

A LAW OF THE ITERATED LOGARITHM FOR STABLE SUMMANDS

BY

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ABSTRACT. Let X_1, X_2, \dots be a sequence of independent identically distributed stable random variables with parameters α ($0 < \alpha < 2$) and β ($|\beta| < 1$). Let $S_n = \sum_{i=1}^n X_i$. Suppose that $(S_{1,n})$ and $(S_{2,n})$ are independent copies of the sequence (S_n) . In this paper we obtain the set of all limit points in the plane of the sequence

$$\left\{ |n^{-1/\alpha}(S_{1,n} - a_n)|^{1/(\log \log n)}, |n^{-1/\alpha}(S_{2,n} - a_n)|^{1/(\log \log n)} \right\}$$

where (a_n) is zero if $\alpha \neq 1$ and is $(2\beta n \log n)/\pi$ if $\alpha = 1$.

Introduction. Let X_1, X_2, \dots be a sequence of independent and identically distributed random variables with characteristic function $g(t)$ given by

$$g(t) = \exp\left\{-|t|^\alpha(1 + i\beta(t/|t|)w(t, \alpha))\right\}, \quad 0 < \alpha < 2,$$

where

$$w(t, \alpha) = \begin{cases} \tan(\pi\alpha/2) & \text{if } \alpha \neq 1, \\ (2\beta/\pi)n \log n & \text{if } \alpha = 1. \end{cases}$$

Write $S_n = \sum_{i=1}^n X_i$. Denote by $(S_{1,n})$ and $(S_{2,n})$ two independent copies of the sequence (S_n) and by (T_n) and (U_n) the sequences $\{n^{-1/\alpha}(S_{1,n} - a_n)\}$ and $\{n^{-1/\alpha}(S_{2,n} - a_n)\}$, respectively, where a_n is zero if $\alpha \neq 1$ and is $(2\beta n \log n)/\pi$ if $\alpha = 1$. Throughout the paper let θ_n stand for $(\log \log n)^{-1}$.

When $\beta = 0$ Chover [2] has established that $P(\limsup_{n \rightarrow \infty} |T_n|^{\theta_n} = e^{1/\alpha}) = 1$ and that almost every point in $[1, e^{1/\alpha}]$ is a limit point of $|T_n|^{\theta_n}$. Chover raises the question whether for this sequence points in $[0, 1)$ are also limit points. Using a result of Chung and Fuchs [3] he notes that when $\alpha > 1$, zero is a limit point. The results obtained below include a complete answer to Chover's question. To be precise we show that the set of all the limit points of $|T_n|^{\theta_n}$ coincides with the following intervals in the various cases: $[e^{-1/(1-\alpha)}, e^{1/\alpha}]$ if $\alpha < 1$, $|\beta| < 1$; $[1, e^{1/\alpha}]$ if $\alpha < 1$ and $\beta = \pm 1$ and $[0, e^{1/\alpha}]$ if $1 < \alpha < 2$. Independently of us J. L. Mijneer [7] has also obtained the limit sets in the cases $\alpha < 1$, $\beta = \pm 1$ and $1 < \alpha < 2$; in the case $\alpha < 1$, $|\beta| < 1$

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he states that he does not know whether $[e^{-1/(1-\alpha)}, e^{1/\alpha}]$ are all the limit points.

The object of this paper is to obtain the set K of all the limit points of the sequence $\xi_n = \{|T_n|^{\theta_n}, |U_n|^{\theta_n}\}$ for all values of α and β . This is presented in §4. The description of the set K is achieved from Lemmas 3 to 9 of §3. In §2, some planar sets and some special integer sequences are defined.

