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Δ-DOMAINS AND THE CORONA

BY

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ABSTRACT. Let D be a bounded domain in the complex plane $\mathbf{C}$. Let $H^\infty(D)$ denote the usual Banach algebra of bounded analytic functions on D. The Corona Conjecture asserts that D is weak* dense in the space $\mathfrak{M}(D)$ of maximal ideals of $H^\infty(D)$. In [2] Carleson proved that the unit disk Δ_0 is dense in $\mathfrak{M}(\Delta_0)$. In [7] Stout extended Carleson's result to finitely connected domains. In [4] Gamelin showed that the problem is local. In [1] Behrens reduced the problem to very special types of infinitely connected domains and established the conjecture for a large class of such domains.

In this paper we extract some of the crucial ingredients of Behrens' methods and extend his results to a broader class of infinitely connected domains.

1. Introduction. By a Δ-domain we mean a domain obtained from the open unit disc Δ_0 by deleting the origin and a sequence of disjoint closed discs $\Delta_n = \Delta(c_n, r_n) = \{z: |z - c_n| \leqslant r_n\}$ with $c_n \to 0$. Behrens [1] indicated that if the Corona Conjecture fails for some bounded domain, then it fails for a Δ-domain. Moreover, he established if the excised discs are "hyperbolically rare" (see §3), then the Corona Conjecture obtains. Zalcman [8] imposed the additional requirement that $\sum r_n/|c_n| < \infty$ and used it to describe a distinguished homomorphism in the fiber of the maximal ideal space at the origin. This homomorphism plays an important (but not essential) role in Behrens' techniques. His proof employs a continuous linear isomorphism of $H^\infty(D)$ into $H^\infty(\Delta_0 \times N)$, the Banach algebra of uniformly bounded functions which are analytic on each "slice" of the product of the unit disk Δ_0 with the set N of nonnegative integers. We will follow the general scheme employed by Behrens in [1] and extend his results to a broader class of Δ-domains. The precise description of the fiber algebra and the Gleason parts in the fiber at the origin given by Behrens in the "hyperbolically rare" case extend to the setting of Theorem 1 below. In particualr, the fiber algebra at the origin is isometrically isomorphic to a restriction algebra of $ZH^\infty(\Delta_0 \times N) + \mathbf{C}$ (see Lemma 5). We also utilize a number of the basic ideas in [3]. The examples in §3 illustrate the broader generality of the results.

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2. Main results.

THEOREM 1. *Let D be a Δ-domain. Suppose*

$$(*) \qquad \lim_{k\to\infty} \sum_{\substack{n=1 \\ n\neq k}}^{\infty} \frac{r_n}{|c_n|} \frac{|c_k|}{|c_k - c_n|} = 0.$$

Then D is dense in $\mathfrak{M}(D)$.

Preliminary to the proof of the theorem consider the following definitions, notation and lemmas. Note that $(*)$ implies $\sum r_n/|c_n| < \infty$.

DEFINITION. The distinguished homomorphism ϕ_0 is

$$\phi_0(f) = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_{\partial\Delta_n} \frac{f(w)}{w} dw \quad \text{(see [8])}.$$

Notation. $\mathfrak{M}_0(D)$ denotes the fiber in $\mathfrak{M}(D)$ at the origin;
A_0 denotes the restriction function algebra $H^\infty(D)|_{\mathfrak{M}_0}$;

$$a_n(f) = \frac{1}{2\pi i} \int_{\partial\Delta_n} \frac{f(w)}{w - c_n} dw, \qquad f \in H^\infty(D).$$

$$P_n f(z) = \frac{1}{2\pi i} \int_{\partial\Delta_n} \frac{f(w)}{w - z} dw,$$

where $z \in \Delta_0$, if $n = 0$ and $z \in \Delta_n^c = \mathbf{C}\backslash\Delta_n$, if $n = 1, 2, 3, \ldots$;

$D_n = \{z: |z - c_n| \leqslant R_n\}$, where R_n is chosen larger than r_n but such that the discs $\{D_n\}$ are still pairwise disjoint. For notational convenience take $c_0 = 0$, $r_0 = R_0 = 1$.

For $f \in H^\infty(D)$, write

$$f(z) = P_n f(z) + F_n(z) + a_n(f), \quad \text{for } z \in D_n\backslash\Delta_n,$$

where F_n is analytic in D_n and $F_n(c_n) = 0$.

