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GENERALIZED HANKEL CONJUGATE TRANSFORMATIONS ON REARRANGEMENT INVARIANT SPACES

BY

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ABSTRACT. The boundedness properties of the generalized Hankel conjugate transformations H_λ on certain weighted Lebesgue spaces are studied. These are used to establish a boundedness criterion for the H_λ on the more general class of rearrangement invariant spaces. The positive operators in terms of which the criterion is given are used to construct pairs of spaces between which the H_λ are continuous; in particular, a natural analogue of a well-known result of Zygmund concerning the classical conjugate function operator is obtained for the H_λ.

1. **Introduction.** Let $M(0, \infty)$ denote the class of Lebesgue-measurable functions on $(0, \infty)$. To each $\lambda > -1$ there is associated a generalized Hankel conjugate transformation H_λ, defined at $f \in M(0, \infty)$ by

$$(H_\lambda f)(y) = \lim_{x\to 0+} \int_0^\infty Q_\lambda(x, y, z) f(z) z^{2\lambda}\, dz, \tag{1.1}$$

provided there is a set E of Lebesgue measure zero so that when $y \notin E$ the integral is defined for all $x > 0$ and the indicated limit exists. Here

$$Q_\lambda(x, y, z) = -(yz)^{-\lambda+1/2} \int_0^\infty e^{-xt} J_{\lambda+1/2}(yt) J_{\lambda-1/2}(zt) t\, dt, \tag{1.2}$$

J_ν denoting the Bessel function of order ν. For X and Y Banach spaces, denote by $[X, Y]$ the space of bounded linear operators from X to Y; in case $X = Y$, abbreviate $[X, X]$ to $[X]$. We will be interested in when $H_\lambda \in [X, Y]$ given that X and Y are certain function spaces with common underlying measure μ_α defined by $d\mu_\alpha(t) = t^{\alpha-1}\, dt$, α any real number. Thus, for example, we will investigate the continuity of the H_λ on the Lebesgue spaces $L^p(\mu_\alpha)$, $1 \leqslant p < \infty$, where, as usual, the function $f \in M(0, \infty)$ is in $L^p(\mu_\alpha)$ if

$$\|f\|_{p,\alpha} = \left[\int_0^\infty |f(x)|^p\, d\mu_\alpha(x)\right]^{1/p} < \infty. \tag{1.3}$$

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Muckenhoupt and Stein [8] considered the H_λ, $\lambda > 0$, in the same relation to Fourier-Bessel (i.e. Hankel) transformations as holds between the classical Hilbert and Fourier transformations (hence the name Hankel conjugate transformation). They showed that for $\lambda > 0$,

$$H_\lambda \in \left[L^p(\mu_{2\lambda+1}) \right], \tag{1.4}$$

when $1 < p < \infty$, with H_λ of weak-type (1, 1) on $L^1(\mu_{2\lambda+1})$. In addition, they gave certain weighted inequalities by modifying $\mu_{2\lambda+1}$ with suitable weight factors.

Rooney [10] considerably extended these results to encompass, in connection with each H_λ, $\lambda > -\frac{1}{2}$, a family of measures of the type μ_α. Thus, he proved that for $\lambda > -\frac{1}{2}$ and $1 < p < \infty$,

$$H_\lambda \in \left[L^p(\mu_\alpha) \right], \tag{1.5}$$

provided $-p < \alpha < (2\lambda + 1)p$. The weighted inequalities given for the H_λ in [8] yield (1.5) in case $\lambda > 0$ and $0 < \alpha < (2\lambda + 1)p$.

In another direction, Andersen [1] generalized (1.4) to the larger context of rearrangement invariant spaces, obtaining a boundedness criterion first established in Boyd [3] for the Hilbert transformation H. To describe the criterion we first define the operators P and P' as follows:

If $f \in M(0, \infty)$, then, for $t > 0$,

$$(Pf)(t) = t^{-1}\int_0^t f(u)\,du \quad \text{and} \quad (P'f)(t) = \int_t^\infty f(u)\,\frac{du}{u}, \tag{1.6}$$

whenever the required integrals exist a.e. The result of [1] now reads:

If ρ_1 and ρ_2 are $\mu_{2\lambda+1}$-rearrangement invariant norms on $M(0, \infty)$, generated by σ_1 and σ_2, respectively, then, in order that $H_\lambda \in [L^{\rho_1}, L^{\rho_2}]$, $\lambda > 0$, it is both necessary and sufficient that

$$P + P' \in \left[L^{\sigma_1}, L^{\sigma_2} \right]. \tag{1.7}$$

One of the two principal results of the second section, Theorem 2.1, features a real variable proof of (1.5) which permits a further extension to all $\lambda > -1$. Theorem 2.2 gives the weak-type inequalities for the case $p = 1$. As pointed out in §5, the methods of proof will be applied in other contexts.

In Theorem 3.1 we obtain the criterion (1.7) for values of λ and α arising from (1.5).

Theorem 3.1 is first applied in §4 to obtain, in the manner of [3], boundedness criteria for the H_λ on special classes of rearrangement invariant spaces–the Lorentz and Orlicz spaces. Following this, pairs of m-rearrangement invariant spaces are constructed between which $P + P'$ is bounded. (Here m is the usual Lebesgue measure on $(0, \infty)$.) In view of Theorem 3.1, these yield continuous pairs for the H_λ as well as for other similar operators, for example H. In particular, the analogue of a well-known inequality of Zygmund relating to the classical conjugate function operator C is obtained for such operators as the H_λ and H.

For the principal definitions and facts, as well as most of the notation, concerning rearrangement invariant spaces, see [4]. To distinguish Lebesgue measure on $(0, \infty)$ we consistently use the standard notation m rather than μ_1. For the same reason we denote the m-rearrangement of a function $f \in M(0, \infty)$ by $f^{\#}$ instead of f^*. Finally, when studying H_λ with respect to a fixed μ_α it will be convenient to write (1.1) as

$$(1.8) \qquad (H_\lambda f)(y) = \lim_{x \to 0+} \int_0^\infty Q_{\lambda,\alpha}(x, y, z) f(z)\, d\mu_\alpha(z),$$

where

$$(1.9) \qquad Q_{\lambda,\alpha}(x, y, z) = z^{2\lambda+1-\alpha} Q_\lambda(x, y, z).$$

2. **H_λ on the Lebesgue spaces.** The proofs of Theorems 2.1 and 2.2 require a number of preliminary results. In the following lemma we extend, and refine somewhat, the estimates (e) given on p. 87 of [8] for $\lambda > 0$ and $\alpha = 2\lambda + 1$.

LEMMA 2.1. *If* $\lambda > -1$ *and* α *is any real number, then,*

$$\text{(i)} \quad Q_{\lambda,\alpha}(x, y, z) = O(y^{-2\lambda-1} z^{2\lambda+1-\alpha}), \quad \text{if } 0 < z \leqslant y/2,$$
$$= O(yz^{-\alpha-1}), \quad \text{if } z \geqslant 2y,$$

$$(2.1) \qquad \text{(ii)} \quad Q_{\lambda,\alpha}(x, y, z) = C_\lambda y^{-\lambda} z^{\lambda+1-\alpha} \frac{y - z}{x^2 + (y - z)^2} + O\left[(yz)^{-\alpha/2}\left(1 + \log^+ \frac{yz}{(y - z)^2}\right)\right],$$
$$\text{if } y/2 \leqslant z \leqslant 2y.$$

Also, the α*th associate kernel* $Q'_{\lambda,\alpha}(x, y, z) = Q_{\lambda,\alpha}(x, z, y)$ *satisfies*

$$\text{(i)}' \quad Q'_{\lambda,\alpha}(x, y, z) = O(y^{-\alpha-1} z), \quad \text{if } 0 < z \leqslant y/2,$$
$$= O(y^{2\lambda+1-\alpha} z^{-2\lambda-1}), \quad \text{if } z \geqslant 2y,$$

$$(2.2) \qquad \text{(ii)}' \quad Q'_{\lambda,\alpha}(x, y, z) = C_\lambda y^{\lambda+1-\alpha} z^{-\lambda} \frac{z - y}{x^2 + (y - z)^2} + O\left[(yz)^{-\alpha/2}\left(1 + \log^+ \frac{yz}{(y - z)^2}\right)\right],$$
$$\text{if } y/2 \leqslant z \leqslant 2y.$$

PROOF. To begin, we obtain sharper estimates for $Q_\lambda(x, y, z)$. From [11, p.

