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OF REGULATED AND STEPLIKE FUNCTIONS

BY

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ABSTRACT. Let C denote the class of regulated real-valued functions on the unit interval vanishing at the origin, whose positive and negative jumps sum to infinity in every nontrivial subinterval of I. Goffman [2] showed that every f in C is (essentially) a sum $g + s$ where g is continuous and s is steplike. In this sense, a function in C is like a function of bounded variation, that has a *unique* such g and s. The import of this paper is that for f in C the representation $f = g + s$ is not only not unique, but by far the opposite holds: g can be chosen to be *any* continuous function on I vanishing at 0, at the expense of a rearrangement of s.

1. **Introduction.** Let $I = [0, 1]$ denote the closed unit interval. A real function $f\colon I \to R$ is called *regulated* [1, II, §1] if

(1) $\quad f(0+), f(1-)$ exist, as do $f(x+), f(x-)$ for all $0 < x < 1$.

Here $f(x+), f(x-)$ denote as usual the right and left limits of f at x. It will be convenient to add:

$$f(0) = f(0+) = 0, \qquad f(1) = f(1-). \tag{2}$$

Let $\mathfrak{R}(I)$ denote the class of all functions $f\colon I \to R$ satisfying (1) and (2). $\mathfrak{R}(I)$ endowed with the supremum norm is a real Banach space, containing the set $\mathcal{C}(I)$ of continuous functions on I vanishing at 0 as a Banach subspace, and the set of all step functions vanishing at 0 and continuous at 0 and 1 as a dense subspace.

Regulated functions are important in the theory of integration, as every regulated function possesses a primitive function [1]. They play a role in the theory of everywhere convergence of Fourier series and of stochastic processes.

Every function of bounded variation is regulated, and in [2] it is shown that the class of regulated functions is the union of all classes of functions of bounded Φ-variation [3] for all convex monotone Φ.

A jump function is a function which is zero to the left of a point $0 < x < 1$ and constant to its right. Thus, every step function in $\mathfrak{R}(I)$ is a finite sum of

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jumps. This representation is unique up to the order of the summands if we require that the points of discontinuity of different summands are different. We call a regulated function *steplike* if it is a uniformly convergent sum of jumps, so that any two summands have different points of discontinuity. Every function of bounded variation has a canonical representation as the sum of a continuous function and a steplike function. Goffman discovered that such a representation is possible also for functions in the class of regulated functions whose positive and negative jumps sum to infinity in every nontrivial subinterval of I ([2, Theorem 3]; see Theorem 1 in §2). We shall show that for this class the representation is not anymore canonical. Moreover, every function in that class is essentially steplike (Theorem 2, §2).

The proof of Theorem 2 makes use of the property of nearly uniform continuity of members of $\mathfrak{R}(I)$ (Theorem 3, §2); that is, for $f \in \mathfrak{R}(I)$ and $\varepsilon > 0$ one can find $\delta > 0$ and amend f on a finite set (depending on f and ε) by removing jumps and inessential left or right discontinuities, so that the oscillation of the amended function is at most ε on subintervals of length at most δ.

This property was known already to Lebesgue. For the convenience of the reader we present a proof of it in §4, having failed to find a reference. The argument used involves a reduction to the uniform continuity of a continuous real function on a compact set of reals. We then use the same reduction to show that the range of a regulated function is the difference of a compact set and an at most countable set.

It is our pleasure to express our gratitude to Casper Goffman for introducing the subject to us and for numerous stimulating discussions.

2. **General remarks and statement of results.** Let $\mathcal{C}_0(I)$ denote the Banach subspace of $\mathfrak{R}(I)$ of all functions h satisfying $h(x-) = h(x+) = 0$. It is easy to verify that $\mathcal{C}_0(I)$ is the class of all functions $h\colon I \to R$ satisfying $h(0) = h(1) = 0$ and

$$\{x\colon |h(x)| > \varepsilon\} \text{ is finite for every } \varepsilon > 0. \tag{3}$$

We now choose a direct complement to $\mathcal{C}_0(I)$ in $\mathfrak{R}(I)$ as follows. Let $\mathfrak{N}\mathcal{C}(I)$ denote the class of left-continuous functions in $\mathfrak{R}(I)$; i.e., those satisfying $f(x) = f(x-)$, $0 < x < 1$. We shall refer to functions of this class as *nearly continuous*.