2. Notations and definitions. Define the following sets in the two dimensional plane:

$$K_1 = \{(e^{a/\alpha}, e^{b/\alpha}); 0 < a, b < 1, a + b < 1\},$$

$$K_2 = \{(e^{a/\alpha}, e^{-b}); 0 < a < 1, b > 0, a + (1 - \alpha)b < 1\},$$

$$K_3 = \{(e^{-a}, e^{b/\alpha}); a > 0, 0 < b < 1, b + (1 - \alpha)a < 1\},$$

$$K_4 = \{(e^{-a}, e^{-b}); a, b > 0, a + (1 - \alpha)b < 1\} \quad \text{if } 0 < \alpha < 1,$$

$$= \{(e^{-a}, e^{-b}); 0 < a < (2 - \alpha)^{-1}, b > 0, a + (1 - \alpha)b < 1\}$$

if $1 < \alpha < 2,$

$$K_5 = \{(e^{-a}, e^{-b}), a, b > 0, b + (1 - \alpha)a < 1\} \quad \text{if } 0 < \alpha < 1,$$

$$= \{(e^{-a}, e^{-b}), a > 0, 0 < b < (2 - \alpha)^{-1}, b + (1 - \alpha)a < 1\}$$

if $1 < \alpha < 2.$

Throughout the paper i.o. will stand for infinitely often: $[u]$ will stand for the largest integer $\leq u$; and $n_r (= n_r(a, b); a, b > 0)$ and m_r , respectively, will stand for the integer sequences $[\exp r^{(a+b)^{-1}}]$ and $[\exp r]$. Also $R, C (> 0)$ will denote absolute constants (R an integer), not necessarily the same at each occurrence.

We say that (x, y) is a limit point of (ξ_n) if for every $\delta > 0, P(\xi_n \in N_\delta \text{ i.o.}) = 1$ where $N_\delta = (x - \delta, x + \delta) \times (y - \delta, y + \delta)$.

3. LEMMA 1. Let (D_n) be a sequence of events in a probability space. If

$$(1) \quad (i) \quad \sum_{n=1}^{\infty} P(D_n) = \infty$$

and

$$(2) \quad (ii) \quad \liminf_{n \rightarrow \infty} \frac{\sum_{r=1}^n \sum_{s=1}^n P(D_r \cap D_s)}{(\sum_{r=1}^n P(D_r))^2} < C,$$

then $P(D_r \text{ i.o.}) \geq C^{-1}$.

For proof see [8, p. 317].

LEMMA 2. Let S_1, S_2, \dots, S_n be successive sums of independent random variables and let a_1, a_2, \dots, a_n be a sequence of real constants. Further, let $b (> 0)$ and α be constants such that

$$\sup_{m < j < n} P(|(S_n - a_n) - (S_j - a_j)| > b) = \alpha < 1;$$

then

$$(i) \quad P\left(\sup_{m < j < n} |S_j - a_j| > 2b\right) \leq (1 - \alpha)^{-1} P(|S_n - a_n| > b),$$

$$(ii) \quad P\left(\inf_{m < j < n} |S_j - a_j| < b\right) \leq (1 - \alpha)^{-1} P(|S_n - a_n| < 2b),$$

$$m = 1, 2, 3, \dots, n.$$

The proof follows on the lines of [1, Lemma 3.21, p. 45] and, hence, is omitted.

Below in Lemmas 3 to 9, we assume that $\alpha \neq 1$. When $\alpha = 1$, nonzero subtracting constants a_n 's are present. The lemmas are again true, but the proofs require slightly modified arguments and are omitted.

LEMMA 3. For all $a, b \geq 0$ with $a + b \geq 1$ and for every $\varepsilon > 0$,

$$P(|T_n|^{\theta_n} > e^{(a+\varepsilon)/\alpha}, |U_n|^{\theta_n} > e^{(b+\varepsilon)/\alpha} \text{ i.o.}) = 0.$$

PROOF. Define the events A_n, B_r and C_r through

$$A_n = (|T_n| > (\log n)^{(a+\varepsilon)/\alpha}, |U_n| > (\log n)^{(b+\varepsilon)/\alpha}),$$

$$B_r = (|n^{1/\alpha} T_n| > n_r^{1/\alpha} (\log n_r)^{(a+\varepsilon)/\alpha}, |n^{1/\alpha} U_n| > n_r^{1/\alpha} (\log n_r)^{(b+\varepsilon)/\alpha})$$

for at least one n in $n_r \leq n < n_{r+1}$,

$$C_r = \left(|n_{r+1}^{1/\alpha} T_{n_{r+1}}| > \frac{n_r^{1/\alpha}}{2} (\log n_r)^{(a+\varepsilon)/\alpha}, |n_{r+1}^{1/\alpha} U_{n_{r+1}}| > \frac{n_r^{1/\alpha}}{2} (\log n_r)^{(b+\varepsilon)/\alpha}\right).$$