45], it follows that

$$Q_\lambda(x,y,z) = -2\left(\lambda+\tfrac{1}{2}\right)y(yz)^{-(\lambda+1/2)}\int_0^\infty e^{-xt}J_{\lambda+1/2}(yt)J_{\lambda+1/2}(zt)\,dt \tag{2.3}$$
$$-(yz)Q_{\lambda+1}(x,z,y).$$

In view of the estimates of [8] referred to above

$$-(yz)Q_{\lambda+1}(x,z,y) = O(y^{-2\lambda-1}),\quad \text{if } 0<z\leqslant y/2,$$
$$= O(yz^{-2\lambda-2}),\quad \text{if } z\geqslant y/2;$$
$$-(yz)Q_{\lambda+1}(x,z,y) = C_{\lambda+1}(yz)^{-\lambda}\frac{y-z}{x^2+(y-z)^2} \tag{2.4}$$
$$+(yz)O\left[z^{-2\lambda-3}\left(1+\log^+\frac{yz}{(y-z)^2}\right)\right],$$

if $y/2\leqslant z\leqslant 2y$. The proof will be complete if it can be shown that the first term on the right-hand side of (2.3) is

$$O(y^{-2\lambda-1}),\quad \text{if } 0<z\leqslant y/2,$$
$$O(yz^{-2\lambda-2}),\quad \text{if } y\geqslant 2z, \tag{2.5}$$
$$O\left[y^{-2\lambda-1}\left(1+\log^+\frac{yz}{(y-z)^2}\right)\right],\quad \text{if } y/2\leqslant z\leqslant 2y.$$

According to [6, p. 50],

$$y(yz)^{-(\lambda+1/2)}\int_0^\infty e^{-xt}J_{\lambda+1/2}(yt)J_{\lambda+1/2}(zt)\,dt \tag{2.6}$$
$$= \pi^{-1}y^{-\lambda}z^{-\lambda-1}Q_\lambda^0\left((x^2+y^2+z^2)/2yz\right)$$

for $\lambda > -1$. Here Q_λ^0 is a Legendre function of the second kind. From [11, p. 389],

$$Q_\lambda^0(v) = \int_0^\infty\left[v^2+(v^2-1)^{1/2}\cosh\theta\right]^{-\lambda-1}d\theta,\qquad \lambda>-1, \tag{2.7}$$

and is thus seen to be strictly decreasing for $v>1$. This, together with the fact that $\int_0^\infty J_{\lambda+1/2}(yt)J_{\lambda+1/2}(zt)\,dt$ is convergent for $\lambda>-1$, whenever $y\neq z$, means it is enough to show the estimates (2.5) for the left-hand side of (2.6) with $x=0$. By [11, pp. 406–407],

$$y(yz)^{-(\lambda+1/2)}\int_0^\infty J_{\lambda+1/2}(yt)J_{\lambda+1/2}(zt)\,dt \tag{2.8}$$
$$= \pi^{-1}y\int_0^\pi \frac{\sin^{2\lambda+1}\phi}{\left[y^2+z^2-2yz\cos\phi\right]^{\lambda+1}}\,d\phi.$$

Writing $t = y/z$, (2.8) equals

$$\pi^{-1}yz^{-2\lambda-2}\int_0^\pi \frac{\sin^{2\lambda+1}\phi}{(t^2-2t\cos\phi+1)^{\lambda+1}}\,d\phi. \tag{2.9}$$

This yields the second assertion of (2.5). The first is similarly obtained. Letting $E = (y-z)^2 + yz\phi^2$, and using the facts that, as $\phi \to 0+$, $\cos\phi = 1 - \phi^2/2 + O(\phi^4)$ and $\sin^{2\lambda+1}\phi = \phi^{2\lambda+1} + O(\phi^{2\lambda+3})$, we obtain, as in [8, Lemma 4],

$$y\int_0^{\pi/2} \frac{\sin^{2\lambda+1}\phi}{\left[y^2+z^2-2yz\cos\phi\right]^{\lambda+1}}\,d\phi$$
$$= y\int_0^{\pi/2}\phi^{2\lambda+1}E^{-\lambda-1}\,d\phi + y\int_0^{\pi/2}O(\phi^{2\lambda+3})E^{-\lambda-1}\,d\phi \tag{2.10}$$
$$+ y\int_0^{\pi/2}O(\phi^{2\lambda+5})(yz)E^{-\lambda-2}\,d\phi + y\int_0^{\pi/2}O(\phi^{2\lambda+7})(yz)E^{-\lambda-2}\,d\phi.$$

The treatment of the first term of (2.10) is typical. The analogous proof of Lemma 4 strongly suggests the change of variable $u = \sqrt{\sigma/\Delta}$, where $\sigma = yz$ and $\Delta = (y-z)^2$. This yields

$$y\int_0^{\pi/2}\phi^{2\lambda+1}E^{-\lambda-1}\,d\phi = y\sigma^{-\lambda-1}\int_0^{(\pi/2)\sqrt{\sigma/\Delta}} \frac{u^{2\lambda+1}}{\left[1+u^2\right]^{\lambda+1}}\,du \tag{2.11}$$
$$\leqslant y(yz)^{-\lambda-1}\int_0^{(\pi/2)yz/(y-z)^2} \frac{u^{2\lambda+1}}{\left[1+u^2\right]^{\lambda+1}}\,du,$$

since $y/2 \leqslant z \leqslant 2y$. This is readily seen to be

$$O\left[y^{-2\lambda-1}\left(1+\log^+\frac{yz}{(y-z)^2}\right)\right].$$

Finally,

$$y\int_{\pi/2}^{\pi} \frac{\sin^{2\lambda+1}\phi}{\left[y^2+z^2-2yz\cos\phi\right]^{\lambda+1}}\,d\phi \leqslant y^{-2\lambda-1}\int_{\pi/2}^{\pi}\sin^{2\lambda+1}\phi\,d\phi. \tag{2.12}$$

REMARK. When $\lambda = -\frac{1}{2}$ the first term on the right-hand side of (2.3) vanishes. Applying the methods of Lemma 2.1 yields in this instance the following improvements for some of the estimates (2.1) and (2.1)′:

$$(2.13)\qquad \begin{aligned} Q_{-1/2,\alpha}(x,y,z) &= O(y^{-2}z^{2-\alpha}), \quad \text{if } 0 < z \leqslant y/2,\\ Q'_{-1/2,\alpha}(x,y,z) &= O(y^{2-\alpha}z^{-2}), \quad \text{if } z \geqslant 2y. \end{aligned}$$

As a consequence of (2.13), the $H_{-1/2}$ result corresponding to (1.5), and others, hold under less restrictive conditions than might be expected. This special case will be considered in some detail following Corollary 2.2.1. Until then it will be assumed that $\lambda > -1$, but $\lambda \neq -\frac{1}{2}$.