PROPOSITION 1. *$\mathfrak{R}(I)$ is the direct sum of its Banach subspaces $\mathfrak{N}\mathcal{C}(I)$ and $\mathcal{C}_0(I)$.*

Let $f \in \mathfrak{N}\mathcal{C}(I)$. Define $f^*\colon I \to R$ by

$$f^*(x) = f(x+) - f(x) \quad (x < 1); \qquad f^*(1) = 0.$$

It is easily seen that $f \to f^*$ is a linear mapping of $\mathcal{RC}(I)$ into $\mathcal{C}_0(I)$, and that $\mathcal{C}(I)$ is the kernel of this mapping. Also $\|f^*\| \leqslant 2\|f\|$, so this is a continuous mapping. In [2, Theorem 2] it is shown that every $h \in \mathcal{C}_0(I)$ has an $f \in \mathcal{RC}(I)$ with $f^* = h$. The f constructed there satisfies $\|f\| = \|f^*\|$. Modifying the construction one can have $2\|f\| = \|f^*\|$.

For $h \in \mathcal{C}_0(I)$ let ${}^*h = \{f \in \mathcal{RC}(I): f^* = h\}$. From the previous remarks follows

PROPOSITION 2. *The mapping $h \to 2 \cdot {}^*h$ is a linear isometry of $\mathcal{C}_0(I)$ onto the quotient Banach space $\mathcal{RC}(I)/\mathcal{C}(I)$.*

Let $h \in \mathcal{C}_0(I)$, $0 < x < 1$. Define $J_x^h \in \mathcal{RC}(I)$ by

$$J_x^h(t) = \begin{cases} 0, & t \leqslant x, \\ h(x), & x < t. \end{cases}$$

The functions J_x^h are called *h-jumps*.

$s \in \mathcal{RC}(I)$ is called *h-steplike* iff there is an enumeration $\{x_n\}_{n=1}^{\infty}$ of $\operatorname{spt}(h) = \{x: h(x) \neq 0\}$ with no repetitions, so that

$$s = \sum_{n=1}^{\infty} J_{x_n}^h$$

and the sum converges uniformly (i.e. in the norm in $\mathcal{RC}(I)$). s is *steplike* (J is a *jump*) if it is h-steplike (an h-jump) for some h. Clearly, if s is h-steplike then $s^* = h$, so $s \in \mathcal{RC}(I)$ is steplike iff s is s^*-steplike.

We shall call $f \in \mathcal{RC}(I)$ *representable* if there are a continuous g and a steplike s so that

$$f = g + s. \tag{4}$$

Let $h \in \mathcal{C}_0(I)$ satisfy $\Sigma\{|h(x)|: x \in I\} < \infty$. Then for any enumeration $\{x_n\}_{n=1}^{\infty}$ of $\operatorname{spt}(h)$ the sum $\Sigma_{n=1}^{\infty} J_{x_n}^h$ converges uniformly and the sum function is independent of the enumeration. Thus, every f with $f^* = h$ has a *unique* representation (4). (If f is of bounded variation then $h = f^*$ has this property, f being the difference of two monotone functions.)

PROPOSITION 3. *Let $f \in \mathcal{RC}(I)$, $h = f^*$. The following are equivalent:*
1. *f is representable.*
2. *${}^*h$ contains a steplike function.*

Goffman proposed the problem to determine the representable f's. By Proposition 3 this is actually a question about $\mathcal{C}_0(I)$, namely, determine those h's so that ${}^*h$ contains a steplike function.

Consider the following condition on $h \in \mathcal{C}_0(I)$:

$$\begin{aligned}(5)\qquad & \sum\{h(x): a < x < b, h(x) > 0\} \\ & \qquad = \sum\{-h(x): a < x < b, h(x) < 0\} \\ & \qquad = \infty \quad \text{whenever } 0 \leqslant a < b \leqslant 1.\end{aligned}$$

There are many h's in $\mathcal{C}_0(I)$ satisfying (5) (see e.g. [2, Example 1]). The following theorem is stated differently as Theorem 3 in [2].