Define $\Psi: H^\infty(D) \to H^\infty(\Delta_0 \times N)$ by

$$\Psi f(z,n) = (P_n f \circ E_n^{-1})(z) + a_n(f),$$

with $E_n(z) = r_n/(z - c_n)$ if $n = 1, 2, \ldots$ and $E_0(z) = z$.

Observe that for $f \in H^\infty(D)$,

$$f(z) = \sum_{n=0}^{\infty} P_n f(z),$$

where the integration defining each P_n is appropriately oriented and the convergence is uniform on compact sets in D.

Let

$$X = \mathfrak{M}[H^\infty(\Delta_0 \times N)] \setminus \bigcup_{n=0}^{\infty} \mathfrak{M}[H^\infty(\Delta_0 \times \{n\})]$$

(see [1]) and let $B = H^\infty(\Delta_0 \times N)|_X$.

The scheme of things is to study the map Ψ with reference to the existence of an induced map from A_0 into B, to obtain an isomorphic copy B_0 of A_0 in B, to identify the maximal ideal space of B_0 as a quotient space of X, and to transfer the positive Corona result for $H^\infty(\Delta_0 \times N)$ [1] back to the domain D. Initially the critical factor is the choice of the discs $\{D_n\}$—the bigger the better. The following lemma and proposition appear in [3] as Proposition 1 and Proposition 2, respectively.

LEMMA 1. *If R_n can be chosen so that $r_n/R_n \to 0$, then Ψ induces a continuous algebra isomorphism ρ of A_0 onto a closed subalgebra B_0 of B, as indicated by the following diagram*:

$$\begin{tikzcd} H^\infty(D) \arrow[r, "\Psi"] \arrow[d, "\text{restriction}"'] & H^\infty(\Delta_0 \times N) \arrow[d, "\text{restriction}"] \\ A_0 \arrow[r, "\rho"] & B \end{tikzcd}$$

PROPOSITION. *Let D be a Δ-domain. Suppose*

(1) *R_n can be chosen so that $r_n/R_n \to 0$, and*

(2) *$\mathfrak{M}(B_0)$ is the quotient space of X obtained by identifying points not separated by B_0.*

Then D is dense in $\mathfrak{M}(D)$.

We will prove Theorem 1 as a corollary to this proposition.

LEMMA 2. *R_n can be chosen so that $r_n/R_n \to 0$ if and only if $r_n/|c_n - c_k| \to 0$ as $n, k \to \infty$.*

PROOF. Let $t_n = \inf_k\{|c_n - c_k|: k \neq n\}$. Then R_n can be chosen comparable in magnitude to t_n. Now $r_n/|c_n - c_k| \to 0$ as $n, k \to \infty$ means given $\varepsilon > 0$ $\exists N$ such that n, k $(n \neq k) \geqslant N \Rightarrow r_n/|c_n - c_k| < \varepsilon$. Since $c_n \to 0$, for sufficiently large n, t_n is attained only by $|c_n - c_k|$ where $k \geqslant N$. Hence $r_n/t_n < \varepsilon$ and $r_n/R_n \to 0$. The "only if" proof is obvious.

COROLLARY. *If D is a Δ-domain satisfying $(*)$ in Theorem 1, then R_n can be chosen so that $r_n/R_n \to 0$.*

PROOF. Suppose

$$\sum_{\substack{n=1 \\ n \neq k}}^{\infty} \frac{r_n}{|c_n|} \frac{|c_k|}{|c_n - c_k|} \xrightarrow{k} 0.$$

Let $\varepsilon > 0$ be given. Choose N so that

(i) $$k \geqslant N \Rightarrow \sum_{\substack{n=1\\ n\neq k}}^{\infty} \frac{r_n}{|c_n|}\frac{|c_k|}{|c_n - c_k|} < \frac{\varepsilon}{2};$$

(ii) $$n \geqslant N \Rightarrow r_n/|c_n| < \varepsilon/2.$$

Suppose $n, k \geqslant N$, $n \neq k$. If $|c_k| \leqslant \frac{1}{2}|c_n|$, then $r_n/|c_n - c_k| \leqslant 2r_n/|c_n| < \varepsilon$, by (ii). If $|c_k| \geqslant \frac{1}{2}|c_n|$, it follows from (i) that $r_n/|c_n - c_k| < (\varepsilon/2)|c_n/c_k| < \varepsilon$. Thus $r_n/|c_n - c_k| \to 0$ as $n, k \to \infty$.

