 1
$$

uniformly in j and $u > 0$. Therefore,

$$\sum_{2^{m-1}}^{2^m-1} A_k^\gamma |\Sigma_1| \leqslant \sum_{2^{m-1}}^{2^m-1} A_k^\gamma |\Delta^{\gamma+1}\eta_k| + \sum_{2^{m-1}}^{2^m-1} A_k^\gamma \sum_{j=k+1}^{2^m-1} |\Delta\lambda_k^{(j)}|\,|\Delta^{\gamma+1}\eta_j|$$

$$+ \sum_{2^{m-1}}^{2^m-1} A_k^\gamma \sum_{i=m+1}^{\infty} \sum_{j=2^{i-1}}^{2^i-1} |\Delta\lambda_k^{(j)}|\,|\Delta^{\gamma+1}\eta_j|$$

$$\leqslant \|\eta\|_{\gamma+1,w} + C \sum_{j=2^{m-1}}^{2^m-1} A_j^\gamma |\Delta^{\gamma+1}\eta_j| \sum_{k=2^{m-1}}^{j-1} |\Delta\lambda_k^{(j)}|$$

$$+ C \sum_{i=m+1}^{\infty} (2^{i-1})^{-\gamma} \sum_{j=2^{i-1}}^{2^i-1} A_j^\gamma |\Delta^{\gamma+1}\eta_j| \sum_{k=2^{m-1}}^{2^m-1} A_k^\gamma |\Delta\lambda_k^{(j)}|$$

$$\leqslant C\|\eta\|_{\gamma+1,w}\left\{1 + (2^m)^\gamma \sum_{i=m}^{\infty} (2^i)^{-\gamma}\right\} \leqslant C\|\eta\|_{\gamma+1,w}.$$

For Σ_2 we observe that

$$\sum_{2^{m-1}}^{2^m-1} A_k^\gamma |\Sigma_2|$$

$$= \sum_{2^{m-1}}^{2^m-1} A_k^\gamma \left| \sum_{j=0}^{a} \binom{a+1}{j} \sum_{i=k}^{\infty} A_{i-k}^{a-\gamma} \left\{ (\Delta^j \eta_i) \Delta^{a+2-j} G\left(\frac{i+j}{u}\right) \right.\right.$$

$$\left.\left. + (\Delta^{j+1}\eta_i) \Delta^{a+1-j} G\left(\frac{i+j+1}{u}\right) \right\} \right|$$

$$= \sum_{2^{m-1}}^{2^m-1} A_k^\gamma \left| \sum_{j=0}^{a+1} C_{j,a} \sum_{i=k}^{\infty} A_{i-k}^{a-\gamma} (\Delta^j \eta_i) \Delta^{a+2-j} G\left(\frac{i+j}{u}\right) \right|$$

$$\leqslant C \sum_{k=2^{m-1}}^{2^m-1} A_k^\gamma \sum_{i=k}^{\infty} A_{i-k}^{a-\gamma} \left| \eta_i \Delta^{a+2} G\left(\frac{i}{u}\right) \right|$$

$$+ C \sum_{j=1}^{a+1} \sum_{k=2^{m-1}}^{2^m-1} A_k^\gamma \left(\sum_{i=k}^{2^m-1} + \sum_{s=m+1}^{\infty} \sum_{i=2^{s-1}}^{2^s-1} \right)$$

$$\cdot A_{i-k}^{a-\gamma} \left| (\Delta^j \eta_i) \Delta^{a+2-j} G\left(\frac{i+j}{u}\right) \right|$$

$$\leqslant C\|\eta\|_\infty \sum_{i=2^{m-1}}^{\infty} \left| \Delta^{a+2} G\left(\frac{i}{u}\right) \right| \sum_{k=2^{m-1}}^{i} A_k^\gamma A_{i-k}^{a-\gamma}$$

