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A CLASS OF INFINITELY CONNECTED DOMAINS AND THE CORONA

BY

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ABSTRACT. Let D be a bounded domain in the complex plane. Let $H^\infty(D)$ be the Banach algebra of bounded analytic functions on D. The corona problem asks whether D is weak* dense in the space $\mathfrak{M}(D)$ of maximal ideals of $H^\infty(D)$. Carleson [3] proved that the open unit disc Δ_0 is dense in $\mathfrak{M}(\Delta_0)$. Stout [9] extended Carleson's result to finitely connected domains. Behrens [2] found a class of infinitely connected domains for which the corona problem has an affirmative answer.

In this paper we will use Behrens' idea to extend the results to more general domains. See [11] for further extensions and applications of these techniques.

Introduction. By a Δ-domain we mean a domain D obtained from the open unit disc Δ_0 by deleting the origin and a sequence of disjoint closed discs $\Delta_n = \Delta(c_n, r_n) = \{z\colon\ |z - c_n| \leqslant r_n\}$ with $c_n \to 0$. Under theadditional hypothesis $\Sigma r_n/|c_n| < \infty$, Zalcman showed in [10] that there is a distinguished homomorphism in $\mathfrak{M}(D)$ defined by

$$\phi_0(f) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\xi)}{\xi}\, d\xi.$$

The distinguished homomorphism is always adherent to D [7]. Behrens showed that if the corona fails for any domain it fails for a Δ-domain. He also showed that if there are numbers $R_n > r_n$ such that $\Sigma r_n/R_n < \infty$ and the discs $D_n = \Delta(c_n, R_n)$ are disjoint, then D is dense in $\mathfrak{M}(D)$. To prove this Behrens constructed an isomorphism of $H^\infty(D)$ into $H^\infty(\Delta_0 \times N)$, the algebra of bounded functions which are analytic on each slice of $\Delta_0 \times N$, where N is the nonnegative integers. He then used the fact that $\Delta_0 \times N$ is dense in $\mathfrak{M}(\Delta_0 \times N)$ [1], to obtain D is dense in $\mathfrak{M}(D)$. In this paper we will use Behrens' idea to extend the results to more general domains.

Notation and statement of results. Throughout we assume that there exist numbers $R_n > r_n$ such that the discs $D_n = \Delta(c_n, R_n)$ are disjoint and such that $r_n/R_n \to 0$. For notational convenience let $c_0 = 0$, $r_0 = R_0 = 1$. Let $E_0(z) = z$ and $E_n(z) = r_n/(z - c_n)$, for $z \in \Delta_n^c = \mathbf{C} \setminus \Delta_n$, $n = 1, 2, \ldots$. Choose a

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sequence of positive integers $\{k_n\}$ such that $\Sigma(r_n/R_n)^{k_n} < \infty$. Let $L_n(z) = E_n(z)^{k_n} - E_n(0)^{k_n}$ and $L(z) = \Sigma L_n(z)$. Choose $S_n = \sqrt{r_n R_n}$ so that $r_n/S_n \to 0$ and $S_n/R_n \to 0$, and let $B_n = \Delta(c_n, S_n)$. Let

$$X = \mathfrak{M}(\Delta_0 \times N)\Big\backslash\Big[\bigcup_n \mathfrak{M}(\Delta_0) \times \{n\}\Big].$$

For $f \in H^\infty(D)$ let

$$a_n(f) = \frac{1}{2\pi i}\int_{\partial\Delta_n} \frac{f(\zeta)}{\zeta - c_n}\, d\zeta \quad \text{and} \quad (P_n f)(z) = \frac{1}{2\pi i}\int_{\partial\Delta_n} \frac{f(\zeta)}{\zeta - z}\, d\zeta,$$

where $z \in \Delta_0$ if $n = 0$ and $z \in \Delta_n^c$ if $n = 1, 2, 3, \ldots$. Let $\mathfrak{M}_0$ denote the fiber in $\mathfrak{M}(D)$ at the origin and $A_0 = H^\infty(D)|_{\mathfrak{M}_0}$.

THEOREM 1. *Let D be a Δ-domain with $r_n/R_n \to 0$. If $\phi \in \mathfrak{M}_0$ and $\hat{L}(\phi) \neq 0$, then ϕ is adherent to D.*

THEOREM 2. *Let D be a Δ-domain. Suppose there exists a positive integer m such that $\Sigma(r_n/R_n)^m < \infty$. Let $\phi \in \mathfrak{M}_0$. If, for some $f \in H^\infty(D)$, $\phi(f) \neq 0$ but $a_n(f) \to 0$, then ϕ is adherent to D.*

In [1] Behrens established that $a_n \to \phi_0$ in norm if $\Sigma r_n/R_n < \infty$. Thus Behrens' theorem can be regarded as a consequence of Theorem 2.

