

## ON THE TRANSFORMATION GROUP OF A REAL HYPERSURFACE

BY

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**ABSTRACT.** The group of biholomorphic transformations leaving fixed a strongly pseudoconvex real hypersurface in a complex manifold is a Lie group. In this paper it is shown that the Chern-Moser invariants must vanish if this group is noncompact and the hypersurface is compact. Also considered are transformation groups of flat hypersurfaces and intransitive groups.

**Introduction.** The purpose of this paper is to study the group of structure preserving, or pseudoconformal, transformations of a real hypersurface  $M$  (of dimension  $2n + 1$ ) in a complex  $n + 1$  manifold, or more generally, of a manifold with the same C-R structure. We will always assume that the C-R structure is integrable and that the Levi form is nondegenerate. It follows from the work of Chern and Moser [2] that this group is a Lie transformation group.

There is some similarity between the geometry of  $M$  and the conformal geometry of a Riemannian manifold. Over  $M$  there is a principal fibre bundle  $B$  (the pseudoconformal bundle), a connection on  $B$ , and curvature invariants the most important of which is the fourth order curvature tensor  $S_{\beta\rho\bar{\sigma}}^{\alpha}$  (see [2]). These have as analogues in the Riemannian case the bundle of conformal frames the conformal connection, and the Weyl conformal curvature tensor. If  $n > 2$ ,  $S$  vanishes if and only if  $M$  is pseudoconformally flat, i.e., locally equivalent to the standard sphere  $S^{2n+1}$  in  $\mathbb{C}^{n+1}$  [8].

Locally the structure of  $M$  is given by a real one-form  $\theta$ ,  $n$  complex one-forms  $\theta^{\alpha}$ , and their complex conjugates  $\theta^{\bar{\alpha}} = \overline{\theta^{\alpha}}$ . They satisfy

$$\theta \wedge \theta^1 \wedge \cdots \wedge \theta^n \wedge \theta^{\bar{1}} \wedge \cdots \wedge \theta^{\bar{n}} \neq 0$$

and the equation

$$d\theta \equiv ig_{\alpha\bar{\beta}}\theta^{\alpha} \wedge \theta^{\bar{\beta}} \pmod{\theta}.$$

Here and throughout this paper Greek indices run from 1 to  $n$  and the summation convention is used. The nondegenerate hermitian matrix  $(g_{\alpha\bar{\beta}})$  is

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the Levi form. If  $M$  is strongly pseudoconvex  $\theta$  is chosen so that this matrix is negative definite. The holomorphic tangent bundle  $H(M)$  is the annihilator of  $\theta$  in the real tangent bundle of  $M$ . We will have need of the dual vector fields  $X_\alpha, X_{\bar{\alpha}} = \bar{X}_\alpha$ , and  $X = \bar{X}$ .

The real one-form  $\theta$  is determined up to multiplication by a nonzero function on  $M$ . For a choice of  $\theta$  the pair  $(M, \theta)$ , called a pseudohermitian manifold in [8], is the analogue of a Riemannian structure underlying a conformal structure.

In §1 we show that if  $M$  is compact, connected, and strongly pseudoconvex, and if the identity component of its pseudoconformal transformation group is noncompact, then  $M$  is flat. The proof owes much to that given by Obata [7] in the conformal case.

In §2 we study the unbounded one-parameter groups of pseudoconformal transformations of  $S^{2n+1}$ . The main result is the same as in the conformal case [6]. This is used to show that, if  $M$  is as in the preceding paragraph and has a finite fundamental group, then  $M$  is globally equivalent to  $S^{2n+1}$ . In §3 we consider one-parameter groups with fixed points on a flat space.

We turn to intransitive groups of pseudohermitian transformations in §4. The theorem here is that if  $G$  is an intransitive group of transformations of  $(M, \theta)$ ,  $M$  having dimension  $2n + 1$ , then  $G$  has dimension less than or equal to  $n^2 + 1$ . For  $n = 1$ , Cartan [1] has shown that a three dimensional group of pseudoconformal transformations of a real hypersurface in  $\mathbf{C}^2$  must be transitive.

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**1. Spaces admitting a noncompact connected Lie group of pseudoconformal transformations.** In this section we prove the following:

**THEOREM (1.1).** *Let  $M$  be a compact, connected, strongly pseudoconvex abstract real hypersurface, and let  $G$  be a connected Lie group of pseudoconformal transformations of  $M$ . If  $G$  is noncompact then  $M$  is locally equivalent to the standard real hypersphere  $S^{2n+1}$  in  $\mathbf{C}^{n+1}$  in the pseudoconformal sense.*

This will be proved as follows. Suppose  $M$  is not locally  $S^{2n+1}$ ; then we will show that the closure of every one-parameter subgroup of  $G$  is compact. By a theorem of Montgomery and Zippin [5]  $G$  must itself be compact.

We first assume that  $\dim M = 2n + 1, n > 2$ .