We will have need of five operators of the sort dealt with in [10, p. 1201]. The first two, depending on both $\lambda > -1$ and $\alpha \in (-\infty, \infty)$, are denoted by $T^{(1)}_{\lambda,\alpha}$ and $T^{(2)}_{\lambda,\alpha}$. They correspond to the kernels

$$(2.14)\quad k^{(1)}_{\lambda,\alpha}(x) = x^{-2\lambda-1}\chi_{(2,\infty)}(x) \quad\text{and}\quad k^{(2)}_{\lambda,\alpha}(x) = x^{2\lambda+1-\alpha}\chi_{(0,1/2)}(x).$$

The remaining three, $T^{(n)}_\alpha$, $n = 1, 2, 3$, depending only on α, have the kernels

$$(2.15)\qquad \begin{aligned} &k^{(1)}_\alpha(x) = x\chi_{(0,1/2)}(x), \quad k^{(2)}_\alpha(x) = x^{-\alpha-1}\chi_{(2,\infty)}(x),\\ &k^{(3)}_\alpha(x) = x^{-\alpha/2}\left(1 + \log^+ \frac{x}{(1-x)^2}\right)\chi_{(1/2,2)}(x). \end{aligned}$$

The boundedness properties of these operators are set forth in

LEMMA 2.2. *If* $1 \leqslant p \leqslant \infty$, *then, with the obvious conventions when* $p = 1$ *or* $p = \infty$,

(i) $T^{(1)}_{\lambda,\alpha} \in [L^p(\mu_\alpha)]$, *provided* $\alpha < (2\lambda + 1)p$; $T^{(1)}_{\lambda,2\lambda+1}$ *is of weak-type* (1, 1) *on* $L^1(\mu_{2\lambda+1})$.

(ii) $T^{(2)}_{\lambda,\alpha} \in [L^p(\mu_\alpha)]$, *provided* $\alpha < (2\lambda + 1)p/(p - 1)$.

(iii) $T^{(1)}_\alpha \in [L^p(\mu_\alpha)]$, *provided* $-p < \alpha$; $T^{(1)}_{-1}$ *is of weak-type* (1, 1) *on* $L^1(\mu_{-1})$.

(iv) $T^{(2)}_\alpha \in [L^p(\mu_\alpha)]$, *provided* $-p/(p - 1) < \alpha$.

(v) $T^{(3)}_\alpha \in [L^p(\mu_\alpha)]$, *for all* α *and* p.

PROOF. The results for all five operators follow from [10, p. 1201], when $1 < p < \infty$.

In case $p = \infty$, $T^{(3)}_\alpha$ is the only operator which offers any problem. Given $f \in L^\infty(\mu_\alpha)$,

$$(2.16)\qquad \begin{aligned} |(T^{(3)}_\alpha f)(y)| &\leqslant \int_{y/2}^{2y} (yz)^{-\alpha/2}\left(1 + \log^+ \frac{yz}{(y-z)^2}\right)|f(z)|\, d\mu_\alpha(z)\\ &\leqslant k_\alpha \int_{1/2}^{2}\left(1 + \log^+ \frac{w}{(1-w)^2}\right)|f(yw)|\, \frac{dw}{w}, \end{aligned}$$

where $k_\alpha = \max[2^{\alpha/2}, 2^{-\alpha/2}]$. It follows that $T_\alpha^{(3)} \in [L^\infty(\mu_\alpha)]$.

To illustrate the proof of the weak-type (1, 1) inequalities we consider $T_{\lambda,\alpha}^{(1)}$ with $\alpha = 2\lambda + 1$. For $f \in L^1(\mu_{2\lambda+1})$, $\lambda > -\frac{1}{2}$,

$$E^s = \left\{y : \left|\left(T_{\lambda,2\lambda+1}^{(1)} f\right)(y)\right| > s\right\} \subset \left\{y : y^{-2\lambda-1}\|f\|_{1,-1} > s\right\} \subset \left\{y : y < \left[s^{-1}\|f\|_{1,-1}\right]^{1/(2\lambda+1)}\right\} \tag{2.17}$$

and so

$$\mu_{2\lambda+1}(E^s) \leqslant \int_0^\beta z^{2\lambda}\,dz = \left[(2\lambda+1)^{-1}\|f\|_{1,-1}\right]s^{-1}, \qquad \beta = \left[s^{-1}\|f\|_{1,-1}\right]^{1/(2\lambda+1)}. \tag{2.18}$$

The next result, which concerns the restriction of the Hilbert transformation H to functions defined on $(0, \infty)$, will be useful in the sequel.

LEMMA 2.3. *Suppose* $f \in L^p(m)$, $1 \leqslant p < \infty$. *Then*

$$\lim_{x\to 0+} \int_{y/2}^{2y} \frac{(y-z)f(z)}{x^2+(y-z)^2}\,dz \qquad (y > 0, z > 0) \tag{2.19}$$

exists almost everywhere. If $p > 1$, *the limit also exists in the mean of order* p *and there is a positive constant* A_p, *independent of* $x > 0$, *so that*

$$\int_0^\infty dy \left|\int_{y/2}^{2y} \frac{(y-z)f(z)}{x^2+(y-z)^2}\,dz\right|^p \leqslant A_p \int_0^\infty |f(z)|^p\,dz. \tag{2.20}$$

The operator defined by (2.19) *is of weak-type* (1, 1) *on* $L^1(m)$.

PROOF. As follows immediately from classical results for H, all assertions would hold if the integral in (2.19) were over $(0, \infty)$. It suffices then to prove them for the integrals over $(0, y/2)$ and $(2y, \infty)$. The pointwise limits for the latter integrals exist everywhere. Again,

$$\left|\int_0^{y/2} \frac{(y-z)f(z)}{x^2+(y-z)^2}\,dz\right| \leqslant \int_0^{y/2} \frac{|f(z)|}{y-z}\,dz \leqslant 2\int_0^{y/2} \frac{f^\#(z)}{y+z}\,dz \tag{2.21}$$

and similarly,

$$\left|\int_{2y}^\infty \frac{(y-z)f(z)}{x^2+(y-z)^2}\,dz\right| \leqslant \int_0^\infty \frac{f^\#(z)}{y+z}\,dz.$$

Therefore, for all $x > 0$, the integrals over $(0, y/2)$ and $(2y, \infty)$ are bounded above in absolute value by a constant multiple of $Sf^{\#}$, the Stieltjes transform of $f^{\#}$, where

$$(Sf^{\#})(y) \equiv \int_0^{\infty} \frac{f^{\#}(z)}{y+z}\, dz. \tag{2.22}$$

Since $S \in [L^p(m)]$ when $1 < p < \infty$ and S is of weak-type (1, 1) on $L^1(m)$, the proof is complete.

We denote by $L^1(\mu_\alpha) + L^\infty(\mu_\alpha)$ the class of all $h \in M(0, \infty)$ for which $h = f + g$, with $f \in L^1(\mu_\alpha)$, $g \in L^\infty(\mu_\alpha)$.

LEMMA 2.4. *Suppose* $1 < p < \infty$. *Then*

$$\lim_{x\to 0+} \int_0^{\infty} Q_{\lambda,\alpha}(x, y, z) f(z)\, d\mu_\alpha(z) \tag{2.23}$$

exists almost everywhere for $f \in L^p(\mu_\alpha)$, *provided* $-p < \alpha < (2\lambda + 1)p$. *Also,*

$$\lim_{x\to 0+} \int_0^{\infty} Q'_{\lambda,\alpha}(x, y, z) f(z)\, d\mu_\alpha(z) \tag{2.24}$$

exists almost everywhere for $f \in L^p(\mu_\alpha)$, *provided*

$$-p/(p-1) < \alpha < (2\lambda + 1)p/(p-1). \tag{2.25}$$

When $\lambda > -\frac{1}{2}$ *and* $-1 \leq \alpha \leq 2\lambda + 1$, *the limit* (2.23) *exists almost everywhere for* $f \in L^1(\mu_\alpha) + L^\infty(\mu_\alpha)$; *the limit* (2.24) *exists almost everywhere for those* f *in* $L^1(\mu_\alpha) + L^\infty(\mu_\alpha)$ *satisfying*

$$\int_0^{\infty} \frac{f^*(t)}{t+1}\, dt < \infty. \tag{2.26}$$

PROOF. For (2.24) under the restriction (2.26). If $-1 < \alpha < 2\lambda + 1$, the desired limit in fact exists for $f \in L^1(\mu_\alpha) + L^\infty(\mu_\alpha)$. For such an f we have, by (2.1)′, part (i)′,