Observe that

$$(i) \quad P(A_n \text{ i.o.}) \leq P(B_r \text{ i.o.}), \quad (ii) \quad P(C_r) \leq C r^{-(1+\varepsilon)}$$

for all $r \geq R$ and

$$\begin{aligned} & P\left(|n_{r+1}^{1/\alpha} T_{n_{r+1}} - n^{1/\alpha} T_n| > \frac{n_r^{1/\alpha}}{2} (\log n_r)^{(a+\varepsilon)/\alpha}\right) \\ (iii) \quad & < P\left(|(n_{r+1} - n)^{-1/\alpha} (n_{r+1}^{1/\alpha} T_{n_{r+1}} - n^{1/\alpha} T_n)| > \frac{n_r^{1/\alpha}}{2} (\log n_r)^{(a+\varepsilon)/\alpha}\right) \\ & < P\left(|X_1| > \frac{n_r^{1/\alpha}}{2} (n_{r+1} - n_r)^{-1/\alpha} (\log n_r)^{(a+\varepsilon)/\alpha}\right) < \frac{1}{2} \end{aligned}$$

for all $r > R$ and for all n in $n_r \leq n < n_{r+1}$. Similar to (iii) one may obtain for all $r > R$ and for all n in $n_r \leq n < n_{r+1}$,

$$(iv) \quad P\left(|n_{r+1}^{1/\alpha}U_{n_{r+1}} - n^{1/\alpha}U_n| > \frac{n_r^{1/\alpha}}{2}(\log n_r)^{(b+\epsilon)/\alpha}\right) \leq \frac{1}{2}.$$

Now from (iii), (iv) and Lemma 2, $P(B_r) \leq 4P(C_r)$ for all $r > R$, and hence, $\sum_{r=1}^{\infty} P(B_r) < \infty$. An appeal to the Borel-Cantelli lemma completes the proof.

LEMMA 4. For every $\epsilon > 0$, $P(|T_n|^{\theta_n} > e^{(1+\epsilon)/\alpha} \text{ i.o.}) = 0$.

PROOF. The proof is exactly similar to Lemma 3 by considering the events $(|T_n| > (\log n)^{(1+\epsilon)/\alpha})$, $(\sup_{m_r \leq n < m_{r+1}} |n^{1/\alpha}T_n| > m_r^{1/\alpha}(\log m_r)^{(1+\epsilon)/\alpha})$ and $(|m_{r+1}^{1/\alpha}T_{m_{r+1}}| > (m_r^{1/\alpha}/2)(\log m_r)^{(1+\epsilon)/\alpha})$, respectively, in place of A_n , B_r and C_r .

LEMMA 5. If $0 < \alpha < 1$ and if $\beta = \pm 1$, then for every $\epsilon > 0$, $P(|T_n|^{\theta_n} < e^{-\epsilon} \text{ i.o.}) = 0$.

PROOF. Define

$$V_n = (|T_n| < (\log n)^{-\epsilon})$$

and

$$W_r = \left(\inf_{m_r \leq n < m_{r+1}} |n^{1/\alpha}T_n| < m_{r+1}^{1/\alpha}(\log m_r)^{-\epsilon}\right).$$

Then

$$P(V_n \text{ i.o.}) \leq P(W_r \text{ i.o.}) = P(|m_r^{1/\alpha}T_{m_r}| < m_{r+1}^{1/\alpha}(\log m_r)^{-\epsilon} \text{ i.o.}).$$

Recall $e^{x-\epsilon}G_\alpha(x) \rightarrow 0$ as $x \rightarrow 0$ for the distribution function $G_\alpha(x)$ of a positive stable random variable [4, p. 424]. Therefore

$$P(|m_r^{1/\alpha}T_{m_r}| < m_{r+1}^{1/\alpha}(\log m_r)^{-\epsilon}) \leq c(\exp - r^{\epsilon/\alpha})$$

for all $r > R$. Hence $\sum_{r=1}^{\infty} P(W_r) < \infty$ and the lemma is proved.