Let  $G_1$  be a one-parameter subgroup of  $G$  with infinitesimal generator  $Y$  on  $M$ . Assuming  $M$  is not flat, it follows that the pseudoconformal curvature tensor  $S_{\beta\bar{\rho}\alpha\bar{\sigma}}$  is nonzero somewhere. Let  $U$  be a nonempty connected component of the open set where  $S$  does not vanish.  $U$  is invariant under  $G$ , since  $G$  is connected.

Since  $M$  is strongly pseudoconvex, we can choose a globally defined nonvanishing one-form  $\theta$  annihilating the holomorphic tangent bundle  $H(M)$ . As in [8, §3], the one-form

$$(1.1) \quad \theta^* = \|S\|_{\theta} \cdot \theta$$

is a pseudoconformal invariant on  $U$ . Let  $X^*$  be the corresponding invariant transversal [8]. The following Lie derivatives then vanish

$$(1.2) \quad L_Y \theta^* = 0, \quad L_Y X^* = [Y, X^*] = 0.$$

On  $U$  we can decompose  $Y$  uniquely as

$$(1.3) \quad Y = \eta X^* + \tilde{Y},$$

where

$$(1.4) \quad \eta = \theta^*(Y), \quad \theta^*(\tilde{Y}) = 0.$$

By the following proposition we may assume that  $\eta$  does not vanish identically.

**PROPOSITION (1.2).** *Let  $Y$  be an infinitesimal pseudoconformal transformation on a nondegenerate, integrable C-R manifold  $M$ . If  $Y_p$  belongs to  $H_p(M)$  for all  $p$  in an open subset  $U$  of  $M$ , then  $Y$  vanishes identically.*

**PROOF.** Choose an local frame  $(X, X_{\alpha}, X_{\bar{\alpha}})$  and dual coframe  $(\theta, \theta^{\alpha}, \theta^{\bar{\alpha}})$ . We have  $d\theta = ig_{\alpha\bar{\beta}} \theta^{\alpha} \wedge \theta^{\bar{\beta}} + \theta \wedge \phi$ , and we can put  $Y = \xi^{\alpha} X_{\alpha} + \xi^{\bar{\alpha}} X_{\bar{\alpha}}$  in the open set  $U$ . Since  $Y$  preserves the structure, we have

$$(1.5) \quad L_Y \theta = u\theta.$$

We also have

$$(1.6) \quad L_Y \theta = \iota_Y d\theta + d\iota_Y \theta = ig_{\alpha\bar{\beta}} \xi^{\alpha} \theta^{\bar{\beta}} - ig_{\alpha\bar{\beta}} \xi^{\bar{\beta}} \theta^{\alpha} - \phi(Y)\theta.$$

Comparing (1.5) and (1.6) we see that, for all  $\beta$ ,  $g_{\alpha\bar{\beta}} \xi^{\alpha} = 0$ . Since  $g_{\alpha\bar{\beta}}$  is nondegenerate,  $\xi^{\alpha} = 0$  for all  $\alpha$ , and so  $Y$  vanishes on  $U$ . The flow  $G_1$  of  $Y$  induces a flow  $G'_1$  on the pseudoconformal bundle  $B$  over  $M$  which preserves the connection forms.  $G'_1$  acts trivially on the part of  $B$  over  $U$ , so by Theorem (3.2) of [3],  $G'_1$  consists of the identity alone. Therefore,  $G_1$  is also the identity, and  $Y$  vanishes everywhere on  $M$ .  $\square$

Returning to the proof of Theorem (1.1), we may assume that  $\eta > 0$  somewhere by replacing  $Y$  by  $-Y$ , if necessary. For a sufficiently small constant  $\epsilon > 0$  we define a nonempty closed subset of  $U$  by

$$U_{\epsilon} = \{p \in U: \eta(p) \geq \epsilon\}.$$

By (1.1) and (1.4) and the definition of  $U$  it is seen that  $\eta(p)$  goes to zero as  $p$  approaches the boundary of  $U$ . Therefore,  $U_{\epsilon}$  is a closed subset of  $M$  and so is compact.

$U_{\epsilon}$  is invariant under  $G_1$ , the flow of  $Y$ , because

$$Y\eta = (L_Y\theta^*)(Y) + \theta^*([Y, Y]) = 0.$$

It is also invariant under the closure  $\overline{G_1}$ .

Let  $P$  be the pseudohermitian bundle over  $(U, \theta^*)$  with fibre  $U(n)$  [8, §1].  $P_e$ , the part of  $P$  lying over  $U_e$ , is compact and invariant by  $\overline{G_1}$ . By Theorem (3.2) of [3] and the fact that  $\overline{G_1}$  is a Lie group,  $\overline{G_1}$  imbeds as a closed submanifold of  $P_e$ . Therefore,  $\overline{G_1}$  is compact, and Theorem (1.1) is true for  $n > 2$ .