Lemma 3. *If D is a Δ-domain as in Theorem* 1, *then $a_k \to \phi_0$ in norm.*

Proof. For fixed k, $f(z)/(z - c_k)$ is analytic in D and

$$\int_{\partial D} f(w)/(w - c_k)\,dw = 0,$$

i.e.

$$\sum_{n=0}^{\infty} \frac{1}{2\pi i}\int_{\partial\Delta_n} \frac{f(w)}{w - c_k}\,dw = 0,$$

with appropriate orientation in the integrals. Thus

$$|a_k(f) - \phi_0(f)| \leqslant \left|\sum_{\substack{n=0\\ n\neq k}}^{\infty} \frac{1}{2\pi i}\int_{\partial\Delta_n} \frac{c_k f(w)}{w(w - c_k)}\,dw\right| + \left|\frac{1}{2\pi i}\int_{\partial\Delta_k} \frac{f(w)}{w}\,dw\right|$$

$$\leqslant \|f\| \sum_{\substack{n=0\\ n\neq k}}^{\infty} \left[\frac{|c_k| r_n}{d(c_k, \Delta_n)\, d(0, \Delta_n)}\right] + \|f\|\frac{r_k}{|c_k| - r_k}$$

(d denotes distance function)

$$\leqslant \|f\| \left[\sum_{\substack{n=0\\ n\neq k}}^{\infty} \frac{r_n}{(|c_n| - r_n)}\frac{|c_k|}{(|c_n - c_k| - r_n)} + \frac{r_k}{|c_k| - r_k}\right]$$

$$= \|f\| \left|\sum_{\substack{n=0\\ n\neq k}}^{\infty} \frac{r_n}{|c_n|}\frac{|c_k|}{|c_n - c_k|}\left(1 - \frac{r_n}{|c_n|}\right)^{-1}\left(1 - \frac{r_n}{|c_n - c_k|}\right)^{-1} + \frac{r_k}{|c_k|}\left(1 - \frac{r_k}{|c_k|}\right)^{-1}\right|.$$

Since $r_n/|c_n| \to 0$, $r_n/|c_n - c_k| \to 0$ as $n, k \to \infty$ and

$$\sum_{\substack{n=0\\ n\neq k}}^{\infty} \frac{r_n}{|c_n|}\frac{|c_k|}{|c_n - c_k|} \xrightarrow{k} 0,$$

the lemma follows.

Let $\underline{c}$ denote the space of convergent sequences. Let Z denote the "coordinate" function in $H^\infty(\Delta_0 \times N)$ defined by $Z(\lambda, n) = \lambda$. Lemma 3 and the definition of Ψ yield that whenever $H^\infty(D)$ satisfies (*), then Ψ maps into $ZH^\infty(\Delta_0 \times N) + \underline{c}$ and $B_0 \subset (ZH^\infty + \mathbf{C})|_X$. We actually have

LEMMA 4. *If D is a Δ-domain as in Theorem* 1, *then* $B_0 = (ZH^\infty + \mathbf{C})|_X$.

PROOF. It suffices to show that if f_n is a sequence of functions such that $f_n \in H(\Delta_n^c)$, $f_n(\infty) = 0$, and $\|f_n\| \leqslant M$, $n = 1, 2, 3, \ldots,$ then $\sum_{n=1}^\infty f_n \in H^\infty(D)$. Now

$$f_n(z) = \frac{1}{2\pi i}\int_{\partial\Delta_n} \frac{f_n(w)}{w - z} dw, \qquad z \in \Delta_n^c,$$

and $|f_n(0)| \leqslant \|f_n\| r_n/(|c_n| - r_n)$. Thus $\sum_{n=1}^\infty |f_n(0)| < \infty$. Also

$$\begin{aligned} |f_n(z) - f_n(0)| &\leqslant \left| \frac{1}{2\pi i}\int_{\partial\Delta_n} \frac{f_n(w) z}{(w - z)w} dw \right| \\ &\leqslant \|f_n\|\,|z| \frac{r_n}{d(z, \partial\Delta_n)(|c_n| - r_n)}. \end{aligned}$$

Suppose $z \in \partial\Delta_k$. Then

$$\sum_{\substack{n=1\\ n\neq k}}^{\infty} |f_n(z) - f_n(0)| \leqslant M \sum_{\substack{n=1\\ n\neq k}}^{\infty} \frac{r_n(|c_k| + r_k)}{d(\partial\Delta_k, \partial\Delta_n)(|c_n| - r_n)}.$$

By (*),

$$\lim_{k\to\infty} \sum_{\substack{n=1\\ n\neq k}}^{\infty} |f_n(z) - f_n(0)| = 0.$$

Hence $\sum_{n=1}^\infty f_n \in H^\infty(D)$.