$$|Q'_{\lambda,\alpha}(x, y, z)\chi_{(0,y/2)}(z)|\, |f(z)| z^{\alpha-1} \leq c k_\alpha^{(2)}(y/z)|f(z)| z^{-1}, \tag{2.27}$$

where $c > 0$ is independent of f and x. In view of Lemma 2.2, the right-hand side of (2.27) is in $L^1(m)$ for almost every $y \in (0, \infty)$, whence the dominated convergence theorem yields (2.24) for $Q'_{\lambda,\alpha}(x, y, z)\chi_{(0,y/2)}(z)$ in place of $Q'_{\lambda,\alpha}(x, y, z)$. The same line of reasoning establishes (2.24) for

$$Q'_{\lambda,\alpha}(x, y, z)\chi_{(2y,\infty)}(z)$$

in place of $Q'_{\lambda,\alpha}(x, y, z)$. Indeed, such an argument involving $k_\alpha^{(3)}$ will complete the proof if it can be shown that

$$\lim_{x\to 0+} \int_{y/2}^{2y} \frac{y-z}{x^2 + (y-z)^2} f(z) z^{\alpha-\lambda-1}\, dz \tag{2.28}$$

exists almost everywhere. Let n be an arbitrary integer. When $y \in (2^n, 2^{n+1})$, (2.28) is equal to (2.19) for the function $z^{\alpha-\lambda-1}f(z)\chi_{(2^{n-1},2^{n+2})}(z) \in L^1(m)$, and so (2.28) exists for almost all $y \in (2^n, 2^{n+1})$.

In the cases $\alpha = 2\lambda + 1$ and $\alpha = -1$ we run into problems with the m-integrability of $k^{(2)}_{\lambda,\alpha}(y/z)|f(z)|z^{-1}$ and $k^{(2)}_{\alpha}(y/z)|f(z)|z^{-1}$, respectively. To see how (2.26) ensures this integrability, consider the case $\alpha = -1$. Given $f \in M(0, \infty)$ and satisfying (2.26),

$$\begin{aligned}\int_0^{y/2}|f(z)|z^{-1}\,dz &= \int_0^{\infty}|z\chi_{(0,y/2)}(z)|\;|f(z)|z^{-2}\,dz \\ &\leqslant \int_0^{\infty} f^*(t)\,g^*(t)\,dt,\end{aligned} \tag{2.29}$$

where $g(z) = z\chi_{(0,y/2)}(z)$. A straightforward calculation shows

$$g^*(t) = 1/\left(t + (2/y)\right). \tag{2.30}$$

But (2.26) is equivalent to the same condition with $t + 1$ replaced by $t + a$ for any $a > 0$.

THEOREM 2.1. *Suppose* $1 < p < \infty$. *Then*

$$H_\lambda \in \left[L^p(\mu_\alpha)\right] \quad \textit{for } -p < \alpha < (2\lambda + 1)p. \tag{2.31}$$

Also,

$$H'_{\lambda,\alpha} \in \left[L^p(\mu_\alpha)\right] \quad \textit{for } -p/(p-1) < \alpha < (2\lambda + 1)p/(p-1). \tag{2.32}$$

PROOF. For $H'_{\lambda,\alpha}$. In view of (2.24), Fatou's lemma will yield (2.32) once we have supplied a positive constant C_p with

$$\left\|\int_0^{\infty} Q'_{\lambda,\alpha}(x, y, z) f(z)\,d\mu_\alpha(z)\right\|_{p,\alpha} \leqslant C_p\|f\|_{p,\alpha} \tag{2.33}$$

for all $x > 0$. The estimate

$$\left|\int_0^{\infty} Q'_{\lambda,\alpha}(x, y, z)\chi_{(0,y/2)}(z) f(z)\,d\mu_\alpha(z)\right| \leqslant c\left(T^{(2)}_{\alpha}|f|\right)(y) \tag{2.34}$$

and Lemma 2.2(iv) yield a $C'_p > 0$ such that (2.32) holds with

$$Q'_{\lambda,\alpha}(x, y, z)\chi_{(0,y/2)}(z)$$

instead of $Q'_{\lambda,\alpha}(x, y, z)$ and C'_p instead of C_p. Observe that, in addition, the dominated convergence theorem ensures that the limit corresponding to (2.24) exists in $L^p(\mu_\alpha)$, if $-p/(p-1) < \alpha$. The same conclusions hold true for $Q'_{\lambda,\alpha}(x, y, z)\chi_{(2y,\infty)}(z)$, provided $\alpha < (2\lambda + 1)p/(p-1)$. Similar considerations involving $T^{(3)}_{\alpha}|f|$ will complete the proof if $C''_p > 0$ can be produced satisfying

$$(2.35)\qquad \left\| y^{\lambda+1-\alpha}\int_{y/2}^{2y}\frac{(y-z)f(z)z^{\alpha-\lambda-1}}{x^2+(y-z)^2}\,dz\right\|_{p,\alpha} < C_p''\|f\|_{p,\alpha}$$

for all $\alpha \in (-\infty, \infty)$. Now,

$$(2.36)\qquad \begin{aligned}&\int_0^\infty d\mu_\alpha(y)\left|y^{\lambda+1-\alpha}\int_{y/2}^{2y}\frac{(y-z)f(z)z^{\alpha-\lambda-1}}{x^2+(y-z)^2}\,dz\right|^p\\ &\quad=\int_0^\infty dy\left[\sum_{n=-\infty}^{\infty}\chi_{E_n}(y)y^\beta\left|\int_{y/2}^{2y}\frac{(y-z)f_n(z)z^{\alpha-\lambda-1}}{x^2+(y-z)^2}\,dz\right|\right]^p\end{aligned}$$

where $E_n = (2^n, 2^{n+1})$, $\beta = \alpha - 1 + p(\lambda + 1 - \alpha)$, and $f_n(z) = f(z)\chi_{(2^{n-1},2^{n+2})}(z)$. On interchanging the order of integration and summation and observing that y^β is of order $2^{n\beta}$ on E_n, it is found that (2.36) is bounded above by a constant multiple of

$$(2.37)\qquad \sum_{n=-\infty}^{\infty}2^{n\beta}\int_0^\infty dy\left|\int_{y/2}^{2y}\frac{(y-z)f_n(z)z^{\alpha-\lambda-1}}{x^2+(y-z)^2}\,dz\right|^p.$$

Applying (2.20) of Lemma 2.3 to the function $z^{\alpha-\lambda-1}f_n(z)$ in $L^p(m)$ yields the upper bound

$$(2.38)\qquad A_p'\sum_{n=-\infty}^{\infty}2^{n\beta}\int_{2^{n-1}}^{2^{n+2}}|f(z)|^p z^{p(\alpha-\lambda-1)}\,dz.$$

This, in turn, is dominated by

$$(2.39)\qquad A_p''\sum_{n=-\infty}^{\infty}\int_{2^{n-1}}^{2^{n+2}}|f(z)|^p z^{\alpha-1}\,dz < C_p''\int_0^\infty|f(z)|^p\,d\mu_\alpha(z),$$

since $z^{-\beta}$ is of the order $2^{-n\beta}$ on $(2^{n-1}, 2^{n+2})$.

COROLLARY 2.1.1. *Suppose* $1 < p < \infty$. *Then, for* $f \in L^p(\mu_\alpha)$,

$$(2.40)\qquad \lim_{x\to 0+}\int_0^\infty Q_{\lambda,\alpha}(x,y,z)f(z)\,d\mu_\alpha(z) = (H_\lambda f)(y)$$

in mean of order p, *provided* $-p < \alpha < (2\lambda + 1)p$. *Also, for* $f \in L^p(\mu_\alpha)$,

$$(2.41)\qquad \lim_{x\to 0+}\int_0^\infty Q_{\lambda,\alpha}'(x,y,z)f(z)\,d\mu_\alpha(z) = (H_{\lambda,\alpha}'f)(y)$$

in mean of order p, *provided* $-p/(p-1) < \alpha < (2\lambda+1)p/(p-1)$.