For the case  $n = 1$ , we replace  $\|S\|_\theta$  by the relative invariant  $Q$  on  $M^3$ , the vanishing of which is necessary and sufficient for  $M^3$  to be flat [1]. The same argument shows that Theorem (1.1) holds in this case also.

**2. Unbounded one-parameter groups on the real hypersphere.** To study further compact, strongly pseudoconvex spaces  $M$  admitting a noncompact, connected Lie group of pseudoconformal transformations, we can restrict ourselves to the pseudoconformally flat spaces. It follows from the structure equations of [2, §5] that  $M$  can be developed locally onto the real hypersphere  $S^{2n+1}$ . If  $M$  is also simply connected, then the same monodromy argument as in the conformal case [4] shows that  $M$  can be developed globally onto an open subset of  $S^{2n+1}$ . This development may not be one-to-one.

A pseudoconformal vector field on  $M$  will under this development be mapped to a pseudoconformal vector field on an open subset of  $S^{2n+1}$ . Because  $S^{2n+1}$  has such a large group of transformations, any such local vector field extends to a unique global vector field. We therefore study one-parameter groups, particularly those with noncompact closure, on  $S^{2n+1}$ . This is a matter of linear algebra.

In this section we will prove the following:

**THEOREM (2.1).** *Let  $G_1$  be a one-parameter group of pseudoconformal transformations of  $S^{2n+1}$  having a noncompact closure. Let  $F_t$  be its flow. Then either*

(1)  $F_t$  has precisely two fixed points,  $p_-$  and  $p_+$ , such that for any other point  $p$  in  $S^{2n+1}$ ,

$$\lim_{t \rightarrow +\infty} F_t(p) = p_+, \quad \lim_{t \rightarrow -\infty} F_t(p) = p_-;$$

or

(2)  $F_t$  has precisely one fixed point  $p'$ , such that for any point  $p$  in  $S^{2n+1}$ ,

$$\lim_{t \rightarrow \pm\infty} F_t(p) = p'.$$

Before giving the proof we will derive the following corollary.

**COROLLARY (2.2).** *Let  $M$  be a compact, connected, strongly pseudoconvex C-R manifold with finite fundamental group. If  $M$  admits a noncompact, connected Lie group  $G$  of pseudoconformal transformations, then  $M$  is globally equivalent to  $S^{2n+1}$ .*

PROOF. By Theorem (1.1),  $M$  and hence its universal covering space  $\tilde{M}$  are flat. Since  $\tilde{M}$  is compact, the development map is a covering. Thus,  $\tilde{M}$  is globally equivalent to  $S^{2n+1}$ . Since  $G$  is noncompact, it has a closed, noncompact one-parameter subgroup  $G_1$ ; see [5].  $G_1$  lifts to a closed noncompact one-parameter subgroup  $\tilde{G}_1$  on  $\tilde{M}$ . By Theorem (2.1),  $\tilde{G}_1$  has either one or two fixed points, which must cover the fixed points of  $G_1$ . Therefore,  $\tilde{M}$  is a one or two sheeted covering. If it were a two sheeted covering, both fixed points of  $\tilde{G}_1$  would lie over the same fixed point of  $G_1$ . However, from case (1) of Theorem (2.1) these two fixed points have different character as  $t \rightarrow +\infty$ , one attracting and one repelling; hence, they could not cover the same fixed point of  $G_1$ . Therefore,  $M \simeq \tilde{M} \simeq S^{2n+1}$ .  $\square$

We go on to prove Theorem (2.1). As in [2] we view  $S^{2n+1}$  as a real hyperquadric in complex projective space. A pseudoconformal transformation of  $S^{2n+1}$  is induced by a projective transformation carrying  $S^{2n+1}$  into itself. Such a transformation is represented by a linear isomorphism  $A$  of  $C^{n+2}$ , the space of homogeneous coordinates, which leaves invariant  $S$ , those coordinates of points in  $S^{2n+1}$ . We denote by  $[Z]$  the point in projective space with homogeneous coordinates  $Z$ .

$S$  is defined by

$$(2.1) \quad \tilde{S} = \{ Z \in C^{n+2}: (Z, \bar{Z}) = 0 \},$$

where  $(, \bar{\phantom{z}})$  is a nondegenerate hermitian form of signature  $(n + 1, 1)$ . We can view the pseudoconformal group of  $S^{2n+1}$  as  $SU(n + 1, 1)$ .

A one-parameter subgroup  $A(t)$  of  $SU(n + 1, 1)$  is given by

$$(2.2) \quad A(t) = e^{tB}$$

where

$$(2.3) \quad (BZ, \bar{W}) + (Z, \overline{BW}) = 0, \quad Z, W \in C^{n+2}.$$

Eigenvectors  $Z$  of  $B$  on  $S$  correspond to fixed points  $[Z]$  of the group  $A(t)$  acting on  $S^{2n+1}$ .