Assuming $r_n/R_n \to 0$, it seems that (*) should characterize when $B_0 = (ZH^\infty + \mathbf{C})|_X$.

PROOF OF THEOREM 1. According to the proposition and the corollary to Lemma 2, all that remains is to identify $\mathfrak{M}(B_0)$. But since

$$B_0 = (ZH^\infty + \mathbf{C})|_X,$$

$\mathfrak{M}(B_0)$ is the quotient space of X obtained by identifying the zero set of Z on X to a point. This is immediate since $J = ZH^\infty|_X$ is an ideal in $H^\infty|_X$ whose hull is the zero set of Z and $\mathfrak{M}(H^\infty|_X) = X$.

3. Examples and applications. First let us deduce Behrens' result.

COROLLARY. *Suppose R_n can be chosen so that $\sum_{n=1}^{\infty} r_n/R_n < \infty$ ("hyperbolically rare"). Then D is dense in $\mathfrak{M}(D)$.*

PROOF.

$$\sum_{\substack{n=1\\ n\neq k}}^{\infty} \frac{r_n}{|c_n|} \frac{|c_k|}{|c_n - c_k|} = \sum_{\substack{n=1\\ n\neq k}}^{\infty} \left| \frac{r_n}{c_k - c_n} + \frac{r_n}{c_n} \right|$$

$$\leqslant \sum_{n=1}^{\infty} \left| \frac{r_n}{R_n} + \frac{r_n}{|c_n|} \right| < \infty.$$

Let

$$f_k(n) = \frac{r_n}{|c_n|} \frac{|c_k|}{|c_n - d_k|}.$$

Then $f_k(n) \leqslant r_n/R_n + r_n/|c_n|$. By dominated convergence, $(*)$ obtains.

We introduce a class of examples where $c_n = 1/n$. In this case, no matter what r_n is, R_n can be chosen at most comparable in size to $1/n^2$. Thus, for hyperbolical rareness, one needs $\sum_{n=1}^{\infty} n^2 r_n < \infty$. Zalcman's condition becomes $\sum_{n=1}^{\infty} n r_n < \infty$.

Theorem 1 allows the following:

THEOREM 2. *Let $c_n = 1/n$. Suppose $\sum_{n=1}^{\infty} n r_n < \infty$. If*
(A) *$n^2 r_n \searrow 0$, or*
(B) *$n^2 (\log n) r_n \to 0$, or*
(C) *for some $p > 0$, $\sum (n^2 r_n)^p < \infty$,*
then D is dense in $\mathfrak{M}(D)$.

REMARK. In condition (C), $0 < p \leqslant 1$, the conclusion is immediate from Behrens, or the above corollary. Moreover, $\sum_{n=1}^{\infty} n r_n < \infty$ is not necessary in (C). Indeed, as the proof below indicates, (C) implies $\sum_{n=1}^{\infty} n r_n < \infty$.

PROOF OF THEOREM 2.

$$\sum_{\substack{n=1\\ n\neq k}}^{\infty} \frac{r_n}{|c_n|} \frac{|c_k|}{|c_n - c_k|} = \sum_{\substack{n=1\\ n\neq k}}^{\infty} \frac{n^2 r_n}{|n-k|}.$$

(A) follows from the following:

LEMMA A. *Suppose $b_n \searrow 0$ and $\sum_{n=1}^{\infty} b_n/n < \infty$. Then $\sum_{n=1; n\neq k}^{\infty} b_n/|n-k| \xrightarrow{k} 0$.*

PROOF. By monotone convergence, $\sum_{j=1}^{\infty} b_{N+j}/j \xrightarrow{N} 0$. Let $\varepsilon > 0$. Choose N so that $\sum_{j=1}^{\infty} b_{N+j}/j < \varepsilon$. Let $k > N$. Then $\sum_{n=1}^{N} b_n/|n-k| \xrightarrow{k} 0$, and

$$\sum_{\substack{n=N+1\\n\neq k}}^{\infty}\frac{b_n}{|n-k|}=\sum_{n=N+1}^{k-1}\frac{b_n}{k-n}+\sum_{n=k+1}^{\infty}\frac{b_n}{n-k}$$

$$\leqslant \frac{b_{N+1}}{1}+\frac{b_{N+2}}{2}+\cdots+\frac{b_{k-1}}{k-1-N}+\frac{b_{k+1}}{1}+\frac{b_{k+2}}{2}+\cdots$$

$$\leqslant 2\sum_{j=1}^{\infty}\frac{b_{N+j}}{j}<2\varepsilon.$$

To prove (B), we break the sum $\sum_{n=0;n\neq k}^{\infty} n^2 r_n/|n-k|$ into four pieces, as follows: Let M_k = greatest integer in $\sqrt{k}$. Then

$$\sum_{\substack{n=0\\n\neq k}}^{\infty}\frac{n^2 r_n}{|n-k|}=\sum_{n=0}^{M_k}+\sum_{n=M_k+1}^{k-1}+\sum_{n=k+1}^{k^2-1}+\sum_{n=k^2}^{\infty}.$$