PROOF. A familiar argument based on the analogue of (2.33) for

$$Q_{\lambda,\alpha}(x, y, z)$$

will complete the proof of (2.40) once (2.40) can be demonstrated for all f which are bounded, measurable, and compact-supported. An examination of the proof of Theorem 2.1 reveals it is enough to show

$$\lim_{x \to 0+} y^{-\lambda} \int_{y/2}^{2y} \frac{(y - z) f(z) z^{\lambda}}{x^2 + (y - z)^2} \, dz \tag{2.42}$$

exists in $L^p(\mu_\alpha)$, $1 < p < \infty$. Suppose supp $f \subset [a, b]$ and let $h(y)$ denote the pointwise almost everywhere limit of (2.42). Then

$$\begin{aligned} \int_0^\infty d\mu_\alpha(y) \left| y^{-\lambda} \int_{y/2}^{2y} \frac{(y - z) f(z) z^{\lambda}}{x^2 + (y - z)^2} \, dz - h(y) \right|^p \\ < \int_{a/2}^{2b} y^{\alpha - 1 - \lambda p} \, dy \left| \int_{y/2}^{2y} \frac{(y - z) f(z) z^{\lambda}}{x^2 + (y - z)^2} \, dz - y^{\lambda} h(y) \right|^p \\ < k \int_0^\infty dy \left| \int_{y/2}^{2y} \frac{(y - z) f(z) z^{\lambda}}{x^2 + (y - z)^2} \, dz - y^{\lambda} h(y) \right|^p . \end{aligned} \tag{2.43}$$

But $f(z) z^{\lambda} \in L^p(m)$ and so Lemma 2.3 yields (2.42).

The proof of (2.41) is similar.

COROLLARY 2.1.2. *The operators H_λ and $H'_{\lambda,\alpha}$ satisfy the relation*

$$\int_0^\infty (H_\lambda f)(y) g(y) \, d\mu_\alpha(y) = \int_0^\infty f(y)(H'_{\lambda,\alpha} g)(y) \, d\mu_\alpha(y) \tag{2.44}$$

for all f and g which are bounded, measurable, and vanish outside a set of μ_α-finite measure.

PROOF. Formula (1.2) for $Q_\lambda(x, y, z)$ shows that, for fixed $x > 0$,

$$Q_{\lambda,\alpha}(x, y, z)$$

is continuous on any compact subset of $(0, \infty) \times (0, \infty)$. If the functions f and g are compact-supported, then, by Fubini's theorem,

$$\begin{aligned} \int_0^\infty g(y) \, d\mu_\alpha(y) \int_0^\infty Q_{\lambda,\alpha}(x, y, z) f(z) \, d\mu_\alpha(z) \\ = \int_0^\infty f(y) \, d\mu_\alpha(y) \int_0^\infty Q'_{\lambda,\alpha}(x, y, z) g(z) \, d\mu_\alpha(z). \end{aligned} \tag{2.45}$$

Applying (2.40) and (2.41) yields (2.44) for such f and g. The corollary now follows by a well-known argument from Theorem 2.1.

THEOREM 2.2. *Suppose* $\lambda > -\frac{1}{2}$ *and* $-1 \leqslant \alpha \leqslant 2\lambda + 1$. *Then both* H_λ *and* $H'_{\lambda,\alpha}$ *are of weak-type* (1, 1) *on* $L^1(\mu_\alpha)$.

PROOF. Arguing in much the same way as in Theorem 2.1 the proof for H_λ is readily reduced to establishing the assertion for the operator defined on $L^1(\mu_\alpha)$ by

$$\lim_{x\to 0+} y^{-\lambda}\int_{y/2}^{2y} \frac{(y-z)f(z)z^\lambda}{x^2+(y-z)^2}\,dz. \tag{2.46}$$

Keeping in mind the corresponding part of Lemma 2.3, the latter result may be obtained by an argument which closely parallels that of Theorem 4 of [8].

The proof for $H'_{\lambda,\alpha}$ is similar.

REMARK. As pointed out following Lemma 2.1, when $\lambda = -\frac{1}{2}$ the results hold for a wider range of α than the other cases suggest. Indeed, from Lemma 2.4 onwards the factor $2\lambda + 1$ (equal to 0) in the restrictions on α may be replaced by the factor 2. This follows from the improved estimates (2.13). These estimates give rise to the operators $T^{(1)}_{-1/2,\alpha}$ and $T^{(2)}_{-1/2,\alpha}$ with kernels

$$k^{(1)}_{-1/2,\alpha}(x) = x^{-2}\chi_{(2,\infty)}(x) \quad\text{and}\quad k^{(2)}_{-1/2,\alpha}(x) = x^{2-\alpha}\chi_{(0,1/2)}(x). \tag{2.47}$$

The boundedness properties of these operators are

(i) For $1 \leqslant p < \infty$, $T^{(1)}_{-1/2,\alpha} \in [L^p(\mu_\alpha)]$, provided $\alpha < 2p$; $T^{(1)}_{-1/2,\alpha}$ is of weak-type (1, 1) on $L^1(\mu_\alpha)$; $T^{(1)}_{-1/2,\alpha} \in [L^\infty(\mu_\alpha)]$ for all α.

(ii) For $1 \leqslant p < \infty$, $T^{(2)}_{-1/2,\alpha} \in [L^p(\mu_\alpha)]$ provided $\alpha < 2p/(p-1)$; $T^{(2)}_{-1/2,\alpha} \in [L^\infty(\mu_\alpha)]$ for $\alpha < 2$.

Using these together with Lemma 2.2, parts (iii), (iv), and (v), the assertion made concerning the case $\lambda = -\frac{1}{2}$ may be established.

3. H_λ on general rearrangement invariant spaces. Throughout this section we assume that either $\lambda > -\frac{1}{2}$ and $-1 \leqslant \alpha \leqslant 2\lambda + 1$ or $\lambda = -\frac{1}{2}$ and $-1 \leqslant \alpha \leqslant 2$.

The following result was first proved in O'Neil and Weiss [9] for the classical conjugate transformations.

LEMMA 3.1. *If* $f \in M(0, \infty)$ *and*

$$\int_0^\infty f^*(t)\sinh^{-1}(1/t)\,dt < \infty, \tag{3.1}$$

then

$$(H_\lambda f)^{**}(t) \leqslant c\int_0^\infty \frac{f^{**}(u)}{\sqrt{t^2+u^2}}\,du, \tag{3.2}$$

where c is a positive constant independent of f.

PROOF. Observe that, by Lemma 2.4 and the comments at the end of §2, H_λ and $H'_{\lambda,\alpha}$ are defined for all f satisfying (3.1). They are associate operators in the sense of (2.44). Morover, by the result of Rooney and Theorem 2.2 they are both of weak-types (1, 1) and (2, 2). Therefore, according to Calderón [5, p. 299], there is a positive constant c so that (3.2) holds for both H_λ and $H'_{\lambda,\alpha}$ when $f \in L^1(\mu_\alpha)$. This meansH_λ and $H'_{\lambda,\alpha}$ are bounded from a dense subset of the Lorentz space $\Lambda(\sinh^{-1}(1/t))$ into the μ_α-rearrangement invariant space L_{1oc} discussed in §4. Hence H_λ and $H'_{\lambda,\alpha}$ are bounded on all of $\Lambda(\sinh^{-1}(1/t))$. In particular, f_n converging to f in $\Lambda(\sinh^{-1}(1/t))$ implies $H_\lambda f_n$ converges to $H_\lambda f$ in measure on sets which are μ_α-finite. Calderón's method of proof then yields (3.2) for all f satisfying (3.1).

DEFINITION 3.1. The dilation operators E_s, $0 < s < \infty$, are given by

$$(E_s f)(t) = f(st), \qquad 0 < t < \infty, f \in M(0, \infty). \tag{3.3}$$

Let σ be an m-rearrangement invariant norm on $M(0, \infty)$. We denote by $h_\sigma(s)$ the norm of E_s as an element of $[L^\sigma]$. Two facts concerning $h_\sigma(s)$ will be needed:

(3.4)
(i) h_σ is nonincreasing on $(0, \infty)$.
(ii) $h_\sigma(st) \leqslant h_\sigma(s)h_\sigma(t)$ $(s > 0, t > 0)$.