THEOREM 1. *Let $h \in \mathcal{C}_0(I)$ satisfy (5). Then $^*h$ contains a steplike function.*

We shall prove

THEOREM 2. *Let $h \in \mathcal{C}_0(I)$ satisfy (5). Then every $f \in {}^*h$ is steplike.*

If h satisfies (5) then the closure of spt(h) is I, so it has the cardinality of the continuum. Using Theorem 2 one can show that whenever $A \subset I$ is a countable set of uncountable closure, then some h with spt(h) $= A$ will satisfy: $^*h$ has a continuum of steplike functions. On the other hand, if spt(h) has a countable closure then $^*h$ contains at most one steplike function. The proof will appear elsewhere.

Theorem 2 could be restated as follows.

THEOREM 2′. *Let $s = \sum_{n=1}^{\infty} J_n$, where J_n is a jump, the sum converges uniformly and, for $n \neq m$, J_n and J_m have different points of discontinuity. Assume that s^* satisfies (5). Then for every continuous g: $I \to R$ satisfying $g(0) = 0$ there is a permutation θ of the natural numbers so that $s + g = \sum_{n=1}^{\infty} J_{\theta(n)}$.*

In this form it is a generalization of Riemann's theorem that a conditionally, but not absolutely, convergent series can be rearranged to converge to any prescribed number.

For a finite set $F \subset (0, 1)$, h: $I \to R$ let $J_F^h = \sum_{x \in F} J_x^h$. For $f \in \mathcal{NC}(I)$, $F \subset (0, 1)$ finite, let $f_F = f - J_F^{f^*}$. We shall say that f is *nearly uniformly continuous* if for every $\varepsilon > 0$ there is a $\delta > 0$ and a finite $F \subset I$ so that for every $x', x'' \in I$ we have

$$|x' - x''| \leqslant \delta \Rightarrow |f_F(x') - f_F(x'')| \leqslant \varepsilon.$$

In §4 we shall prove

THEOREM 3. *Every nearly continuous function on I is nearly uniformly continuous.*

This fact is used in the proof of Theorem 2, and its proof also yields

THEOREM 4. *Let f: $I \to R$ be regulated. Then there is a compact set $C \subset R$ and an at most countable set A so that the range of f is $C - A$.*

3. **Proof of Theorem 2.** The following easy proposition is left to the reader.

PROPOSITION 4. *Let $h \in \mathcal{C}_0(I)$. The following are equivalent properties of s:*
(i) *s is an h-steplike function.*
(ii) *$s = \sum_{n=1}^{\infty} J_{F_n}^h$, where $\{F_n\}_{n=1}^{\infty}$ is a partition of* spt(h) *into finite sets and the sum converges uniformly.*

LEMMA. *Let $f \in \mathfrak{N}\mathcal{C}(I)$. If f^* satisfies* (5) *then for every $\varepsilon > 0$ there is a finite $F \subset$* spt(h) *so that $\|f_F\| \leqslant \varepsilon$.*

PROOF. By Theorem 3 f is nearly uniformly continuous on I, so we can find a $\delta > 0$ and a finite F_0 so that for all $x', x'' \in I$ we have

$$|x' - x''| \leqslant \delta \Rightarrow |f_{F_0}(x') - f_{F_0}(x'')| \leqslant \varepsilon/3.$$