Condition (2.3) readily implies the following:

LEMMA (2.3). *Let  $Z$  be an eigenvector of  $B$  with eigenvalue  $\lambda$ . If  $Z$  is not in  $\tilde{S}$ , then  $\lambda$  is purely imaginary.*  $\square$

LEMMA (2.4). *Let  $(, \bar{\phantom{z}})$  be an hermitian form on  $C^{N+1}$ , and  $B$  be any operator satisfying (2.3). If  $B$  has no eigenvectors on  $\tilde{S}$ , then  $C^{N+1}$  has a basis of eigenvectors with purely imaginary eigenvalues.*

PROOF.  $B$  has some eigenvector,  $BZ_0 = \lambda_0 Z_0$ , and  $\lambda_0 + \bar{\lambda}_0 = 0$  by Lemma (2.3). By assumption  $Z_0 \notin (Z_0)^\perp$ ; hence,  $C^{N+1} = (Z_0) \oplus (Z_0)^\perp$ .  $(Z_0)^\perp$  is invariant under  $B$  by (2.3). We restrict  $B$  and the hermitian form to  $(Z_0)^\perp$ ,

where, by induction, we have a basis  $Z_1, \dots, Z_N$  of eigenvectors with purely imaginary eigenvalues.  $Z_0, Z_1, \dots, Z_N$  is the required basis of  $C^{N+1}$ .  $\square$

For the operator  $B$  of Lemma (2.4) it follows that the corresponding one-parameter group  $A(t)$  is bounded. Therefore, we assume that  $B$  has at least one eigenvector on  $\tilde{S}$ . There are two cases.

Case (1).  $B$  has at least two distinct eigenvectors  $Z_0$  and  $Z_{n+1}$  on  $S$ .

We must have  $(Z_0, \bar{Z}_{n+1}) \neq 0$ . Otherwise,  $S$  would contain the complex 2-plane of all  $aZ_0 + bZ_{n+1}$ ,  $a, b \in C$ , and  $S^{2n+1}$  would contain a complex line, which it does not. We may assume that  $(Z_0, \bar{Z}_{n+1}) = -i/2$ . From (2.3),  $Z_0$  and  $Z_{n+1}$  have eigenvalues  $\lambda$  and  $-\bar{\lambda}$ , respectively.

Choose  $Z_1, \dots, Z_n$ , a basis of  $(Z_0, Z_{n+1})^\perp$ . If relative to the basis  $(Z_0, Z_\alpha, Z_{n+1})$  of  $C^{n+2}$  we have

$$(2.4) \quad Z = \zeta^a Z_\alpha, \quad 0 \leq a \leq n + 1,$$

then

$$(2.5) \quad (Z, \bar{Z}) = \zeta h^i \bar{\zeta} = h_{\alpha\bar{\beta}} \zeta^\alpha \bar{\zeta}^\beta + \frac{i}{2} (\zeta^{n+1} \bar{\zeta}^0 - \zeta^0 \bar{\zeta}^{n+1}),$$

where  $h_{\alpha\bar{\beta}}$  is positive definite hermitian. (2.3) is now written

$$(2.6) \quad Bh + h^i \bar{B} = 0.$$

From (2.6) it is seen that  $B$  has the matrix form

$$(2.7) \quad \begin{pmatrix} \lambda & b^\beta & p \\ c_\alpha & B_\alpha^\beta & 2ib_\alpha \\ q & \frac{1}{2i} c^\beta & -\bar{\lambda} \end{pmatrix},$$

where

$$(2.8) \quad \begin{aligned} p &= \bar{p}, \quad q = \bar{q}, \quad B_\alpha^\gamma h_{\gamma\bar{\beta}} + h_{\alpha\bar{\gamma}} B_\beta^\gamma = 0, \\ c_\alpha &= h_{\alpha\bar{\gamma}} c^\gamma, \quad b_\alpha = h_{\alpha\bar{\gamma}} b^\gamma, \quad \text{trace } B = B_\alpha^\alpha + \lambda - \bar{\lambda} = 0. \end{aligned}$$

Since  $Z_0$  and  $Z_{n+1}$  are eigenvectors, we have

$$p = q = c_\alpha = c^\beta = b_\alpha = b^\beta = 0.$$

Since  $h_{\alpha\bar{\beta}}$  is positive definite hermitian and the second relation of (2.8) holds, we may choose  $Z_1, \dots, Z_n$  so that both  $(Z_\alpha, \bar{Z}_\beta) = \delta_{\alpha\bar{\beta}}$  and  $BZ_\alpha = \lambda_\alpha Z_\alpha$  where  $\lambda_\alpha + \bar{\lambda}_\alpha = 0$ . As  $A(t)$  is unbounded, we must have  $\lambda = \mu + i\nu$ ,  $\mu \neq 0$ .