For proofs see Boyd [3].

THEOREM 3.1. *Let ρ_1 and ρ_2 be μ_α-rearrangement invariant norms on $M(0, \infty)$, generated by σ_1 and σ_2, respectively. Then, in order that $H_\lambda \in [L^{\rho_1}, L^{\rho_2}]$ it is both necessary and sufficient that*

$$P + P' \in [L^{\sigma_1}, L^{\sigma_2}]. \tag{3.5}$$

PROOF. The sufficiency follows from Lemma 3.1 as in Boyd [3, Theorem 2.1].

To prove the necessity, observe that since $P + P'$ has a nonnegative kernel one need only show there exists an $A > 0$ such that

$$\sigma_2([P + P']f) \leqslant A\sigma_1(f), \tag{3.6}$$

for all nonnegative $f \in L^{\sigma_1}$. Indeed, it follows from the Fatou property of σ_1 and σ_2 that attention may be restricted to those f which also have compact support.

For n a positive integer define $g(z) = f(n - z)$ if $n - \sqrt{n} < z < n$ and $g(z) = 0$ otherwise.

We find, as in [1, Theorem 3.1], that if $H_\lambda \in [L^{\rho_1}, L^{\rho_2}]$ then there exists a positive constant c, independent of f, so that

$$(3.7)\qquad \sigma_2\big([(P+P')f]\chi_{[0,|\bar{n}]}\big) \leq \begin{cases} c\sigma_1\big(E_{n^{\alpha-1}}(g^*)\big), & \alpha > 1, \\ c\sigma_1\big(E_{(n+|\bar{n})^{\alpha-1}}(g^*)\big), & \alpha \leq 1, \end{cases}$$

for sufficiently large n. Similarly, we obtain

$$(3.8)\qquad \sigma_1(f) \geq \begin{cases} \sigma_1\big(E_{n^{\alpha-1}}(g^*)\big), & \alpha > 1, \\ \sigma_1\big(E_{(n-|\bar{n})^{\alpha-1}}(g^*)\big), & \alpha \leq 1. \end{cases}$$

The case $\alpha > 1$ then proceeds as in [1]. As for $\alpha \leq 1$, we have

$$(3.9)\qquad \begin{aligned} \sigma_1\big(E_{(n+|\bar{n})^{\alpha-1}}(g^*)\big) &\leq \sigma_1\big(E_{(2n)^{\alpha-1}}(g^*)\big) \\ &\leq h_{\sigma_1}(4^{\alpha-1})\sigma_1\big(E_{(n/2)^{\alpha-1}}(g^*)\big) \\ &\leq h_{\sigma_1}(4^{\alpha-1})\sigma_1\big(E_{(n-|\bar{n})^{\alpha-1}}(g^*)\big). \end{aligned}$$

Hence,

$$(3.10)\qquad \sigma_2\big([(P+P')f]\chi_{[0,|\bar{n}]}\big) \leq A\sigma_1(f),$$

where $A = ch_{\sigma_1}(4^{\alpha-1})$. The result now follows using the Fatou property of σ_2.

COROLLARY 3.1.1. *Let the notation and hypotheses be those of Theorem* 3.1, *where, in addition,* $\sigma_1 = \sigma_2 = \sigma$ *and, hence,* $\rho_1 = \rho_2 = \rho$. *Then, in order that* $H_\lambda \in [L^\rho]$ *it is both necessary and sufficient that*

$$(3.11)\qquad \lim_{s\to 0+} sh_\sigma(s) = \lim_{s\to\infty} h_\sigma(s) = 0.$$

PROOF. See Boyd [3, Lemmas 3.4 and 3.6].

REMARKS.

1. Condition (3.5) is necessary when $\lambda > -1$ and α is any real number.
2. Criterion (3.5) is self-dual. This means that if ρ_1 and ρ_2 are μ_α-rearrangement invariant norms, then $H_\lambda \in [L^{\rho_1}, L^{\rho_2}]$ if and only if $H_\lambda \in [L^{\rho_2'}, L^{\rho_1'}]$, where ρ_1' and ρ_2' are the norms associate to ρ_1 and ρ_2, respectively.

4. Applications.

THEOREM 4.1. *Let* σ_1 *and* σ_2 *be m-rearrangement invariant norms on* $M(0, \infty)$. *Then* $P \in [L^{\sigma_1}, L^{\sigma_2}]$ *implies* $L^{\sigma_1} \subset L^{\sigma_2}$.

PROOF. We show $f^\# \in L^{\sigma_1}$ implies $f^\# \in L^{\sigma_2}$. It is always true that

$$(4.1)\qquad f^\#(t) \leq (Pf^\#)(t), \qquad t > 0,$$

and, hence, that

$$(4.2)\qquad f^{\#\#}(t) \leq (Pf^\#)^{\#\#}(t), \qquad t > 0.$$

But, $Pf^\# \in L^{\sigma_2}$ and so $f^\# \in L^{\sigma_2}$.

COROLLARY 4.1.1. *Suppose that* $\lambda > -1$ *and that* α *is any real number. Let* ρ_1 *and* ρ_2 *be* μ_α*-rearrangement invariant norms. Then* $H_\lambda \in [L^{\rho_1}, L^{\rho_2}]$ *implies* $L^{\rho_1} \subset L^{\rho_2}$.

In the next two theorems $L^{\rho_1} = L^{\rho_2}$ as sets. This means there is no essential loss of generality in assuming $\rho_1 = \rho_2$, since the embedding of each in the other will be continuous. Here, as in the rest of this section, we assume, unless otherwise stated, that either $\lambda > -\frac{1}{2}$ and $-1 \leqslant \alpha \leqslant 2\lambda + 1$ or $\lambda = -\frac{1}{2}$ and $-1 < \alpha \leqslant 2$.

Suppose ϕ is a nonnegative, nonincreasing function on $(0, \infty)$ for which

$$\Phi(t) \equiv \int_0^t \phi(u)\, du < \infty, \qquad t > 0. \tag{4.3}$$

Let $\rho_p, p \geqslant 1$, be the μ_α-rearrangement invariant norm on $M(0, \infty)$ generated by σ_p, where

$$\sigma_p(f) = \left\{ \int_0^\infty \phi(t) \left[f^\#(t) \right]^p dt \right\}^{1/p} \tag{4.4}$$

for nonnegative f in $M(0, \infty)$. The μ_α-rearrangement invariant space $\Lambda(\phi, p)$ determined by ρ_p is a so-called Lorentz space. For $p = 1$ the notation $\Lambda(\phi)$ is also used. The space $M(\phi)$ associate to $\Lambda(\phi)$ has its norm generated by σ' given at nonnegative f in $M(0, \infty)$ by

$$\sigma'(f) = \sup_{u>0} \left\{ \int_0^u f^\#(t)\, dt / \Phi(u) \right\}. \tag{4.5}$$

THEOREM 4.2. *Suppose* $1 < p < \infty$. *Then* $H_\lambda \in [\Lambda(\phi, p)]$ *if and only if* $\Lambda(\phi, p)$ *is uniformly convex.*

PROOF. Follows from Corollary 3.1.1 together with the substance of Theorem 4.1 of [3].

For the definition of the Orlicz norms discussed below see [3, p. 611].

THEOREM 4.3. *Let* σ *be an Orlicz norm on* $M(0, \infty)$ *and let* ρ *be the* μ_α*-rearrangement invariant norm on* $M(0, \infty)$ *generated by* σ. *Then* $H_\lambda \in [L^\rho]$ *if and only if* L^ρ *is reflexive.*

PROOF. Follows from Corollary 3.1.1 and the substance of Theorem 5.8 of [3].

The following results explain the conditions given in Theorem 4.5 and Theorem 4.5′ for a given space to have a continuous partner with respect to $P + P'$.