$$+ C \sum_{j=1}^{a+1} \sum_{i=2^{m-1}}^{2^m-1} |\Delta^j \eta_i|$$

$$\cdot \sum_{k=2^{m-1}}^{i} A_{i-k}^{a-\gamma} A_k^{\gamma+j-a-2} \left\{ k^{a+2-j} \left| \Delta^{a+2-j} G\left(\frac{i+j}{u}\right) \right| \right\}$$

$$+C\sum_{j=1}^{a+1}\sum_{s=m+1}^{\infty}\sum_{i=2^{s-1}}^{2^s-1}|\Delta^j\eta_i|\cdot\left|\Delta^{a+2-j}G\left(\frac{i+j}{u}\right)\right|\sum_{k=2^{m-1}}^{2^m-1}A_k^\gamma A_{2^m-1-k}^{a-\gamma}$$

$$\leqslant C\|\eta\|_\infty\sum_{i=0}^{\infty}A_i^{a+1}\left|\Delta^{a+2}G\left(\frac{i}{u}\right)\right|+C\sum_{j=1}^{a+1}\sum_{i=2^{m-1}}^{2^m-1}A_i^{j-1}|\Delta^j\eta_i|$$

$$+C\sum_{j=1}^{a+1}(2^m)^j\sum_{s=m+1}^{\infty}(2^{s-1})^{-j}$$

$$\cdot\sum_{i=2^{s-1}}^{2^s-1}A_i^{j-1}|\Delta^j\eta_i|\left\{i^{a+2-j}\left|\Delta^{a+2-j}G\left(\frac{i+j}{u}\right)\right|\right\}$$

$$\leqslant C\|\eta\|_{\gamma+1,w},$$

by our previous observations, which completes the proof.

4. Proof of Theorem 1. In view of the Marcinkiewicz multiplier theorem and Lemma 1 we may assume that $\alpha \geqslant \beta \geqslant -\frac{1}{2}$, $\alpha > -\frac{1}{2}$, and $\alpha+\frac{1}{2}<\gamma<\alpha+\frac{3}{2}$. Since the sums

$$S_n(\theta)=\sum_{k=0}^{n}c_kR_k(\cos\theta)$$

are dense in L^p, $1<p<\infty$, it suffices to show that

$$\|S_n^\eta\|_p\leqslant C_p\|S_n\|_p,\qquad 1<p<\infty, \tag{4.1}$$

with C_p independent of S_n. To prove this we will need the following convolution structure for Jacobi series.

It was proved in [9], [10] that if $\alpha\geqslant\beta\geqslant-\frac{1}{2}$ and $\alpha>-\frac{1}{2}$, then there is a nonnegative function $K(\theta,\varphi,\psi)$, symmetric in θ, φ, ψ, such that

$$R_n(\cos\theta)R_n(\cos\varphi)=\int R_n(\cos\psi)K(\theta,\varphi,\psi)\,d\mu(\psi) \tag{4.2}$$

and

$$\int K(\theta,\varphi,\psi)\,d\mu(\psi)=1 \tag{4.3}$$

for $0<\theta,\varphi<\pi$. From this it follows that the generalized translation operator T_φ defined by

$$T_\varphi f(\theta)=\int f(\psi)K(\theta,\varphi,\psi)\,d\mu(\psi)$$

$$\sim\sum_{k=0}^{\infty}f^\wedge(k)h_kR_k(\cos\theta)R_k(\cos\varphi)$$

is a bounded operator on L^1, and that with the convolution of two functions in L^1 defined by

$$(f*g)(\theta) = \int f(\varphi)(T_\varphi g(\theta))\,d\mu(\varphi)$$
$$= \iint f(\varphi)g(\psi)K(\theta,\varphi,\psi)\,d\mu(\psi)\,d\mu(\varphi)$$

we have $(f*g)\hat{}(k) = f\hat{}(k)g\hat{}(k)$.