Let  $Z \neq 0$  given by (2.4) represent a point  $[Z]$  in  $S^{2n+1}$  other than  $[Z_0]$  and  $[Z_{n+1}]$ . From (2.5) it is seen that  $\zeta^0 \neq 0$  and  $\zeta^{n+1} \neq 0$ . The action of  $A(t)$  is given by

$$(\zeta^0, \zeta^\alpha, \zeta^{n+1}) \rightarrow (e^{t(\mu+i\nu)} \zeta^0, e^{t\lambda_\alpha} \zeta^\alpha, e^{-t(\mu-i\nu)} \zeta^{n+1}).$$

Let us assume  $\mu > 0$ , so that as  $t \rightarrow +\infty$  we have

$$|e^{t(\mu+iv)}\xi^0| \rightarrow +\infty, \quad |e^{-t(\mu-iv)}\xi^{n+1}| \rightarrow 0,$$

and the other components remain bounded. Therefore, as  $t \rightarrow +\infty$ ,  $[Z] \rightarrow [Z_0]$ . Also, as  $t \rightarrow -\infty$ ,  $[Z] \rightarrow [Z_{n+1}]$ .

Case (2).  $B$  has only one eigenvector  $Z_0$  in  $\tilde{S}$  with eigenvalue  $\lambda_0$ .

LEMMA (2.5). *If  $B$  satisfies (2.6) and has  $\lambda$  as an eigenvalue, it also has  $-\bar{\lambda}$  as an eigenvalue.*

PROOF.

$$\begin{aligned} 0 &= \det(B - \lambda I)\det(h) = \det(Bh - \lambda h) = \det(-h'\bar{B} - \lambda h) \\ &= \det(-h)\det(\overline{B + \bar{\lambda}I}). \quad \square \end{aligned}$$

Let  $W$  be an eigenvector with eigenvalue  $-\bar{\lambda}_0$ . We are assuming  $(W, \bar{W}) \neq 0$ , so that  $\lambda_0$  is purely imaginary, as are all the other eigenvalues. Let  $\lambda_0, \lambda_1, \dots, \lambda_r$  be the distinct eigenvalues of  $B$  with eigenspaces  $K_0, K_1, \dots, K_r$ , respectively.

LEMMA (2.6).  $K_0 + K_1 + \dots + K_r \subseteq (Z_0)^\perp$ .

PROOF. It follows from (2.3) that if  $BZ_j = \lambda_j Z_j$  for  $j > 0$ , then  $(Z_j, \bar{Z}_0) = 0$ . Suppose  $Z' \in K_0$  and  $(Z', \bar{Z}) \neq 0$ , say  $(Z', \bar{Z}_0) = 1$ . Put  $Z = aZ_0 + Z'$ ,  $a = \bar{a}$ . Then  $(Z, \bar{Z}) = 2a + (Z', \bar{Z}')$ , and we can choose  $a$  so that  $Z$  lies in  $\tilde{S}$ , contradicting the fact that  $Z_0$  is the only eigenvector in  $S$ . Hence  $K_0 \subseteq (Z_0)^\perp$ .  $\square$

LEMMA (2.7). *For any  $Z$  in  $C^{n+2}$  the complex line determined by the origin and  $A(t)Z$  approaches the subspace  $K_0 + \dots + K_r$  as  $t \rightarrow \pm\infty$ .*

This will finish the proof of Theorem (2.1) as follows: since  $S$  is invariant by  $A(t)$  and  $S \cap (K_0 + \dots + K_r) = (Z_0)$ , we have, for any  $[Z]$  in  $S^{2n+1}$ ,  $[A(t)Z] \rightarrow [Z_0]$  as  $t \rightarrow \pm\infty$ .

PROOF. This will follow from the Jordan canonical form for  $B$ , according to which  $C^{n+2} = V^0 \oplus \dots \oplus V^r$ . The dimension  $n_j$  of  $V^j$  is the multiplicity of  $\lambda_j$  in the characteristic polynomial of  $B$ , and each  $V^j$  is invariant under  $B$  and under  $A(t)$ .  $B$  induces on each  $V^j$  an operator  $B_j$  of the form

$$B_j = N_j + D_j, \quad N_j D_j = D_j N_j, \quad D_j = \lambda_j I_{n_j},$$

where  $N_j$  is nilpotent.  $A(t)$  decomposes as  $A(t) = e^{tB_0} \oplus \dots \oplus e^{tB}$ , and

$$e^{tB_j} = e^{t\lambda_j} \left( I + tN_j + \dots + \frac{t^k}{k!} (N_j)^k \right), \quad |e^{t\lambda_j}| = 1,$$

where  $k < n_j$  is such that  $(N_j)^k \neq 0$  and all higher powers of  $N$  vanish.