LEMMA 6. For every $\epsilon > 0$, $P(|T_n|^{\theta_n} < e^{-(a+\epsilon)}, |U_n|^{\theta_n} < e^{-(b+\epsilon)} \text{ i.o.}) = 0$. Here a, b satisfy (i) $a, b > 0$ and (ii) $a + (1 - \alpha)b > 1$ and $b + (1 - \alpha)a > 1$.

PROOF. Define

$$A_n = (|T_n| < (\log n)^{-(a+\epsilon)}, |U_n| < (\log n)^{-(b+\epsilon)}),$$

$$B_r = (|n^{1/\alpha}T_n| < n_{r+1}^{1/\alpha}(\log n_r)^{-(a+\epsilon)}, |n^{1/\alpha}U_n| < n_{r+1}^{1/\alpha}(\log n)^{-(b+\epsilon)})$$

for at least one n in $n_r \leq n < n_{r+1}$

and

$$C_r = (|T_{n_{r+1}}| < 2(\log n_r)^{-(a+\epsilon)}, |U_{n_{r+1}}| < 2(\log n_r)^{-(b+\epsilon)}).$$

As in Lemma 3 we see that for all $r \geq R$ and for all n in $n_r < n < n_{r+1}$,

$$P\left(|n_r^{1/\alpha} T_{n_{r+1}} - n^{1/\alpha} T_n| > n_r^{1/\alpha} (\log n_r)^{-(a+\varepsilon)}\right) \leq \frac{1}{2}$$

and

$$P\left(|n_r^{1/\alpha} U_{n_{r+1}} - n^{1/\alpha} U_n| > n_r^{1/\alpha} (\log n_r)^{-(b+\varepsilon)}\right) \leq \frac{1}{2}.$$

In claiming these, use is made of the fact that

$$n_{r+1}^{1/\alpha} (n_{r+1} - n_r)^{-1/\alpha} (\log n_r)^{-(\lambda+\varepsilon)} \rightarrow \infty \quad \text{as } r \rightarrow \infty,$$

if $\lambda = a$ and $a + (1 - \alpha)b > 1$ or if $\lambda = b$ and $b + (1 - \alpha)a > 1$. Further, as the density of X_1 exists, $P(C_r) \leq Cr^{-(1+\varepsilon)}$ for all $r \geq R$. An application of Lemma 2 now implies that $\sum_{r=1}^{\infty} P(B_r) < \infty$. Hence by the Borel-Cantelli lemma and by the fact that $P(A_n \text{ i.o.}) \leq P(B_r \text{ i.o.})$, the proof is complete.

LEMMA 7. For any $\varepsilon > 0$ and for $0 < \alpha < 1$,

$$(a) \quad P(|T_n|^{\theta_n} > e^{(a+\varepsilon)/\alpha}, |U_n|^{\theta_n} < e^{-(b+\varepsilon)} \text{ i.o.}) = 0$$

when $a, b \geq 0$ satisfy $a + (1 - \alpha)b > 1$,

$$(b) \quad P(|T_n|^{\theta_n} < e^{-(a+\varepsilon)}, |U_n|^{\theta_n} > e^{(b+\varepsilon)/\alpha} \text{ i.o.}) = 0$$

when $a, b \geq 0$ satisfy $b + (1 - \alpha)a > 1$.

Proof is on the lines of Lemmas 3 and 6. The details are omitted.

LEMMA 8. For every $\varepsilon > 0$ and for all $a, b \geq 0$

$$(1.1) \quad (i) \quad P(|T_{n_r}|^{\theta_{n_r}} > e^{(a+\varepsilon)/\alpha}, |U_{n_r}|^{\theta_{n_r}} > e^{b/\alpha} \text{ i.o.}) = 0,$$

$$(1.2) \quad (ii) \quad P(|T_{n_r}|^{\theta_{n_r}} > e^{a/\alpha}, |U_{n_r}|^{\theta_{n_r}} > e^{(b+\varepsilon)/\alpha} \text{ i.o.}) = 0,$$