Preliminary lemmas and proofs.

LEMMA 1. *Let $\varepsilon > 0$ be given. Then there exists a positive integer N such that, for all $f \in H^\infty(D)$, $\|f\| \leq 1$,*

$$|f(z) - (P_n f)(z) - a_n(f)| < \varepsilon, \quad \textit{for } z \in B_n \setminus \Delta_n \textit{ and } n \geq N.$$

PROOF. Choose N so that $S_n/(R_n - S_n) < \varepsilon$, $n \geq N$. Write $f(z) = (P_n f)(z) + a_n(f) + F_n(z)$, for $z \in D_n \setminus \Delta_n$, where

$$F_n(z) = \sum_{j=1}^{\infty} b_{nj}(z - c_n)^j \quad \text{and} \quad b_{nj} = \frac{1}{2\pi i}\int_{\partial D_n} \frac{f(\zeta)}{(\zeta - c_n)^{j+1}}\, d\zeta.$$

Then $|b_{nj}| \leq 1/R_n^j$ and, for $z \in B_n \setminus \Delta_n$, $n \geq N$,

$$|f(z) - (P_n f)(z) - a_n(f)| = |F_n(z)| \leq \sum_{j=1}^{\infty} \frac{S_n^j}{R_n^j} = \frac{S_n}{R_n - S_n} < \varepsilon.$$

LEMMA 2. *Let $f_n \in H^\infty(\Delta_n^c)$ with $\|f_n\| \leq M$, $n = 1, 2, \ldots$. Let*

$$f(z) = \sum_{n=0}^{\infty} L_n(z) f_n(z).$$

Then $f \in H^\infty(D)$ and, given $\varepsilon > 0$, there is a positive integer N such that

$$\sum_{n=0;\, n\neq m}^{\infty} |L_n(z)f_n(z)| < \varepsilon \quad \textit{for } z \in D_m,\ m \geqslant N.$$

Moreover, $a_n(f) \to 0$.

PROOF. Observe that $a_m(f) = \sum_{n=0;n\neq m}^{\infty}(L_nf_n)(c_m)$. Incorporating this observation the proof is essentially the same as the analogous lemma in [2].

Define Ψ: $H^\infty(D) \to H^\infty(\Delta_0 \times N)$ by

$$\Psi(f)(z, n) = (P_nf) \circ E_n^{-1}(z) + a_n(f), \quad f \in H^\infty(D).$$

LEMMA 3. Ψ *is continuous.*

PROOF. An elementary computation shows there exists $M > 0$ such that $\|P_nf\| \leqslant M\|f\|_{B_n \setminus \Delta_n}$, for all n, whenever $\limsup(r_n/R_n) < 1$.

LEMMA 4. $\|f\|_{\mathfrak{M}_0} = \|\Psi(f)\|_X$, *all* $f \in H^\infty(D)$.

PROOF. $\|f\|_{\mathfrak{M}_0} = \limsup\|f\|_{B_n\setminus\Delta_n} = \limsup\|\Psi(f)\|_{\Delta_0\times\{n\}} = \|\Psi(f)\|_X$.

The middle equality follows from Lemma 1, the left equality from a well-known theorem concerning the Shilov boundary of the fiber algebra [6], and the right equality from the definition of X and elementary properties of $H^\infty(\Delta_0 \times N)$.

LEMMA 5. *If* $f, g \in H^\infty(D)$ *then*

$$\Psi(fg)|_X = \Psi(f)|_X \cdot \Psi(g)|_X.$$

PROOF. Assume $\|f\| \leqslant 1$, $\|g\| \leqslant 1$. Let $\varepsilon > 0$ be given. Then there exists a positive integer N such that, for $z \in B_n \setminus \Delta_n$, $n \geqslant N$,

(1) $|f(z) - (P_nf)(z) - a_n(f)| < \varepsilon/3$;

(2) $|g(z) - (P_ng)(z) - a_n(g)| < \varepsilon/3$;

(3) $|(fg)(z) - (P_nfg)(z) - a_n(fg)| < \varepsilon/3$.

Multiplying (1) and (2) and simplifying we get

$$\left| -(fg)(z) + \left[(P_nf)(z) + a_n(f)\right]\left[(P_ng)(z) + a_n(g)\right]\right| \leqslant \varepsilon,$$

for $z \in B_n \setminus \Delta_n$, $n \geqslant N$.