Hence, with $G(t)$ as in Lemma 2,

$$S_n^\eta(\theta) = (S_n * \chi_u)(\theta), \qquad u \geqslant n,$$

where

$$\chi_u(\theta) = \sum_{k=0}^{\infty} \eta_k G\left(\frac{k}{u}\right) h_k R_k(\cos\theta), \qquad u > 0, \tag{4.4}$$

and so in proving (4.1) we may restrict ourselves to a (uniform in u) discussion of the kernel (4.4). On account of the observations in Coifman and Weiss [5, pp. 74, 75] it suffices to show that the generalized translation $T_\varphi \chi_u(\theta)$ satisfies the following version of Hörmander's condition [12] (see also Connett and Schwartz [6], [7]):

$$\int_E |T_\varphi \chi_u(\theta) - T_{\varphi_0}\chi_u(\theta)|\,d\mu(\theta) \leqslant C \tag{4.5}$$

where $E = \{\theta: |\theta - \varphi_0| > 2|\varphi - \varphi_0|\}$ and C is independent of u.

By a change in order of summation (cf. [17, p. 22]), for each $u > 0$ we can represent $T_\varphi \chi_u(\theta)$ by

$$T_\varphi \chi_u(\theta) = \sum_{n=0}^{\infty} A_n^\gamma \left(\Delta^{\gamma+1}\left(\eta_n G\left(\frac{n}{u}\right)\right)\right)\sigma_n^\gamma(\theta,\varphi),$$

where $\sigma_n^\gamma(\theta,\varphi)$ is the Cesàro kernel (1.2). Hence,

$$\begin{aligned}
&\int_E |T_\varphi \chi_u(\theta) - T_{\varphi_0}\chi_u(\theta)|\,d\mu(\theta)\\
&\leqslant \sum_{m=1}^{\infty}\sum_{n=2^{m-1}}^{2^m-1} A_n^\gamma \left|\Delta^{\gamma+1}\left(\eta_n G\left(\frac{n}{u}\right)\right)\right| \int_E |\sigma_n^\gamma(\theta,\varphi) - \sigma_n^\gamma(\theta,\varphi_0)|\,d\mu(\theta)\\
&\leqslant \left\|\left\{\eta_k G\left(\frac{k}{u}\right)\right\}\right\|_{\gamma+1,w} \sum_{m=1}^{\infty} \sup_{2^{m-1}\leqslant n<2^m} \int_E |\sigma_n^\gamma(\theta,\varphi) - \sigma_n^\gamma(\theta,\varphi_0)|\,d\mu(\theta)\\
&\leqslant C\|\eta\|_{\gamma+1,w}\left(\sum_{m=0}^{m_0} + \sum_{m=m_0+1}^{\infty}\right) \sup_{2^m\leqslant n<2^{m+1}} \int_E |\sigma_n^\gamma(\theta,\varphi) - \sigma_n^\gamma(\theta,\varphi_0)|\,d\mu(\theta),
\end{aligned} \tag{4.6}$$

where we have chosen m_0 such that

$$2^{m_0-1}|\varphi - \varphi_0|/\pi < 1 \leqslant 2^{m_0}|\varphi - \varphi_0|/\pi.$$

Then

$$2^{m-m_0} \leqslant n|\varphi - \varphi_0|/\pi < 2^{m-m_0+2} \quad \text{when } 2^m \leqslant n < 2^{m+1},$$

and so to complete the proof one need but observe that by Proposition 1 (proved below) both sums in (4.6) are bounded independent of the choice of φ and φ_0.

Corresponding to Connett and Schwartz [7], it also follows from (4.5), under the conditions of Theorem 1, that η is of weak type (1, 1).