$$(2.1) \quad (iii) \quad P(|T_{n_r}|^{\theta_{n_r}} > e^{(a+\varepsilon)/\alpha}, |U_{n_r}|^{\theta_{n_r}} < e^{-b} \text{ i.o.}) = 0,$$

$$(2.2) \quad (iv) \quad P(|T_{n_r}|^{\theta_{n_r}} > e^{a/\alpha}, |U_{n_r}|^{\theta_{n_r}} < e^{-(b+\varepsilon)} \text{ i.o.}) = 0,$$

$$(3.1) \quad (v) \quad P(|T_{n_r}|^{\theta_{n_r}} < e^{-(a+\varepsilon)}, |U_{n_r}|^{\theta_{n_r}} > e^{b/\alpha} \text{ i.o.}) = 0,$$

$$(3.2) \quad (vi) \quad P(|T_{n_r}|^{\theta_{n_r}} < e^{-a}, |U_{n_r}|^{\theta_{n_r}} > e^{(b+\varepsilon)/\alpha} \text{ i.o.}) = 0,$$

$$(4.1) \quad (vii) \quad P(|T_{n_r}|^{\theta_{n_r}} < e^{-(a+\varepsilon)}, |U_{n_r}|^{\theta_{n_r}} < e^{-b} \text{ i.o.}) = 0,$$

$$(4.2) \quad (viii) \quad P(|T_{n_r}|^{\theta_{n_r}} < e^{-a}, |U_{n_r}|^{\theta_{n_r}} < e^{-(b+\varepsilon)} \text{ i.o.}) = 0.$$

PROOF. A direct application of the Borel-Cantelli lemma yields (i) through (viii).

Lemmas 3 and 7 above help us identify the points which are not limit points of (ξ_n) , while Lemma 8 identifies the points which are not the limit

points of the subsequence (ξ_{n_r}) for specific values of a, b of interest. Lemma 9 below helps identify the points which are the limit points of (ξ_{n_r}) . This identification is done by showing that with positive probability the sequence (ξ_{n_r}) lies i.o. in every neighbourhood of a point specified by a and b and by appealing to the Hewitt-Savage zero-one law.

LEMMA 9.

$$(1.3) \text{ (i) } P(|T_{n_r}|^{\theta_{n_r}} > e^{a/\alpha}, |U_{n_r}|^{\theta_{n_r}} > e^{b/\alpha} \text{ i.o.}) = 1$$

if $a, b > 0$ and $a + b < 1$.

$$(2.3) \text{ (ii) } P(|T_{n_r}|^{\theta_{n_r}} > e^{a/\alpha}, |U_{n_r}|^{\theta_{n_r}} < e^{-b} \text{ i.o.}) = 1$$

if $0 \leq a < 1, b > 0$ and $a + (1 - \alpha)b < 1$.

$$(3.3) \text{ (iii) } P(|T_{n_r}|^{\theta_{n_r}} < e^{-a}, |U_{n_r}|^{\theta_{n_r}} > e^{b/\alpha} \text{ i.o.}) = 1$$

if $a > 0, 0 < b < 1$ and $b + (1 - \alpha)a \leq 1$.

$$(4.3) \text{ (iv) } P(|T_{n_r}|^{\theta_{n_r}} < e^{-a}, |U_{n_r}|^{\theta_{n_r}} < e^{-b} \text{ i.o.}) = 1$$

*if $a, b > 0$ for $0 < \alpha < 1, 0 < a < (2 - \alpha)^{-1},$
 $b > 0$ for $1 < \alpha < 2$ and $a + (1 - \alpha)b < 1$.*

$$(5.3) \text{ (v) } P(|T_{n_r}|^{\theta_{n_r}} < e^{-a}, |U_{n_r}|^{\theta_{n_r}} < e^{-b} \text{ i.o.}) = 1$$

*if $a, b > 0$ for $0 < \alpha < 1, a > 0, 0 < b < (2 - \alpha)^{-1}$
for $1 < \alpha < 2$ and $b + (1 - \alpha)a < 1$.*

PROOF. We prove (4.3) below. Similar proofs for (1.3) and (2.3) can be given. Since T_{n_r} and U_{n_r} are interchangeable, (3.3) and (5.3) then follow by reasons of symmetry. The details are omitted.