THEOREM 4.4. *Let* σ_1 *be an m-rearrangement invariant norm on* $M(0, \infty)$. *If there exists another such norm* σ_2 *so that* $P + P' \in [L^{\sigma_1}, L^{\sigma_2}]$, *then* L^{σ_1} *is contained in the m-rearrangement invariant Lorentz space* $\Lambda(\sinh^{-1}(1/t))$.

PROOF. Shown as part of Theorem 3.1.

COROLLARY 4.4.1. *Suppose that* $\lambda > -1$ *and that* α *is any real number. Let* ρ_1 *be a* μ_α*-rearrangement invariant norm on* $M(0, \infty)$*. If there exists another such norm* ρ_2 *so that* $H_\lambda \in [L^{\rho_1}, L^{\rho_2}]$*, then* L^{ρ_1} *is contained in the* μ_α*-rearrangement invariant Lorentz space* $\Lambda(\sinh^{-1}(1/t))$.

PROOF. See the first remark following Corollary 3.1.1.

The duals of the above theorem and its corollary read:

THEOREM 4.4′. *Let* σ_2 *be an m-rearrangement invariant norm on* $M(0, \infty)$*. If there exists another such norm* σ_1 *so that* $P + P' \in [L^{\sigma_1}, L^{\sigma_2}]$*, then* L^{σ_2} *contains the m-rearrangement invariant Lorentz space* $M(\sinh^{-1}(1/t))$.

COROLLARY 4.4.1′. *Suppose that* $\lambda > -1$ *and that* α *is any real number. Let* ρ_2 *be a* μ_α*-rearrangement invariant norm on* $M(0, \infty)$*. If there exists another such norm* ρ_1 *so that* $H_\lambda \in [L^{\rho_1}, L^{\rho_2}]$*, then* L^{ρ_2} *contains the* μ_α*-rearrangement invariant Lorentz space* $M(\sinh^{-1}(1/t))$.

THEOREM 4.5. *Let* σ *be an m-rearrangement invariant norm on* $M(0, \infty)$ *with* $L^\sigma \supset M(\sinh^{-1}(1/t))$*. Then there exists another such norm* $\boldsymbol{\sigma}$ *so that* $P + P' \in [L^{\boldsymbol{\sigma}}, L^\sigma]$.

PROOF. Define $\boldsymbol{\sigma}$ for nonnegative $f \in M(0, \infty)$ by

$$\boldsymbol{\sigma}(f) = \sigma([P + P']f^{\#}). \tag{4.6}$$

In verifying that $\boldsymbol{\sigma}$ is an m-rearrangement invariant norm we refer to Definition 1.1 of [3]. To see (i) observe that for nonnegative $f, g \in M(0, \infty)$

$$\begin{aligned} \boldsymbol{\sigma}(f + g) &= \sigma\big([P + P'](f + g)^{\#}\big) \\ &= \sigma\big(P'(f + g)^{\#\#}\big) \leq \sigma\big(P'(f^{\#\#} + g^{\#\#})\big) \\ &\leq \sigma\big(P'f^{\#\#}\big) + \sigma\big(P'g^{\#\#}\big) = \boldsymbol{\sigma}(f) + \boldsymbol{\sigma}(g). \end{aligned} \tag{4.7}$$

Property (ii) follows from the Fatou property of σ since

$$f_n \uparrow f \Rightarrow f_n^{\#} \uparrow f^{\#} \text{ a.e.} \Rightarrow (P + P')f_n^{\#} \uparrow (P + P')f^{\#}. \tag{4.8}$$

If E is a Lebesgue measurable subset of $(0, \infty)$ with $m(E) < \infty$, then

$$\big[(P + P')\chi_{(0,m(E))}\big](t) \leq 2\big[S\chi_{(0,m(E))}\big](t) \leq 2\ln(1 + m(E)/t). \tag{4.9}$$

Applying L'Hôpital's rule we find there exists a positive M so that for all $u > 0$

$$\int_0^u \big[(P + P')\chi_{(0,m(E))}\big](t)\, dt \Big/ \int_0^u \sinh^{-1}(1/t)\, dt \leq M. \tag{4.10}$$

Hence $\boldsymbol{\sigma}(\chi_E) < \infty$, since $L^\sigma \supset M(\sinh^{-1}(1/t))$.

Finally, suppose E is a Lebesgue-measurable subset of $(0, \infty)$ with $m(E) <$

∞. Let $c > 0$ be such that

$$\int_E |g|\, dm \leqslant c\sigma(|g|) \tag{4.11}$$

for all $g \in L^{\boldsymbol{\sigma}}$. Now $\boldsymbol{\sigma}(|g|) < \infty$ implies $\sigma(|g|) < \infty$ with

$$\sigma(|g|) \leqslant \sigma\big((P + P')g^{\#}\big) = \boldsymbol{\sigma}(|g|). \tag{4.12}$$

From (4.11) we get

$$\int_E |g|\, dm \leqslant c\boldsymbol{\sigma}(|g|). \tag{4.13}$$

The very definition of $\boldsymbol{\sigma}$ ensures that $P + P' \in [L^{\boldsymbol{\sigma}}, L^{\sigma}]$.

COROLLARY 4.5.1. *Let ρ be a μ_α-rearrangement invariant norm on* $M(0, \infty)$, *generated by σ, with L^ρ containing the μ_α-rearrangement invariant space* $M(\sinh^{-1}(1/t))$. *Then there exists another such norm $\boldsymbol{\rho}$ so that $H_\lambda \in [L^{\boldsymbol{\rho}}, L^\rho]$.*

PROOF. Follows from Theorem 3.1.

The results dual to Theorem 4.5 and its corollary follow.

THEOREM 4.5′. *Let σ be an m-rearrangement invariant norm on* $M(0, \infty)$ *with $L^\sigma \subset \Lambda(\sinh^{-1}(1/t))$. Then there exists another such norm $\tilde{\sigma}$ so that $P + P' \in [L^\sigma, L^{\tilde{\sigma}}]$.*

PROOF. Letting a prime indicate the associate of a given norm, take $\tilde{\sigma} = (\boldsymbol{\sigma'})'$.

COROLLARY 4.5.1′. *Let ρ be a μ_α-rearrangement invariant norm on* $M(0, \infty)$ *with L^ρ contained in the μ_α-rearrangement invariant space* $\Lambda(\sinh^{-1}(1/t))$. *Then there exists another such norm $\tilde{\rho}$ so that $H_\lambda \in [L^\rho, L^{\tilde{\rho}}]$.*

REMARKS. 1. One may give $\tilde{\sigma}$, and hence $\tilde{\rho}$, a somewhat more explicit form than in the above results by using a construction similar to that in Bennett [2]. Thus, firstly, $\tilde{\sigma}^0$ is given at nonnegative $g \in M(0, \infty)$ by

$$\tilde{\sigma}^0(g) = \inf\Big\{\sigma(|f|) : g^{\#\#} \leqslant \big[(P + P')f^{\#}\big]^{\#\#}, f \in L^\sigma\Big\} \tag{4.14}$$

with the convention that $\tilde{\sigma}^0(g) = \infty$ if no such f exists. Then, $\tilde{\sigma}$ is defined at nonnegative $g \in M(0, \infty)$ by

$$\tilde{\sigma}(g) = \sup \tilde{\sigma}^0(g\chi_E), \tag{4.15}$$

the supremum being taken over all Lebesgue-measurable subsets E of $(0, \infty)$ with $m(E) < \infty$.

2. It is clear from the constructions of $\boldsymbol{\sigma}$ and $\tilde{\sigma}$ that, with respect to $P + P'$, $L^{\boldsymbol{\sigma}}$ is the largest domain space having L^σ as range, while $L^{\tilde{\sigma}}$ is the smallest range space having L^σ as domain. In particular, if $P + P' \in [L^\sigma]$, then

$L^\sigma = L^{\tilde{\sigma}} = L^\sigma$, the norms being equivalent. Similar remarks hold for H_λ relative to ρ and $\tilde{\rho}$.