5. **Proof of Proposition 1.** Let us first consider the easier inequality (1.4) for $\alpha \geqslant \beta \geqslant -\frac{1}{2}$, $\alpha > -\frac{1}{2}$. Then, by (4.2),

$$\sigma_n^\gamma(\theta,\varphi) = \int \sigma_n^\gamma(\psi) K(\theta,\varphi,\psi)\, d\mu(\psi),$$

where $\sigma_n^\gamma(\psi) = \sigma_n^\gamma(\psi, 0)$. Using a standard method and the fact that $K(\theta,\varphi,\psi) = 0$ unless $|\theta - \varphi| < \psi$ (see [10, p. 268]) we have, for $0 < \varepsilon < \gamma - \alpha - \frac{1}{2} \leqslant 1$,

$$\begin{aligned} &\int |\sigma_n^\gamma(\theta,\varphi)| (n|\theta - \varphi|)^\varepsilon\, d\mu(\theta) \\ &\qquad \leqslant \iint |\sigma_n^\gamma(\psi)| K(\theta,\varphi,\psi)(n|\theta-\varphi|)^\varepsilon\, d\mu(\psi)\, d\mu(\theta) \\ &\qquad \leqslant \int |\sigma_n^\gamma(\psi)|(n\psi)^\varepsilon \left\{ \int K(\theta,\varphi,\psi)\, d\mu(\theta) \right\} d\mu(\psi) \\ &\qquad = \int |\sigma_n^\gamma(\psi)|(n\psi)^\varepsilon\, d\mu(\psi) \leqslant C_\varepsilon \end{aligned} \tag{5.1}$$

by means of (4.3) and the Bonami and Clerc [3, p. 230] estimates

$$|\sigma_n^\gamma(\psi)| \leqslant \begin{cases} Cn^{2\alpha+2}, & 0 \leqslant \psi \leqslant \pi/2, \\ Cn^{\alpha+1/2-\gamma}\psi^{-\alpha-\gamma-3/2}, & 2/n \leqslant \psi \leqslant \pi/2, \\ Cn^{\alpha+1/2-\gamma}(\pi-\psi)^{-\beta-1/2}, & \pi/2 \leqslant \psi \leqslant \pi - 2/n, \\ Cn^{\alpha+\beta+1-\gamma}, & \pi/2 \leqslant \psi \leqslant \pi, \end{cases} \tag{5.2}$$

which hold for $\gamma \leqslant \alpha + \frac{3}{2}$, $\alpha \geqslant \beta \geqslant -\frac{1}{2}$. Similarly, the case $\alpha = \beta = -\frac{1}{2}$ of (1.4) follows by using the identity

$$\begin{aligned} &R_n^{(-1/2,-1/2)}(\cos\theta) R_n^{(-1/2,-1/2)}(\cos\varphi) \\ &\qquad = \tfrac{1}{2} R_n^{(-1/2,-1/2)}(\cos(\theta+\varphi)) + \tfrac{1}{2} R_n^{(-1/2,-1/2)}(\cos(\theta-\varphi)), \end{aligned} \tag{5.3}$$

in place of the product formula (4.2).

We shall now prove (1.3). In [13] Koornwinder showed that if $\alpha > \beta > -\frac{1}{2}$, then for the right side of (4.2) we have that

$$\int_0^\pi R_n(\cos\psi)K(\theta,\varphi,\psi)\,d\mu(\psi) = \int_0^\pi \int_0^1 R_n(\cos\Psi)\,dm(r,t), \tag{5.4}$$

where

$$dm(r,t) = \frac{2\Gamma(\alpha+1)}{\Gamma(\frac{1}{2})\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})}(1-r^2)^{\alpha-\beta-1}r^{2\beta+1}(\sin t)^{2\beta}\,dr\,dt, \tag{5.5}$$

$$\begin{aligned}\cos\Psi &= \cos\theta\cos\varphi + r\sin\theta\sin\varphi\cos t\\ &\quad+\tfrac{1}{2}(r^2-1)(1-\cos\theta)(1-\cos\varphi),\end{aligned} \tag{5.6}$$

and he also showed that (5.4) holds with R_n replaced by an arbitrary continuous function.