Define

$$D_r = (|T_{n_r}| < (\log n_r)^{-a}, |U_{n_r}| < (\log n_r)^{-b}).$$

In view of our earlier remarks, to establish (4.3) it is sufficient to show that $P(D_r \text{ i.o.}) > 0$.

Observe that $\sum_{r=1}^n P(D_r) \sim c(\log n)$ and, consequently, as $n \rightarrow \infty$,

$$\left(\sum_{r=1}^n P(D_r) \right)^{-2} \sum_{r=1}^n \sum_{s=1}^n P(D_r \cap D_s)$$

is asymptotically

$$2 \left(\sum_{r=1}^n P(D_r) \right)^{-2} \sum_{r=1}^{n-1} \sum_{s=r+1}^n P(D_r \cap D_s).$$

Below we find a suitable upper bound for

$$\sum_{r=1}^{n-1} \sum_{s=r+1}^n P(D_r \cap D_s) \left(\sum_{r=1}^n P(D_r) \right)^{-2}.$$

If $(a + b) < 1$, then $n_r/n_s < e^{-1} + \delta < 1$ for some $\delta > 0$ and, hence,

$$\begin{aligned} P(|T_{n_r}| < (\log n_r)^{-a}, |T_{n_s}| < (\log n_s)^{-a}) \\ &< P \left[|T_{n_r}| < (\log n_r)^{-a}, \left| \frac{S_{1, n_s} - S_{1, n_r}}{(n_s - n_r)^{1/\alpha}} \right| < c(\log n_s)^{-a} \right] \\ &< P(|T_{n_r}| < (\log n_r)^{-a}) P(|X_1| < c(\log n_s)^{-a}) \quad \text{for all } r > R \\ &< cP(|T_{n_r}| < (\log n_r)^{-a}) P(|T_{n_s}| < (\log n_s)^{-a}) \quad \text{for all } r > R. \end{aligned}$$

Repeating the steps for (U_n) we can write $P(D_r \cap D_s) < cP(D_r)P(D_s)$ for $r > R$. Hence (1) and (2) readily hold.

Suppose now $(a + b) > 1$; two cases arise.

Case 1. $(a + b) > 1$ but $a < 1$.

Define $q = (a + b - 1)(a + b)^{-1}$ and $\varphi = \min(n, [r + r^q])$ and write

$$\sum_{r=1}^{n-1} \sum_{s=r+1}^n P(D_r \cap D_s) < \sigma_1 + \sigma_2$$

where

$$\sigma_1 = \sum_{r=1}^{n-1} \sum_{s=r+1}^{[r+r^q]} P(D_r \cap D_s) \quad \text{and} \quad \sigma_2 = \sum_{r=1}^{n-1} \sum_{s=\varphi}^n P(D_r \cap D_s).$$

For $(r + 1) < s < [r + r^q]$ and for all $r > R$,

$$P(|U_{n_r}| < (\log n_r)^{-b}, |U_{n_s}| < (\log n_s)^{-b}) < c(1 - n_r/n_s)^{-1/\alpha} (rs)^{-b/(a+b)}.$$

Hence

$$\begin{aligned} P(D_r \cap D_s) &\leq c(1 - n_r/n_s)^{-1/\alpha} (rs)^{-b/(a+b)} P(|T_{n_r}| < (\log n_r)^{-a}) \\ &\leq c(1 - n_r/n_s)^{-1/\alpha} r^{-1/\alpha} s^{-b/(a+b)}. \end{aligned}$$

Easy calculations lead to $(1 - n_r/n_s)^{-1/\alpha} \leq cr^{q/\alpha} (s - r)^{-1/\alpha}$. Thus

$$(3) \quad \sigma_1 \leq \text{Const} + c \sum_{r=1}^{n-1} r^{(q/\alpha - (a+2b)/(a+b))} \sum_{k=1}^{r^q} k^{-1/\alpha} \leq c(\log n).$$

When $s > [r + r^q]$ it is trivial to see that $n_r/n_s < e^{-1} + \delta < 1$ for some $\delta > 0$. Consequently for all $r > R$, $P(D_r \cap D_s) < cP(D_r)P(D_s)$. Since

$$(4) \quad \sum_{r=1}^R \sum_{s=\varphi}^n P(D_r \cap D_s) \leq R \sum_{s=1}^n P(D_s) \leq c(\log n),$$

$$\sigma_2 \leq c \log n + c \sum_{r=R}^n \sum_{s=1}^n P(D_r)P(D_s) \leq c(\log n)^2.$$

(3), (4) and Lemma 1 now guarantee that $P(D_r \text{ i.o.}) > 0$.