As a final application of the criterion of Theorem 3.1 we obtain a natural analogue of a well-known result of Zygmund concerning the classical function operator C, and given in Theorem 2.8 of [13]. His result may be considered as saying that C is bounded from the Orlicz space $L_{M\Phi}([0, 2\pi])$ to $L^1([0, 2\pi])$, where the Young's function Φ is given by $\Phi(u) = u \log^+ u$. Now, for m on $[0, 2\pi]$, the space $L_{M\Phi}$ is the same, up to equivalence of norms, as the m-rearrangement invariant Lorentz space $\Lambda(\sinh^{-1}(1/t))$. It will be shown below, subject to our restrictions on λ and α, that H_λ is bounded from the μ_α-rearrangement invariant space $\Lambda(\sinh^{-1}(1/t))$ on $(0, \infty)$ to the space L_{loc} of functions in $M(0, \infty)$ which are locally μ_α-integrable. Observe that, by Corollaries 4.4.1 and 4.4.1′, L^1 cannot be a domain space or a range space of H_λ.

To begin, we make

DEFINITION 4.1. The norm $\sigma_{1,\infty}$ is given at nonnegative $f \in M(0, \infty)$ by

$$\sigma_{1,\infty}(f) = \|f\|_1 + \|f\|_\infty. \tag{4.16}$$

REMARK. One easily verifies that $\sigma_{1,\infty}$ is an m-rearrangement invariant norm. Moreover, if (X, M, μ) is any σ-finite measure space and if $\rho_{1,\infty}$ denotes the ν-rearrangement invariant norm generated by $\sigma_{1,\infty}$, then, as a set, the space determined by $\rho_{1,\infty}$ is just $L^1(\nu) \cap L^\infty(\nu)$. Since every ν-rearrangement invariant space is contained in $L^1(\nu) + L^\infty(\nu)$, duality considerations show that the norm $\rho_{1,\infty}$ determines the smallest such space. Finally, as a set, the space determined by the associate norm $\rho'_{1,\infty}$ is equal to $L^1(\nu) + L^\infty(\nu)$.(2)

THEOREM 4.6. *Let (X, M, ν) be a σ-finite, nonatomic measure space. The set $L^1(\nu) + L^\infty(\nu)$ then consists of the functions which are locally ν-integrable.*

PROOF. Clearly $f \in L^1(\nu) + L^\infty(\nu)$ implies f is locally ν-integrable. To prove the converse it will be sufficient to show that f locally ν-integrable implies $\int_0^a f^*(t)\,dt < \infty$ for some, and hence all, $a > 0$. For suppose this to be the case. Then $f^*(t) < \infty$ for all $t > 0$. Defining

$$f_1(t) = \begin{cases} f(t), & |f(t)| > f^*(1), \\ 0, & \text{otherwise, and} \end{cases} \tag{4.17}$$

$$f_2(t) = f(t) - f_1(t)$$

we get $f = f_1 + f_2$ with $f \in L^1(\nu), f_2 \in L^\infty(\nu)$.

Now, since ν is nonatomic,

(2) On p. 184 of Butzer and Berens, *Semi-groups of operators and approximation*, Die Grundlehren der math. Wissenschaften, Bd. 145, Springer-Verlag, New York, 1967, it is shown that an equivalent rearrangement invariant norm is given at ν-measurable f by $\int_0^1 f^*(t)\,dt$.

(4.18) $$\int_0^t f^*(u)\,du = \sup \int_E |f|\,d\nu,$$

the supremum being taken over all $E \in M$ with $\nu(E) \leq t$. If $\int_0^a f^*(t)\,dt$ were infinite for all $a > 0$, then to each positive integer n there would correspond a set $E_n \in M$ with $\nu(E_n) \leq 2^{-n}$ and $\int_{E_n} |f|\,d\nu > n$. Taking $E = \bigcup_{n=1}^{\infty} E_n$ we would obtain $\nu(E) \leq 1$ while $\int_E |f|\,d\nu = \infty$, thereby contradicting the local ν-integrability.

In view of Theorem 4.6 we denote the space determined by $\sigma'_{1,\infty}$ by L_{loc}; the one determined by $\sigma_{1,\infty}$ by L'_{loc}.

THEOREM 4.7. *Let Λ_s denote the m-rearrangement invariant Lorentz space $\Lambda(\sinh^{-1}(1/t))$; M_s the associate space $M(\sinh^{-1}(1/t))$. Then*

(4.19)
$$\text{(i)}\quad P + P' \in [\Lambda_s, L_{\text{loc}}].$$
$$\text{(ii)}\quad P + P' \in [L'_{\text{loc}}, M_s].$$

Moreover, up to equivalence of norms, Λ_s is the space associated with L_{loc} by Theorem 4.5; *M_s the space associated with L'_{loc} by Theorem* 4.5′.

PROOF. Theorem 4.5′ and the fact that every m-rearrangement invariant space is continuously embedded in L_{loc} ensure $P + P' \in [\Lambda_s, L_{\text{loc}}]$. By duality (ii) follows.

The remaining two assertions will be established once it has been shown $f \in M(0, \infty)$ and $(P + P)f^{\#} \in M_s$ imply $f \in L'_{\text{loc}}$.

Suppose, if possible, that $f \in M(0, \infty)$ and $(P + P')f^{\#} \in M_s$, but $f \notin L^1(m)$. Using Fubini's theorem we obtain

(4.20) $$\int_0^u [(P + P')f^{\#}](t)\,dt = \int_0^{\infty} [(P + P')\chi_{[0,u]}](t) f^{\#}(t)\,dt.$$

Then (4.9) with u in place of $m(E)$ shows the M_s norm of $(P + P')f^{\#}$ (possibly infinite) to be bounded below by the supremum over $u > 0$ of $\int_0^{\infty} f^{\#}(t) g_u(t)\,dt$ where

(4.21) $$g_u(t) = \tfrac{1}{2}\ln(1 + u/t) \Big/ \int_0^u \sinh^{-1}(1/t)\,dt.$$

An application of L'Hôpital's rule gives $\lim_{u\to\infty} g_u(t) = 1$ for all $t > 0$. Invoking Fatou's lemma reveals the M_s norm of $(P + P')f^{\#}$ larger than $\frac{1}{4}\int_0^{\infty} f^{\#}(t)\,dt = \infty$–a contradiction.

Suppose now, if possible, that $f \in M(0, \infty)$ and $(P + P')f^{\#} \in M_s$, but $f \notin L^{\infty}(m)$. Then given arbitrary $B > 0$ there exists $\varepsilon > 0$ such that $f^{\#}(t) \geq B$ when $0 < t \leq b$. As before we have the M_s norm of $(P + P')f^{\#}$ bounded below by the supremum over $u > 0$ of $\frac{1}{2}\int_u^b f^{\#}(t) g_u(t)\,dt$. But, for $t \geq u$, $\ln(1 + u/t) \leq u/t$ and so the M_s norm of $(P + P')f^{\#}$ will be no smaller than the limit as $u \to 0+$ of $\frac{1}{2} Bu \ln(b/u) / \int_0^u \sinh^{-1}(1/t)\,dt$, that is by $\frac{1}{2} B$.

Since $B > 0$ was arbitrary, a contradiction has been reached.

The analogue of Zygmund's result is readily obtained from Theorem 4.7 in view of Theorem 3.1.

The assertions of Corollaries 4.5.1 and 4.5.1′, as well as the result mentioned above, hold for many operators other than the H_λ-operators for which criterion (3.5) is valid. In particular, they hold for the Hilbert transformation H on R and the singular integral operators with Calderón-Zygmund kernels on R^n.

Observe, finally, that the pairs in (4.19) are optimal in the sense that the domains cannot be increased nor the ranges decreased.

5. **Further developments.** In [7] the study of the H_λ on general μ_α-rearrangement invariant spaces is completed for all possible values of λ and α. The methods used to investigate the H_λ may also be applied to ultraspherical conjugate transformations. This will be taken up in a future paper.

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