Our proof of (1.3) depends at a crucial step on the observation that with $\cos\Psi$ defined by (5.6) and $0 \leqslant r \leqslant 1$, $0 \leqslant t, \theta, \varphi \leqslant \pi$, we have the inequality

$$\left|\frac{\partial\cos\Psi}{\partial\varphi}\right| \leqslant C(1-\cos\Psi)^{1/2}. \tag{5.6}$$

Clearly,

$$\begin{aligned}1-\cos\Psi &= 1-\cos(\theta-\varphi) + (1-r\cos t)\sin\theta\sin\varphi\\ &\quad+\tfrac{1}{2}(1-r^2)(1-\cos\theta)(1-\cos\varphi),\end{aligned} \tag{5.7}$$

and

$$\frac{\partial\cos\Psi}{\partial\varphi} = \sin(\theta-\varphi) + (r\cos t-1)\sin\theta\cos\varphi + \tfrac{1}{2}(r^2-1)(1-\cos\theta)\sin\varphi.$$

Hence, by (5.7),

$$\begin{aligned}|\sin(\theta-\varphi)| &= (1+\cos(\theta-\varphi))^{1/2}(1-\cos(\theta-\varphi))^{1/2}\\ &\leqslant 2^{1/2}(1-\cos\Psi)^{1/2},\\ |\tfrac{1}{2}(r^2-1)(1-\cos\theta)\sin\varphi| &\leqslant C\{(1-r^2)(1-\cos\theta)(1-\cos\varphi)\}^{1/2}\\ &\leqslant C(1-\cos\Psi)^{1/2}.\end{aligned}$$

If $\varphi \leqslant \theta/2$ then $\sin|(\theta-\varphi)/2| \geqslant \sin\theta/4$, while if $\varphi \geqslant (\pi+\theta)/2$ then $\sin|(\theta-\varphi)/2| \geqslant \sin(\pi-\theta)/4$, and so in either case

$$|(r\cos t-1)\sin\theta\cos\varphi| \leqslant 2\sin\theta\left(\frac{1-\cos\Psi}{1-\cos(\theta-\varphi)}\right)^{1/2}$$
$$= C(1-\cos\Psi)^{1/2}\frac{\sin\theta}{\sin|(\theta-\varphi)/2|} \leqslant C(1-\cos\Psi)^{1/2}.$$

If $\theta/2 < \varphi < (\pi+\theta)/2$, then $\sin\varphi \geqslant \sin\theta/2$ when $\varphi \leqslant \pi/2$ and $\sin\varphi \geqslant \sin(\pi+\theta)/2 = \sin(\pi-\theta)/2$ when $\varphi \geqslant \pi/2$, and so

$$|(r\cos t-1)\sin\theta\cos\varphi|$$
$$\leqslant (1-r\cos t)\sin\theta\left(\frac{1-\cos\Psi}{(1-r\cos t)\sin\theta\sin\varphi}\right)^{1/2}$$
$$\leqslant 2^{1/2}(1-\cos\Psi)^{1/2}\left(\frac{\sin\theta}{\sin\varphi}\right)^{1/2} \leqslant C(1-\cos\Psi)^{1/2}.$$

Thus inequality (5.6) holds.

From (4.2), (5.4), (5.6) it follows that if $\alpha > \beta > -\frac{1}{2}$ and $\Phi_0 < \Phi$ then