Let $h = f^*_{F_0}$. Pick $x_0, \dots, x_n \in I$ so that $0 = x_0 < x_1 < \cdots < x_n = 1$, $x_i \not\in$ spt(h) and $|x_i - x_{i-1}| \leqslant \delta$, $i = 1, \dots, n$. Let $y_i = f_{F_0}(x_i)$, $i = 0, \dots, n$. Then $y_0 = 0$ and for $x_i \leqslant x \leqslant x_{i+1}$,

$$\text{(a)} \qquad |f_{F_0}(x) - y_i| \leqslant \varepsilon/3.$$

Clearly $h(x) = f^*(x)$ for $x \not\in F_0$, and so h satisfies (5). Thus we can pick for $0 \leqslant i < n$, $x_{ij} \in [x_i, x_{i+1})$, $j = 0, \dots, n_i$, so that $x_i = x_{i0} < x_{i1} < \cdots < x_{in_i}$, $0 < n_i$, $x_{ij} \in$ spt(h) for $0 < j \leqslant n_i$, and

$$\text{(b)} \qquad h(x_{ij}) \cdot h(x_{ij+1}) \geqslant 0 \qquad (0 \leqslant j < n_i),$$

$$\text{(c)} \qquad \left| (y_{i+1} - y_i) - \sum_{j=1}^{n_i} h(x_{ij}) \right| \leqslant \frac{1}{3n} \cdot \varepsilon.$$

Let $z_{ik} = \sum_{j=1}^{k} h(x_{ij})$. By (b) $|z_{ik}| = \sum_{j=1}^{k} |h(x_{ij})|$, and so by (c),

$$\text{(d)} \qquad |z_{ik}| < |z_{ik+1}| \leqslant |z_{in_i}| \leqslant \varepsilon/3 + \varepsilon/3n \qquad (0 \leqslant k < n_i).$$

Let $F_1 = \{x_{ij}: 0 < j \leqslant n_i, i = 0, \dots, n-1\}$. Then $F_1 \subset$ spt(h). Since $h(x) = 0$ for $x \in F_0$, we have $F_1 \cap F_0 = \varnothing$. Let $F = F_0 \cup F_1$. We shall show that $\|f_F\| \leqslant \varepsilon$.

First note that $J_F^{f^*} = J_{F_0}^{f^*} + J_{F_1}^h$, so

$$f_F = f - J_F^{f^*} = \left(f - J_{F_0}^{f^*}\right) - J_{F_1}^h = f_{F_0} - J_{F_1}^h.$$

Let $0 < i \leqslant n$. It is easily seen that $J_{F_1}^h(x_i) = \sum_{t=0}^{i-1} z_{tn_t}$, and so we have by (c):

$$\text{(e)} \quad |f_{F_0}(x_i) - J_{F_1}^h(x_i)| = \left| y_i - \sum_{k=0}^{i-1} z_{kn_k} \right| \leqslant \sum_{k=0}^{i-1} |(y_{k+1} - y_k) - z_{kn_k}| \leqslant \frac{i}{3n} \varepsilon.$$

Now let $0 < x < 1$, say $x_{ij} < x \leqslant x_{ij+1}$. Then $J_{F_1}^h(x) - J_{F_1}^h(x_i) = z_{ij}$, and so by (a), (d), (e):

$$|f_F(x)| = |f_{F_0}(x) - J^h_{F_1}(x)|$$
$$\leq |f_{F_0}(x) - f_{F_0}(x_i)| + |f_{F_0}(x_i) - J^h_{F_1}(x_i)| + |J^h_{F_1}(x_i) - J^h_{F_1}(x)|$$
$$\leq \frac{1}{3}\varepsilon + \frac{i}{3n}\varepsilon + \frac{1}{3}\varepsilon + \frac{1}{n}\varepsilon = \frac{2n+i+1}{3n}\varepsilon \leq \varepsilon.$$

It follows that $\|f_F\| \leq \varepsilon$. □

Proof of Theorem 2. Let $h \in \mathcal{C}_0(I)$ satisfy (5) and let $f \in \mathfrak{N}\mathcal{C}(I)$ satisfy $f^* = h$. Let $A = \operatorname{spt}(h)$. By the lemma we can define a sequence $\{F_n\}_{n=0}^{\infty}$ of pairwise disjoint finite subsets of A and a sequence $\{f^n\}_{n=0}^{\infty}$ so that $f^0 = f$, $f^{n+1} = f^n_{F_n}$ and $\|f^n\| \leq 1/n$. Since $f^{n+1} = f - \sum_{i=1}^n J^h_{F_i}$, we see that $\sum_{i=1}^n J^h_{F_i}$ converges to f in the norm, and $A = \bigcup_{i=1}^{\infty} F_i$. By Proposition 4 f is steplike. □