Since any  $Z$  in  $C^{n+2}$  has the decomposition  $Z = Z^0 + \dots + Z^r$ ,  $Z^j \in V^j$ , it suffices to show that the complex line determined by the origin and  $e^{tB}Z^j$  approaches  $K_j$  as  $t \rightarrow \pm\infty$ . To simplify the notation assume  $n_j = k + 1$ , so that  $N_j$  has all ones below the main diagonal, and  $V^j$  contains a unique (up to multiple) eigenvector  $W^j = (1, 0, \dots, 0)$ . Putting  $Z^j = (\zeta^0, \dots, \zeta^k)$ , we have

$$\zeta \cdot e^{tN} = \left( \zeta^0 + \zeta^1 t + \dots + \zeta^k \frac{t^k}{k!}, \zeta^1 + \dots + \zeta^k \frac{t^{k-1}}{(k-1)!}, \dots, \zeta^{k-1} + \zeta^k t, \zeta^k \right).$$

As  $t \rightarrow \pm\infty$  the dominant term is in the first coordinate; hence, the line of  $\zeta \cdot e^{tN}$  approaches the line of  $W^j$ .  $\square$

Theorem (2.1) is proved.

**3. One parameter groups on pseudoconformally flat spaces.** In this section we will prove the following theorem.

**THEOREM (3.1).** *Let  $M$  be connected, strongly pseudoconvex, and pseudoconformally flat, and suppose  $M$  admits a one-parameter Lie group  $G_1$  of transformations which has a fixed point  $p_0$ . Then either  $G_1$  is compact or  $M$  is globally equivalent to  $S^{2n+1}$  or to  $S^{2n+1}$  with a point deleted.*

Let  $Y$  be the infinitesimal generator of  $G_1$  on  $M$ . Develop a small neighborhood  $U$  of  $p_0$  onto an open subset  $U'$  of  $S^{2n+1}$ ,  $p_0$  corresponding to  $p'_0$ .  $Y|_U$  corresponds to a pseudoconformal vector field on  $U'$ , which extends to a unique global  $Y'$  on  $S^{2n+1}$ . There are three cases according to whether the one-parameter group  $G'_1$  of  $Y'$  is

- (1) bounded, i.e., has compact closure  $\overline{G'_1}$  in  $SU(n + 1, 1)$ ,
- (2) unbounded with behavior (1) of Theorem (2.1), or
- (3) unbounded with behavior (2) of Theorem (2.1).

*Case (1).* By integration over  $\overline{G'_1}$ ,  $\overline{G'_1}$  leaves invariant some  $\theta'$  and also the corresponding positive definite Riemannian metric

$$ds^2 = \theta'^2 - \text{Re} \left( g'_{\alpha\bar{\beta}} \theta'^\alpha \otimes \theta'^{\bar{\beta}} \right)$$

on  $S^{2n+1}$ . A small geodesic ball  $B'$  in  $U'$  centered at  $p'_0$  is mapped isometrically by each  $g'$  in  $G'_1$ . Each  $g$  in  $G_1$  leaves the corresponding  $\theta$  on  $U$  invariant and maps the corresponding  $B \subseteq U$  diffeomorphically.

**LEMMA (3.2).** *In Case (1),  $G_1$  is compact.*

**PROOF.** We introduce the following notation:

$\text{PsC}(M, p_0)$  = the group of global pseudoconformal transformations of  $M$  leaving  $p_0$  fixed.

$\text{PsC}(B, p_0)$  = the group of pseudoconformal transformations of  $B$  leaving  $p_0$  fixed.

$\text{PsC}'(B, p_0) = \text{PsC}(B, p_0) \cap \text{PsC}(M, p_0)$ .

$\text{PsH}(B, \theta, p_0)$  = the group of pseudohermitian transformations leaving  $p_0$  fixed.

$\text{PsH}(B, \theta, p_0)$  is compact. We have

$$G_1 \text{ closed} \subseteq \text{PsC}(M, p_0),$$

since  $G_1$  is a Lie subgroup; therefore,

$$G_1 \text{ closed} \subseteq \text{PsC}'(B, p_0).$$

Since

$$G_1 \subseteq \text{PsH}(B, \theta, p_0) \subseteq \text{PsC}(B, p_0),$$

it will suffice to show that  $\text{PsC}'(B, p_0)$  is a closed subset of  $\text{PsC}(B, p_0)$ .

Let  $\{\phi_j\}$  be a sequence in  $\text{PsC}'(B, p_0)$  and suppose  $\phi_j$  converges to  $\phi$  in  $\text{PsC}(B, p_0)$ . We must show that  $\phi$  is globally defined on  $M$ . Choose a pseudoconformal frame  $e_0$  at  $p_0$ . We have  $\phi_j \cdot e_0 \rightarrow \phi \cdot e_0$ . By Theorem (3.2) of [3] the map  $g \rightarrow g \cdot e_0$  embeds  $\text{PsC}(M)$  into the bundle  $B$  of pseudoconformal frames as a closed submanifold. So there exists some  $\phi_0$  in  $\text{PsC}(M)$  with  $\phi_0 \cdot e_0 = \phi \cdot e_0$ . Since  $\phi$  and  $\phi_0$  agree at  $e_0$ , we have  $\phi = \phi_0$  on  $B$ .  $\square$