$$\int |\sigma_n^\gamma(\theta,\Phi) - \sigma_n^\gamma(\theta,\Phi_0)|\,d\mu(\theta)$$
$$= \int \left|\int_{\Phi_0}^{\Phi} \frac{\partial}{\partial\varphi}\sigma_n^\gamma(\theta,\varphi)\,d\varphi\right| d\mu(\theta)$$
$$= \int \left|\int_{\Phi_0}^{\Phi}\int_0^\pi\int_0^1 \frac{\partial\sigma_n^\gamma(\Psi)}{\partial\varphi}\,dm(r,t)\,d\varphi\right| d\mu(\theta)$$
$$(5.8)\qquad = \int \left|\int_{\Phi_0}^{\Phi}\int_0^\pi\int_0^1 \frac{d\sigma_n^\gamma(\Psi)}{d\cos\Psi}\frac{\partial\cos\Psi}{\partial\varphi}\,dm(r,t)\,d\varphi\right| d\mu(\theta)$$
$$\leqslant C\iint_{\Phi_0}^{\Phi}\int_0^\pi\int_0^1 (1-\cos\Psi)^{1/2}\left|\frac{d\sigma_n^\gamma(\Psi)}{d\cos\Psi}\right| dm(r,t)\,d\varphi\,d\mu(\theta)$$
$$= C\iint_{\Phi_0}^{\Phi}\int (1-\cos\psi)^{1/2}\left|\frac{d\sigma_n^\gamma(\psi)}{d\cos\psi}\right| K(\theta,\varphi,\psi)\,d\mu(\psi)\,d\varphi\,d\mu(\theta)$$
$$= C\iint_{\Phi_0}^{\Phi} (1-\cos\psi)^{1/2}\left|\frac{d\sigma_n^\gamma(\psi)}{d\cos\psi}\right| \left\{\int K(\theta,\varphi,\psi)\,d\mu(\theta)\right\} d\varphi\,d\mu(\psi),$$

which by means of (4.3) gives

$$(5.9)\qquad \int |\sigma_n^\gamma(\theta,\varphi) - \sigma_n^\gamma(\theta,\varphi_0)|\,d\mu \leqslant C|\varphi-\varphi_0| \int (1-\cos\psi)^{1/2}\left|\frac{d\sigma_n^\gamma(\psi)}{d\cos\psi}\right| d\mu(\psi)$$

for $0 \leqslant \varphi, \varphi_0 \leqslant \pi$.

Similarly, the case $\alpha = \beta = -\frac{1}{2}$ of (5.9) follows by using (5.3), and the cases $\alpha = \beta > -\frac{1}{2}$ and $\beta = -\frac{1}{2} < \alpha$ follow by using the fact [13, p. 131] that (5.4) also holds in these two cases with the measure $dm(r, t)$ replaced by the measures

$$\frac{\Gamma(\alpha+1)}{\Gamma(\frac{1}{2})\Gamma(\alpha+\frac{1}{2})}\delta(1-r)(\sin t)^{2\alpha}\,dr\,dt,$$

and

$$\frac{\Gamma(\alpha+1)}{\Gamma(\frac{1}{2})\Gamma(\alpha+\frac{1}{2})}(1-r^2)^{\alpha-1/2}(\delta(t)+\delta(\pi-t))\,dr\,dt,$$

respectively, where $\delta(t)$ represents Dirac's delta function. Note, in particular, that when $\alpha = \beta > -\frac{1}{2}$ this is equivalent to using the Gegenbauer product formula (see [8, 3.15 (20)])

$$\begin{aligned} &R_n^{(\alpha,\alpha)}(\cos\theta)R_n^{(\alpha,\alpha)}(\cos\varphi) \\ &\quad = \frac{\Gamma(\alpha+1)}{\Gamma(\frac{1}{2})\Gamma(\alpha+\frac{1}{2})}\int_0^\pi R_n^{(\alpha,\alpha)}(\cos\theta\cos\varphi+\sin\theta\sin\varphi\cos t)(\sin t)^{2\alpha}\,dt \end{aligned}$$

in place of the Koornwinder product formula.