4. **Proofs of Theorems 3 and 4.** Let $\{x_n\}_{n=1}^{\infty}$ be an enumeration without repetitions of a countable subset X of I. Let $g(x_n) = 2^{-n}$ and define U: $I \to [0, 2]$ by

$$U(x) = x + \sum_{n=1}^{\infty} J^g_{x_n}(x).$$

Then U is a regulated monotone function. Let $t_n = U(x_n)$. Then the range B of U is the set

$$B = [0, 2] - \bigcup_{n=1}^{\infty} (t_n, t_n + 2^{-n}]$$

and the union on the right is a disjointed one. Let C = closure of B. Then C is the compact set

$$C = [0, 2] - \bigcup_{n=1}^{\infty} (t_n, t_n + 2^{-n}).$$

Let $A = C - B = \{t_n + 2^{-n}: 1 \leq n < \infty\}$.

Let $f \in \mathfrak{N}\mathcal{C}(I)$ and assume $\operatorname{spt}(f^*) \subseteq X$. We associate with f a continuous function $\tilde{f}$: $C \to R$ as follows:

$$\tilde{f}(U(x)) = f(x), \qquad 0 \leq x \leq 1,$$
$$\tilde{f}(t_n + 2^{-n}) = f(x_n +), \qquad 1 \leq n < \infty.$$

Proof of Theorem 3. Let $\varepsilon > 0$ be given. We have to show that for suitable $\delta > 0$ and finite F, $|f_F(x') - f_F(x'')| \leq \varepsilon$ whenever $|x' - x''| \leq \delta$. Since $\tilde{f}$ is a continuous real function on the compact metric space C, it is uniformly continuous. Let $\delta_0 > 0$ satisfy

(a) $$|t' - t''| \leq \delta_0 \Rightarrow |\tilde{f}(t') - \tilde{f}(t'')| \leq \varepsilon/2.$$

Let n satisfy $\sum_{m=n+1}^{\infty} 2^{-m} \leq \delta_0/2$. Let $F = \{x_1, \ldots, x_n\}$ and

$$\delta_1 = \min\{|x_i - x_j| : 1 \leq i < j \leq n\}.$$

Finally, let $\delta = \min\{\delta_0/2, \delta_1\}$.

We shall show that for $0 \leq x' < x'' \leq 1$ with $x'' - x' \leq \delta$, $|f_F(x'') - f_F(x')| \leq \varepsilon$ holds.

Claim. Let $0 \leq a < b \leq 1$.

(i) If $[a, b) \cap F = \emptyset$, then

$$U(b) - U(a) \leq b - a + \delta_0/2.$$

(ii) If $a = x_i$ and $(a, b) \cap F = \emptyset$, then

$$U(b) - \left(U(a) + 2^{-i}\right) \leq b - a + \delta_0/2.$$

PROOF. We have

$$U(b) - U(a) = b - a + \sum\left\{2^{-m}: x_m \in [a, b)\right\}.$$

(i) If $[a, b) \cap F = \emptyset$, then

$$\sum\left\{2^{-m}: x_m \in [a, b)\right\} \leq \sum_{m=n+1}^{\infty} 2^{-m} \leq \tfrac{1}{2}\delta_0.$$

(ii) If $[a, b) \cap F = \{a\} = \{x_i\}$, then

$$\sum\left\{2^{-m}: x_m \in [a, b)\right\} = 2^{-i} + \sum\left\{2^{-m}: x_m \in (a, b)\right\}$$
$$\leq 2^{-i} + \sum_{m=n+1}^{\infty} 2^{-m} \leq 2^{-i} + \tfrac{1}{2}\delta_0. \quad \square$$

Case 1. $[x', x'') \cap F = \emptyset$.