Case (2). Pass to the universal covering  $\tilde{M}$  of  $M$ ,  $\tilde{Y}$  covering  $Y$  and  $p_1$  lying over  $p_0$ . Let  $f$  be development of  $\tilde{M}$  over  $S^{2n+1}$ . Let  $p_1$  correspond to  $p_-$  of Theorem (2.1) and  $\tilde{Y}$  correspond to  $Y'$  under  $f$ . Define

$$V = \{p_1\} \cup \{\text{all orbits of } \tilde{Y} \text{ leaving } p_1\}.$$

Since  $Y$  and hence  $\tilde{Y}$  are complete vector fields,  $f$  maps  $V$  one-to-one onto  $S^{2n+1} - (p_+)$ . Suppose  $p_2$  is in  $\text{bdry}(V)$ ; then  $f(p_2) = p_+$ . Since  $f$  is a local diffeomorphism,  $\text{bdry}(V)$  is discrete, and so  $V$  has no exterior. Furthermore,  $\tilde{Y}$  has exactly one zero of type  $p_-$ . Interchanging  $p_2$  and  $p_1$  in this argument shows that  $\tilde{Y}$  has exactly one zero of type  $p_+$  also. So, either  $\text{bdry}(V) = \{p_2\}$  and  $M \simeq S^{2n+1}$ , or  $\text{bdry}(V) = \emptyset$  and  $M \simeq S^{2n+1} - \{p_+\}$ . In the latter case  $\tilde{Y}$  has only one zero, so  $\tilde{M}$  is a one sheeted covering. In the former case  $\tilde{Y}$  has two zeroes of opposite character, so again  $\tilde{M}$  is a one sheeted covering.

Case (3). We proceed as in Case (2) except that there is only one zero  $p'$  of  $Y'$  and  $p' = f(p_1)$ . Let

$$U = \{\text{all orbits of } Y \text{ leaving } p_1\}.$$

Again, since  $\tilde{Y}$  is complete,  $f$  maps  $U$  one-to-one onto  $S^{2n+1} - \{p'\}$ ,  $\text{bdry}(U) \subseteq f^{-1}(p')$  is discrete, and  $U$  has no exterior. For any  $p_2$  in  $U$   $\lim_{t \rightarrow -\infty} \phi_t(p_2) = p_1$ , where  $\phi_t$  is the flow of  $\tilde{Y}$ . Suppose also  $f(p_1^*) = p'$ . Then for any  $p_2^*$  in

$U^*$ ,  $\lim_{r \rightarrow -\infty} \phi_r(p_2^*) = p_1^*$ . As  $M$  is connected,  $U \cap U^* \neq \emptyset$ , so we have  $p_1 = p_1^*$ . Therefore,  $f$  maps  $\tilde{M} = U \cup \{p_1\}$  one-to-one and onto  $S^{2n+1}$ . Since  $\tilde{Y}$  has only one zero,  $M \simeq \tilde{M} \simeq S^{2n+1}$ .

This proves Theorem (3.1).

In view of Theorem (3.1) it follows that if  $M$  satisfies the hypothesis of Theorem (1.1), then either  $M \cong S^{2n+1}$  or every closed noncompact one-parameter subgroup on  $M$  has no fixed points. In this respect let us note the following:

**PROPOSITION (3.3).** *Suppose  $G_1$  is a one-parameter Lie group of pseudoconformal transformations of  $M$  as in Theorem (1.1), and let  $Y$  be its infinitesimal generator. If  $Y$  is never tangent to  $H(M)$ , then  $G_1$  is compact.*

**PROOF.** Choose some  $\theta$  defining  $H(M)$ . Let  $\eta = \theta(Y)$ .  $\eta$  is never zero, since  $Y$  is always transversal. We have  $L_Y\theta = u\theta$ , so that  $L_Y\eta = u\eta$ , and  $L_Y(\eta^{-1}\theta) = 0$ .  $G_1$  leaves  $\theta^* = \eta^{-1}\theta$  invariant, so must be compact by Theorem (1.2) of [8].  $\square$

**4. Intransitive groups.** In this section we prove the following theorem.

**THEOREM (4.1).** *Let  $G$  be an effective Lie group of pseudohermitian transformations of a strongly pseudoconvex, connected,  $C$ - $R$  manifold  $(M, \theta)$  of dimension  $2n + 1$ . If  $G$  is not transitive then the dimension of  $G$  is less than or equal to  $n^2 + 1$ .*

**EXAMPLE (4.2).** To see that the estimate on  $\dim G$  is the best possible, consider  $S^{2n+1} \subseteq C^{n+1}$  with  $\theta = i\bar{\partial}r$ , where  $r = z^1\bar{z}^1 + \dots + z^n\bar{z}^n + w\bar{w} - 1$ . The group is  $G = U(n) + U(1)$  acting on  $C^{n+1}$  by

$$(z^\alpha, w) \rightarrow (z^\beta U_\beta^\alpha, w\mu), \quad U^i\bar{U} = I, \mu\bar{\mu} = 1.$$

This group preserves  $r$  and hence  $\theta$  and is clearly intransitive.