Now

$$\sum_{n=k^2}^{\infty}\frac{n^2 r_n}{|n-k|}=\sum_{n=k^2}^{\infty} n r_n\frac{n}{n-k}\leqslant 2\sum_{n=k^2}^{\infty} n r_n \xrightarrow{k} 0;$$

$$\sum_{n=0}^{M_k}\frac{n^2 r_n}{|n-k|}=\sum_{n=0}^{M_k}\frac{n^2 r_n}{k-n}\leqslant \max_{0\leqslant n\leqslant M_k}\{n^2 r_n\}\sum_{n=0}^{M_k}\frac{1}{k-n}$$

$$\leqslant\left(\max_{0\leqslant n\leqslant M_k}\{n^2 r_n\}\right)\frac{\sqrt{k}}{k-\sqrt{k}}\xrightarrow{k} 0;$$

$$\sum_{n=M_k+1}^{k-1}\frac{n^2 r_n}{|n-k|}\leqslant\left(\max_{M_k<n<k}\{n^2 r_n\}\right)\sum_{M_k+1}^{k-1}\frac{1}{k-n}\leqslant\left(\max_{M_k<n<k}\{n^2 r_n\}\right)\log k$$

$$=\frac{1}{2}\left(\max_{\sqrt{k}<n<k}\{n^2 r_n\}\right)\log\sqrt{k}\xrightarrow{k} 0;$$

and

$$\sum_{n=k+1}^{k^2-1}\frac{n^2 r_n}{|n-k|}\leqslant\left(\max_{k<n\leqslant k^2}\{n^2 r_n\}\right)\log k^2=2\left(\max_{k<n\leqslant k^2}\{n^2 r_n\}\right)\log k\xrightarrow{k} 0.$$

For the proof of (C) we only need consider $p>1$. Choose q so that $1/p+1/q=1$. Fix $N<k$. Then

$$\sum_{n=0}^{N}\frac{n^2 r_n}{|k-n|}\leqslant \max_{0\leqslant n\leqslant N}\{n^2 r_n\}\frac{N}{k-N}\xrightarrow{k} 0,$$

and

$$\sum_{\substack{n=N+1\\n\neq k}}^{\infty} \frac{n^2 r_n}{|n-k|} \leqslant \left[\sum_{n=N+1}^{\infty} (n^2 r_n)^p\right]^{1/p} \left[\sum_{\substack{n=N+1\\n\neq k}}^{\infty} \frac{1}{|n-k|^q}\right]^{1/q}$$

$$\leqslant \left[\sum_{n=N+1}^{\infty} (n^2 r_n)^p\right]^{1/p} \left[2 \sum_{n=1}^{\infty} \frac{1}{n^q}\right]^{1/q}.$$

Since $\sum_{n=1}^{\infty} 1/n^q < \infty$,

$$\sum_{\substack{n=N+1\\n\neq k}}^{\infty} \frac{n^2 r_n}{|n-k|} < \varepsilon \quad \text{for large } N.$$

Hence in all cases (*) obtains and the Corona follows.

REMARK. It is not difficult to construct an example which indicates that condition (B) is, in a sense, sharp. In particular, in the above case with $c_n = 1/n$, it is not sufficient to obtain (*) under the hypothesis that $n^2 r_n \to 0$, i.e. $r_n/R_n \to 0$.

A further elementary consequence of Theorem 1 is the following:

THEOREM 3. *Suppose* (*) *obtains for a sequence c_n on the positive real axis and radii r_n.*

Let $b_n = e^{i\theta_n} c_n$. Let $\Delta_n = \{z: |z - b_n| \leqslant r_n\}$, and let D be the corresponding Δ-domain. Then D is dense in $\mathfrak{M}(D)$.

PROOF.

$$\sum_{\substack{n=0\\n\neq k}}^{\infty} \frac{r_n}{|b_n|} \frac{|b_k|}{|b_n - b_k|} \leqslant \sum_{\substack{n=0\\n\neq k}}^{\infty} \frac{r_n}{c_n} \frac{c_k}{|c_n - c_k|}.$$

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