Then by the Claim (i), $U(x'') - U(x') \leq x'' - x' + \delta_0/2$. But $x'' - x' \leq \delta \leq \delta_0/2$, so $U(x'') - U(x') \leq \delta_0$, and so $|\tilde{f}(U(x'')) - \tilde{f}(U(x'))| \leq \varepsilon/2$. That is, $|f(x'') - f(x')| \leq \varepsilon/2$. Since $[x', x'') \cap F = \emptyset$, $J_F^f(x') = J_F^f(x'')$ and so $f(x'') - f(x') = f_F(x'') - f_F(x')$. Thus $|f_F(x'') - f_F(x')| \leq \varepsilon/2 < \varepsilon$.

Case 2. $[x', x'') \cap F \neq \emptyset$.

By $x'' - x' \leq \delta$ there is exactly one $1 \leq i \leq n$ so that $x' \leq x_i < x''$. By the claim and $x'' - x' \leq \delta_0/2$ we have

$$t_i - U(x') = U(x_i) - U(x') \leq x_i - x' + \delta_0/2 \leq \delta_0,$$
$$U(x'') - \left(t_i + 2^{-i}\right) = U(x'') - \left(U(x_i) + 2^{-i}\right) \leq x'' - x_i + \delta_0/2 \leq \delta_0.$$

Thus $|\tilde{f}(U(x_i)) - \tilde{f}(U(x'))| \leq \varepsilon/2$ and $|\tilde{f}(U(x'')) - \tilde{f}(t_i + 2^{-i})| \leq \varepsilon/2$.

By the definition of $\tilde{f}$ this means

$$|f(x_i) - f(x')| \leq \varepsilon/2 \quad \text{and} \quad |f(x'') - f(x_i+)| \leq \varepsilon/2.$$

By $[x', x_i) \cap F = \emptyset$ we have $f(x_i) - f(x') = f_F(x_i) - f_F(x')$. Similarly,

$$f(x'') - f(x_i+) = f(x'') - (f(x_i+) - f(x_i)) - f(x_i) = f_F(x'') - f_F(x_i),$$

and so we have

$$|f_F(x_i) - f(x')| \leq \varepsilon/2 \quad \text{and} \quad |f_F(x'') - f_F(x_i)| \leq \varepsilon/2,$$

whence $|f_F(x'') - f_F(x')| \leq \varepsilon$. □

PROOF OF THEOREM 4. We prove it first for $f \in \mathfrak{NC}(I)$. With the notations of this section let $\tilde{A} = \{\tilde{f}(t)\colon t \in A\}$, $\tilde{B} = \{f(t)\colon t \in B\}$ and $\tilde{C} = \{\tilde{f}(t)\colon t \in C\}$. Then $\tilde{A}$ is at most countable (as the image of the countable A), $\tilde{C}$ is compact (as the continuous image of the compact set C), $\tilde{C} = \tilde{B} \cup \tilde{A}$ (as $C = B \cup A$) and $\tilde{B} = \{\tilde{f}(U(x))\colon x \in I\} = \{f(x)\colon x \in I\}$ is the range of f. Since $\tilde{C} - \tilde{A} \subseteq \tilde{B} \subseteq \tilde{C}$ the desired result follows.

We assume next that $f \in \mathfrak{R}(I)$. Let $f = f_0 + f_1$, with $f_0 \in \mathcal{C}_0(I)$ and $f_1 \in \mathfrak{NC}(I)$ (Proposition 1). Let $\tilde{B}$ denote the range of f_1, $\tilde{C}$ the closure of $\tilde{B}$ and $\tilde{A} = \tilde{C} - \tilde{B}$. By the previous case, $\tilde{C}$ is compact and $\tilde{A}$ at most countable. Let

$$X_n = \{x\colon |f_0(x)| > 1/n\}, \qquad X = \bigcup_{n=1}^{\infty} X_n,$$

$$\tilde{X} = \{f(x)\colon x \in X\}, \qquad \tilde{Y} = \{f_1(x)\colon x \in X\}.$$

By (3) X_n is finite for all n, so $\tilde{X}$, $\tilde{Y}$ are at most countable. Also

$$(\tilde{C} \cup \tilde{X}) - (\tilde{X} \cup \tilde{Y} \cup \tilde{A}) = (\tilde{C} - \tilde{A}) - \tilde{Y} = \tilde{B} - \tilde{Y} \subseteq \text{range}(f) \subseteq \tilde{C} \cup \tilde{X}.$$