To prove Theorem (4.1) we recall that relative to an admissible coframe as in [8], we have

$$(4.1) \quad d\theta = ig_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\bar{\beta}}.$$

On  $H(M)$  we have the negative definite hermitian form

$$(4.2) \quad h = g_{\alpha\bar{\beta}}\theta^\alpha\theta^{\bar{\beta}}.$$

This form is invariant under the action of  $G$ . For a point  $p$  in  $M$  we use the notation

$$G(p) = \text{orbit of } p \text{ under } G$$

and

$$G_p = \text{isotropy subgroup of } p,$$

so that  $G(p) = G/G_p$ . We also define  $W_p = T_p(G(p)) \cap H_p(M)$ , and

$$J = \text{complex multiplication on } H_p(M).$$

$W_p$  is a real subspace of  $H_p$ , so  $0 \leq \dim W_p \leq 2n$ , and is invariant under the action of  $G_p$ . We have a faithful representation of  $G_p$  on  $U(n)$ , the unitary group of  $(H_p, h, J)$ .

The proof is divided into four parts.

(1) Suppose  $W_p = 0$  for some  $p$  in  $M$ . Then  $\dim G(p) \leq 1$  and  $\dim G = \dim G(p) + \dim G_p \leq 1 + n^2$ .

(2) *Claim.* For any  $p$  in  $M$ ,  $\dim W_p < 2n$ , i.e.,  $W_p \neq H_p$ . If  $W_p = H_p$ , then two cases are possible:

(a)  $T_p(G(p)) = T_p(M)$ . Then  $G$  is locally transitive at  $p$  and  $G(p)$  is an open subset of  $M$ . Since the pseudohermitian bundle has compact fibres  $U(n)$  and  $G$  is a Lie transformation group,  $G(p)$  is also closed in  $M$ . So  $G(p) = M$ , contradicting the fact that  $G$  is not transitive.

(b)  $T_p(G(p)) = H_p(M)$ . Then  $G(p)$  is a maximal integral submanifold of  $\theta = 0$ . Therefore  $d\theta$  vanishes when restricted to  $G(p)$ , contradicting the fact that the form (4.2) is negative definite on  $H_p(M)$ .

So we may assume that  $0 < \dim W_p < 2n$  for all  $p$  in  $M$ . Let us denote  $V_p = W_p \cap JW_p$ , and  $K_p = W_p + JW_p$ . These are  $G_p$  invariant complex subspaces of  $H_p(M)$ .

(3) Suppose that, for some  $p$  in  $M$ ,  $W_p$  is not a totally real subspace of  $H_p$ , i.e.,  $V_p \neq 0$ . We get a nontrivial  $G_p$  invariant decomposition  $H_p = V_p \oplus V_p^\perp$ . Letting  $l = \dim_{\mathbb{C}} V_p$ , we have  $G_p \subseteq U(l) \oplus U(n - l)$ . So

$$\dim G_p \leq l^2 + (n - l)^2 \leq 1 + (n - 1)^2,$$

and

$$\dim G \leq 2n + 1 + (n - 1)^2 = n^2 + 2.$$

If equality holds in the last estimate, then  $\dim G(p) = 2n$  and  $\dim G_p = 1 + (n - 1)^2$ . The complex dimension of  $V_p$  is then  $n - 1$ , and  $G_p \simeq U(n - 1) \oplus U(1)$ . Since the dimension of  $W_p$  is at most one less than  $\dim G(p)$ , we have  $\dim W_p = 2n - 1$ . We have  $G_p$  invariant decompositions  $W_p = V_p \oplus R$ , and  $H_p = V_p \oplus (R + JR)$ , where  $R$  is a one dimensional subspace. It follows that  $G_p \subseteq U(n - 1) \oplus O(1, R)$ , implying that  $\dim G_p \leq (n - 1)^2$ , a contradiction. Hence  $\dim G < n^2 + 1$ .

(4) Suppose that, for some  $p$  in  $M$ ,  $W_p$  is nontrivial and totally real. We then have  $\dim W_p = k \leq n$ , and  $K_p$  is of complex dimension  $k$ . We have the  $G_p$  invariant decompositions  $H_p = K_p \oplus K_p^\perp$ , and  $K_p = W_p \oplus JW_p$ . Therefore,

$$\dim G_p \leq \frac{1}{2}k(k - 1) + (n - k)^2,$$

and

$$\begin{aligned} \dim G &\leq (k+1) + \frac{1}{2}k(k-1) + (n-k)^2 \\ &\leq n^2 + 1 - \frac{1}{2}k^2 + \frac{1}{2}k \leq n^2 + 1, \end{aligned}$$

since  $k < n$ .

Theorem (4.1) is proved.

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