Adding to (3), we get

$$\left|\left[(P_nf)(z) + a_n(f)\right]\left[(P_ng)(z) + a_n(g)\right] - \left[(P_nfg)(z) + a_n(fg)\right]\right| < 2\varepsilon,$$

for $z \in B_n \setminus \Delta_n$, $n \geqslant N$. Thus

$$|\Psi(f)(z, n) \cdot \Psi(g)(z, n) - \Psi(fg)(z, n)| < \varepsilon,$$

for all $z \in \Delta_0$, $n \geqslant N$. It follows that

$$\Psi(f)(\phi) \cdot \Psi(g)(\phi) = \Psi(fg)(\phi), \quad \text{for } \phi \in X.$$

Summarizing, we have shown:

PROPOSITION 1. *Ψ induces a map $\overline{\Psi}$: $A_0 \to H^\infty(\Delta_0 \times N)|_X$. $\overline{\Psi}$ is an algebra isometric isomorphism of A_0 with a closed subalgebra B_0 of $H^\infty(\Delta_0 \times N)|_X$.*

The map θ: $\mathfrak{M}(B_0) \to \mathfrak{M}_0$ defined by $\theta(\phi)(f) = \phi(\overline{\Psi}(f))$, for $\phi \in \mathfrak{M}(B_0)$, $f \in A_0$, is a homeomorphism. Note that

$$H^\infty(\Delta_0 \times N) \cong ZH^\infty(\Delta_0 \times N) + l^\infty,$$

where Z is the "coordinate" function $Z(\lambda, n) = \lambda$. The sequence $\{a_n(f)\}$ is the l^∞-component of $\Psi(f)$.

LEMMA 6. $\lim_{n\to\infty}\|f - a_n(f)\|_{\partial B_n} = 0$, *for* $f \in H^\infty(D)$.

PROOF. From Lemma 1 it suffices to show $\lim_{n\to\infty}\|P_n f\|_{\partial B_n} \to 0$. But for $z \in \partial B_n$,

$$|(P_n f)(z)| = \left|\frac{1}{2\pi i}\int_{\partial\Delta_n}\frac{f(\xi)}{\xi - z}\,d\xi\right| \leqslant \|f\|\,\frac{r_n}{S_n - r_n} \to 0.$$

LEMMA 7. *Let $T \in H^\infty(\Delta_0 \times N)$ be defined by $T(z, n) = z^{k_n}$. Then $\overline{\Psi}(L) = T$ and*

$$TH^\infty(\Delta_0 \times N)|_X + \mathbf{C} \subset B_0.$$

PROOF. Let $h = \{z^{k_n}h_n\} \in TH^\infty(\Delta_0 \times N)$ and let

$$f = \sum_{n=1}^{\infty} L_n(h_n \circ E_n).$$

Then $f \in H^\infty(D)$ and $a_n(f) \to 0$, by Lemma 2. Also $\overline{\Psi}(f) = h|_X$.

At this point recall some general facts. Let B be a commutative Banach algebra, B_0 a closed subalgebra of B, and J a closed ideal in$B$. If $J \subset B_0$, then every $\phi \in \mathfrak{M}_{B_0}$ which does not annihilate J has a unique extension to $\mathfrak{M}_B$. If furthermore, $B_0 = J + \mathbf{C}$, then $\mathfrak{M}_{B_0}$ is obtained from $\mathfrak{M}_B$ by identifying the hull of J to a point.

If B_0 is as defined earlier, $B = H^\infty(\Delta_0 \times N)|_X$ and $J = TH^\infty(\Delta_0 \times N)|_X$, then $\mathfrak{M}_B = X$ and the following lemma is immediate.

LEMMA 8. *If $\phi \in \mathfrak{M}(B_0)$ and $\phi(T) \neq 0$, then ϕ extends to an evaluation homomorphism at a point of X.*

For the proof of Theorem 1, we need one final lemma.

LEMMA 9. *A homomorphism $\phi \in \mathfrak{M}(B_0)$ is an evaluation homomorphism at a point of X if and only if the corresponding homomorphism $\theta(\phi)$ in $\mathfrak{M}_0$ is adherent to $\bigcup(B_n \setminus \Delta_n)$.*

PROOF. Suppose w_α is a net in $\bigcup(B_n \setminus \Delta_n)$ converging to $\phi \in \mathfrak{M}_0$, where $w_\alpha \in B_{n_\alpha} \setminus \Delta_{n_\alpha}$. Define $(z_\alpha, n_\alpha) \in \Delta_0 \times N$ so that $E_{n_\alpha}(w_\alpha) = z_\alpha$. Then $n_\alpha \to \infty$. Passing to a subnet, we can assume (z_α, n_α) converges to $\phi^* \in X$. If $f \in H^\infty(D)$, then

$$f(\phi) = \lim f(w_\alpha) = \lim(\Psi f)(z_\alpha, n_\alpha) \doteq \Psi f(\phi^*)$$

so that $\theta(\phi^*) = \phi$.