By using [16, (4.21.7)]

$$(d/dx)P_n^{(\alpha,\beta)}(x) = \tfrac{1}{2}(n+\alpha+\beta+1)P_{n-1}^{(\alpha+1,\beta+1)}(x)$$

and the method which Bonami and Clerc [3] used to derive the estimates (5.2) it is easily shown that if $\gamma \leqslant \alpha + \frac{3}{2}$ (and in fact for a larger range) then

$$\left|\frac{d\sigma_n^\gamma(\psi)}{d\cos\psi}\right| \leqslant \begin{cases} Cn^{2\alpha+4}, & 0 \leqslant \psi \leqslant \pi/2, \\ Cn^{\alpha+3/2-\gamma}\psi^{-\alpha-\gamma-5/2}, & 2/n \leqslant \psi \leqslant \pi/2, \\ Cn^{\alpha+3/2-\gamma}(\pi-\psi)^{-\beta-3/2}, & \pi/2 \leqslant \psi \leqslant \pi - 2/n, \\ Cn^{\alpha+\beta+3-\gamma}, & \pi/2 \leqslant \psi \leqslant \pi, \end{cases}$$

when $\alpha \geqslant \beta \geqslant -\frac{1}{2}$. These estimates give

$$\int (1-\cos\psi)^{1/2}\left|\frac{d\sigma_n^\gamma(\psi)}{d\cos\psi}\right| d\mu(\psi) \leqslant Cn, \tag{5.10}$$

which, with (5.9), completes the proof of Proposition 1. Inequality (5.10) and the last inequality in (5.1) can also be proved by repeating the steps in Szegö's proof [16, §9.41] of the boundedness of the sequence of Lebesgue constants for Jacobi series.

6. The Poisson kernel. If $\sum_0^\infty a_n$ is any absolutely convergent series and

$$k(\theta, \varphi) = \sum_{n=0}^{\infty} a_n R_n(\cos\theta) R_n(\cos\varphi),$$

then by the methods in (5.1) and (5.8) it follows for $\alpha \geqslant \beta \geqslant -\frac{1}{2}$ that

$$\int |k(\theta,\varphi)| \cdot |\theta - \varphi|^\varepsilon \, d\mu(\theta) \leqslant \int |k(\psi)| \psi^\varepsilon \, d\mu(\psi), \qquad \varepsilon \geqslant 0, \tag{6.1}$$

and

$$\int |k(\theta,\varphi) - k(\theta,\varphi_0)| \, d\mu(\theta) \leqslant C|\varphi - \varphi_0| \int (1 - \cos\psi)^{1/2} \left| \frac{dk(\psi)}{d\cos\psi} \right| d\mu(\psi), \tag{6.2}$$

where $k(\psi) = k(\psi, 0)$. In particular, for the Poisson kernel $W_r(\theta, \varphi)$, $0 < r \leqslant 1$, we have

$$\int W_r(\theta,\varphi) |\theta - \varphi|^\varepsilon \, d\mu(\theta) \leqslant \int W_r(\psi) \psi^\varepsilon \, d\mu(\psi), \qquad \varepsilon \geqslant 0, \tag{6.3}$$

and

$$\begin{aligned} &\int |W_r(\theta,\varphi) - W_r(\theta,\varphi_0)| \, d\mu(\theta) \\ &\qquad \leqslant C|\varphi - \varphi_0| \int (1 - \cos\psi)^{1/2} \left| \frac{dW_r(\psi)}{d\cos\psi} \right| d\mu(\psi), \end{aligned} \tag{6.4}$$

where $W_r(\psi) = W_r(\psi, 0)$. Thus to prove (1.6) and the inequality (see [7, p. 249])

$$\int W_r(\theta,\varphi) \left(\frac{|\theta - \varphi|}{r} \right)^\varepsilon d\mu(\theta) \leqslant C_\varepsilon, \qquad 0 \leqslant \varepsilon < 1, \tag{6.5}$$

we need but show that

$$\int (1 - \cos\psi)^{1/2} \left| \frac{dW_r(\psi)}{d\cos\psi} \right| d\mu(\psi) \leqslant \frac{C}{r} \tag{6.6}$$

and

$$\int W_r(\psi) \left(\frac{\psi}{r} \right)^\varepsilon d\mu(\psi) \leqslant C_\varepsilon, \qquad 0 \leqslant \varepsilon < 1. \tag{6.7}$$

Note that in (6.3) and (6.7), unlike in the Cesàro kernel case (5.1), it was not necessary to use the absolute value of the Poisson kernel since this kernel is nonnegative (it has recently been proved [11] that $\sigma_n^\gamma(\theta, \varphi) \geqslant 0$, $n = 0, 1, \ldots,$ for $\alpha \geqslant \beta \geqslant -\frac{1}{2}$ if and only if $\gamma \geqslant \alpha + \beta + 2$).