So the result will follow if we show that $\tilde{C} \cup \tilde{X}$ is compact. $\tilde{C} \cup \tilde{X}$ is bounded, as so are $\tilde{C}$ and $\tilde{X}$. To see that $\tilde{C} \cup \tilde{X}$ is closed, note that $|f_0(x)| = |f(x) - f_1(x)| \leq 1/n$ for $x \notin \bigcup_{m=1}^{n} X_m$. Thus $\tilde{X}$ has only finitely many members whose distance from $\tilde{B}$ (hence from $\tilde{C}$) is more than $1/n$. Hence all accumulation points of $\tilde{X}$ are in $\tilde{C}$, and so $\tilde{X} \cup \tilde{C}$ is closed.

Theorem 4 is an immediate corollary. □

5. **Generalization to vector valued functions.** The notions of regulated, nearly continuous and steplike functions have a straightforward generalization for functions from I into any Banach space **B**. (The definition of a regulated function in [1] is actually given for this more general case.) Theorems 3 and 4 and the propositions generalize with their proofs almost word for word. The generalization of Theorem 2 requires a modification of (5) that we describe now.

In the sequel J denotes a nonempty open subinterval of I, F a finite subset of I and h a member of $\mathcal{C}_0(I)$; that is, $h\colon I \to \mathbf{B}$ satisfies $h(0) = h(1) = 0$, and for every $\varepsilon > 0$, $\{t\colon \|h(t)\| > \varepsilon\}$ is finite. We adopt the convention that $\sup(\varnothing) = 0$ and $\inf(\varnothing) = \infty$. We put

$$h((F)) = \sum_{t \in F} h(t), \quad d^h(F) = \sup\Big\{\big\|h((F \cap [0, s]))\big\|\colon s \in I\Big\}.$$

Thus, if $F = \{t_1, \dots, t_n\}$ with $t_1 < \cdots < t_n$, then

$$d^h(F) = \max\left\{\left\| \sum_{j=1}^{i} h(t_j) \right\| : i = 1, \dots, n\right\}.$$

For $x \in \mathbf{B}$, $\delta > 0$ let

$$d_J^h(x, \delta) = \inf\left\{d^h(F): F \subset J, \|h((F)) - x\| \leqslant \delta\right\},$$

$$d_J^h(x) = \sup\left\{d_J^h(x, \delta): \delta > 0\right\}.$$

Clearly, $d_J^h(x) \geqslant \|x\|$. Consider the following condition on h:

(6) For every open nonempty interval $J \subset I$ and every $x \in \mathbf{B}$, $d_J^h(x) = \|x\|$.

THEOREM 5. *Let $h \in \mathcal{C}_0(I)$ satisfy* (6). *Then every $f \in {}^*h$ is steplike.*

The proof is a straightforward generalization of that of Theorem 2.

REMARKS. 1. Consider the following condition on $h \in \mathcal{C}_0(I)$:

(7) For every open nonempty interval $J \subset I$, $\{h((F)): F$ is a finite subset of $J\}$ is dense.

It is easy to see that (6) implies (7). We do not know whether they are equivalent.

2. There is an $h \in \mathcal{C}_0(I)$ satisfying (6) if and only if $\mathbf{B}$ is separable. (5), (6) and (7) are equivalent if $\mathbf{B}$ is the real line.

3. There are several conditions which are equivalent to (6). We shall describe one.

For $x \in \mathbf{B}$ let $d^h(x) = \sup d_J^h(x)$, where the supremum is taken over all open nonempty subintervals J of I. For $\varepsilon > 0$ let $r^h(\varepsilon) = \sup\{d^h(x): \|x\| \leqslant \varepsilon\}$. It can be shown that (6) is equivalent to $\inf\{r^h(\varepsilon): \varepsilon > 0\} = 0$.

Let M be any real function with $\lim_{t\to 0} M(t) = 0$ and $M(t) \geqslant t$ for $t > 0$. It follows that in (6) we can replace the equality $d_J^h(x) = \|x\|$ by the inequality $d_J^h(x) \leqslant M(\|x\|)$.

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