Conversely, suppose $\phi^* \in X$. Since $r_n/S_n \to 0$, X is adherent to

$$V = U\left[E_n(B_n \setminus \Delta_n) \times \{n\}\right] \subset \Delta_0 \times N.$$

Choose a net $(z_\alpha, n_\alpha) \in V$ so that $(z_\alpha, n_\alpha) \to \phi^*$. Then $n_\alpha \to \infty$. Set $w_\alpha = E_{n_\alpha}^{-1}(z_\alpha) \in B_{n_\alpha} \setminus \Delta_{n_\alpha}$. Since $\lim(\Psi f)(z_\alpha, n_\alpha) = \Psi f(\phi^*)$ exists for all $f \in H^\infty(D)$, also $\lim f(w_\alpha)$ exists. Hence w_α converges to some $\phi \in \mathfrak{M}(D)$. Evidently $\phi \in \mathfrak{M}_0$ and $\phi = \theta(\phi^*)$.

PROOF OF THEOREM 1. Let $\phi^* = \theta^{-1}(\phi)$. Then $\phi^*(T) = \phi(L) \neq 0$. By Lemma 8, ϕ^* extends to an evaluation homomorphism at a point of X. Thus $\theta(\phi^*) = \phi$ is adherent to D by Lemma 9.

COROLLARY 1. *D is dense in $\mathfrak{M}(D)$ if and only if $\hat{L}^{-1}(0)$ has no interior.*

Note that Lemma 9 also establishes the following proposition:

PROPOSITION 2. *Suppose D is a Δ-domain with $r_n/R_n \to 0$. If $\mathfrak{M}(B_0)$ is the quotient space of X obtained by identifying points not separated by B_0, then D is dense in $\mathfrak{M}(D)$.*

Since the proof of Theorem 1 works for any choice of positive integers $\{k_n\}$ such that

$$\sum L_n f_n \in H^\infty, \quad \text{for } f_n \in H^\infty(\Delta_n^c),\ \|f_n\| \leqslant M,\ n = 1, 2, \ldots,$$

one might hope to deduce D is dense in $\mathfrak{M}(D)$ merely under the hypothesis $r_n/R_n \to 0$. We have not been able to do this. One difficulty is that the hypothesis $r_n/R_n \to 0$ does not guarantee the existence of a distinguished homomorphism in $\mathfrak{M}_0$. For example, take $c_n = 2^{-n}$, $r_n = \varepsilon_n 2^{-(n+2)}$ and $R_n = 2^{-(n+1)}$, where $0 < \varepsilon_n < 1$, $\varepsilon_n \to 0$ and $\Sigma \varepsilon_n = +\infty$.

Since this is an (L)-domain of the type treated by Zalcman in [10] and $\Sigma r_n/c_n = \infty$, it follows that there is no distinguished homomorphism in $\mathfrak{M}_0$. Moreover a theorem of T. Gamelin and J. Garnett [7] shows that in this case the sequence $\{a_n\}$ has a subsequence which is an interpolating sequence.

PROOF OF THEOREM 2. Let $\phi^* = \theta^{-1}(\phi)$, $L(z) = \Sigma(r_n/(z - c_n))^m - (r_n/-c_n)^m$ and $T(z, n) = z^m$ so $T = \Psi(L) = Z^m|_X$. Then

$$Z^m H^\infty(\Delta_0 \times N)|_X \subset B_0 \quad \text{by Lemma 7,}$$

where Z denotes the coordinate function in $H^\infty(\Delta_0 \times N)$. Since $a_n(f) \to$

0, $\overline{\Psi}(f) \in ZH^{\infty}(\underline{\Delta}_0 \times N)|_X$ and $[\overline{\Psi}(f)]^m = \overline{\Psi}(f^m) \in Z^m H^{\infty}(\Delta_0 \times n)|_X$. Now $0 \neq \phi(f^m) = \phi^*[\overline{\Psi}(f^m)]$. Thus ϕ^* is not zero on $Z^m H^{\infty}(\Delta_0 \times N)|_X$ and $\phi^*(T) \neq 0$. By Theorem 1, $\phi(L) = \phi^*(T) \neq 0$ and ϕ is adherent to D.

COROLLARY 2. *Suppose there exists a positive integer m such that* $\Sigma(r_n/R_n)^m < \infty$ *and suppose, for every* $f \in H^{\infty}(D)$, $a_n(f)$ *converges.*
Then D *is dense in* $\mathfrak{M}(D)$.

PROOF. Define $\phi_0(f) = \lim a_n(f), f \in H^{\infty}(D)$. By Lemma 6, $\phi_0 \in \mathfrak{M}_0(D)$ and ϕ_0 is adherent to D. If $\phi \in \mathfrak{M}_0 \setminus \{\phi_0\}$, Theorem 2 shows ϕ is adherent to D.

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