The estimates needed to prove (6.6) and (6.7) are much easier to derive than the corresponding ones for the Cesàro kernel case since we can use the hypergeometric representation

$$(6.8)\quad \begin{aligned} W_r(\psi) &= \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \cdot \frac{r}{(2-r)^{\alpha+\beta+2}} \\ &\quad \times F\left[\frac{\alpha+\beta+2}{2}, \frac{\alpha+\beta+3}{2}; \beta+1; \frac{2(1-r)(1+\cos\psi)}{(2-r)^2}\right], \end{aligned}$$

which is a consequence of Bailey's formula [1, (2.3)]. From (6.8) we have

$$\begin{aligned} \frac{\partial W_r(\psi)}{\partial \cos\psi} = \frac{Cr(1-r)}{(2-r)^{\alpha+\beta+2}} F\bigg[\frac{\alpha+\beta+4}{2}; \frac{\alpha+\beta+5}{2}; \\ \beta+2; \frac{2(1-r)(1+\cos\psi)}{(2-r)^2}\bigg], \end{aligned}$$

where C depends only on α and β. Thus

$$(6.9)\quad \begin{aligned} W_r(\psi) &= \frac{Cr(2-r)^{\alpha-\beta+1}}{\{r^2+2(1-r)(1-\cos\psi)\}^{\alpha+3/2}} \\ &\quad \times F\left[\frac{\beta-\alpha}{2}, \frac{\beta-\alpha-1}{2}; \beta+1; \frac{2(1-r)(1+\cos\psi)}{(2-r)^2}\right] \end{aligned}$$

and

$$(6.10)\quad \begin{aligned} \frac{\partial W_r(\psi)}{\partial \cos\psi} &= \frac{Cr(1-r)(2-r)^{\alpha-\beta+1}}{\{r^2+2(1-r)(1-\cos\psi)\}^{\alpha+5/2}} \\ &\quad \times F\left[\frac{\beta-\alpha}{2}, \frac{\beta-\alpha-1}{2}; \beta+2; \frac{2(1-r)(1+\cos\psi)}{(2-r)^2}\right] \end{aligned}$$

by means of the transformation formula [8, 2.9(2)]

$$F[a,b;c;z] = (1-z)^{c-a-b} F[c-a, c-b; c; z].$$

Since the series $F[a,b;c;z]$ converges [8, §2.1.1] for $z = 1$ and hence uniformly for $0 \leqslant z \leqslant 1$ when $c-a-b > 0$, $c > -1$, it follows that the hypergeometric series on the right sides of (6.9) and (6.10) are uniformly bounded; so that

$$W_r(\psi) \leqslant \frac{Cr}{(r^2+(1-r)\psi^2)^{\alpha+3/2}}$$

and

$$\left|\frac{\partial W_r(\psi)}{\partial \cos\psi}\right| \leqslant \frac{Cr(1-r)}{(r^2+(1-r)\psi^2)^{\alpha+5/2}}$$

for $\alpha, \beta > -1$, $0 < r \leqslant 1$, $0 \leqslant \psi \leqslant \pi$. By integrating over the intervals $(0, r)$ and (r, π) these estimates give (6.6) and (6.7), and hence (1.6) and (6.5) for $\alpha \geqslant \beta \geqslant -\frac{1}{2}$. Inequality (6.5) can be shown to also hold for $\alpha + \beta \geqslant -1$, $-1 < \beta < -\frac{1}{2}$, by using the integral representation for the product of Jacobi polynomials in [10].

Extensions of Theorem 1 and additional applications of the methods of this paper will be given elsewhere.

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