When $0 < \alpha < 1$, the proof of (4.3) is complete. When $\alpha > 1$, there are other values of a, b which are discussed in Case 2.

Case 2. $(a + b) > 1$ and $1 < a < (2 - \alpha)^{-1}$; $1 < \alpha < 2$. Define

$$p = (1 - a + (\alpha - 1)b) / (a + b)(\alpha - 1)$$

and note that $p > 0$. Take q and φ as given in Case 1 and observe that $p < q$. Now write

$$\sum_{r=1}^{n-1} \sum_{s=r+1}^n P(D_r \cap D_s) \leq \Delta_1 + \Delta_2 + \Delta_3$$

where

$$\Delta_1 = \sum_{r=1}^{n-1} \sum_{s=r+1}^{[r+r^p]} P(D_r \cap D_s),$$

$$\Delta_2 = \sum_{r=1}^{n-1} \sum_{s=[r+r^p]}^{[r+r^q]} P(D_r \cap D_s) \quad \text{and}$$

$$\Delta_3 = \sum_{r=1}^n \sum_{s=\varphi}^n P(D_r \cap D_s).$$

If $s \in (r, [r + r^p])$ and if $r \geq R$, then

$$P(D_r \cap D_s) \leq c(1 - n_r/n_s)^{-1/\alpha} r^{-1} s^{-b/(a+b)},$$

and proceeding as in Case 1 one gets

$$(5) \quad \Delta_1 \leq c(\log n).$$

If $s \in ([r + r^p], [r + r^q])$ and if $r \geq R$, then

$$P(|T_n| < (\log n_r)^{-a}, |T_n| < (\log n_s)^{-a}) \leq c(1 - n_r/n_s)^{-1/\alpha} (rs)^{-a/(a+b)}$$

and

$$P(|U_n| < (\log n_r)^{-b}, |U_n| < (\log n_s)^{-b})$$

$$\leq c(1 - n_r/n_s)^{-1/\alpha} (rs)^{-b/(a+b)}.$$

Hence

$$(6) \quad \Delta_2 \leq c \log n + c \sum_{r=R}^n r^{2(q-p-\alpha)/\alpha} \sum_{s=[r+r^p]}^{[r+r^q]} K^{-2/\alpha} \leq c(\log n).$$

For $s > \varphi$, by arguments similar to the ones given in Case 1 we get

$$(7) \quad \Delta_3 < c(\log n)^2.$$

From (5), (6) and (7) we conclude that a c exists such that (2) holds and, consequently, $P(D_r \text{ i.o.}) > 0$. The proof is now completed.

4. THEOREM. *The set K of all the limit points of the sequence (ξ_n) is given by*

$$K = \begin{cases} K_1 & \text{if } 0 < \alpha < 1 \text{ and } \beta = \pm 1, \\ \bigcup_{j=1}^5 K_j & \text{if } 0 < \alpha < 1, |\beta| < 1 \text{ or } 1 < \alpha < 2. \end{cases}$$

PROOF. When $0 < \alpha < 1$ and $|\beta| = 1$, Lemmas 3 and 5 establish that the limit points of (ξ_n) must be in $K (= k_1)$ only; (1.1), (1.2) and (1.3) establish that every point in K is a limit point.

That no point outside K is a limit point of (ξ_n) follows from Lemmas 3, 4, 6 and 7 in the case $0 < \alpha < 1$, $|\beta| < 1$ and from Lemmas 3, 4 and 6 in the case $1 < \alpha < 2$. In both cases, that every point in K_j is a limit point of (ξ_n) follows from (j.1), (j.2) and (j.3) for $j = 1, 2, 3, 4$, and from (4.1), (4.2) and (5.3) for $j = 5$.

REMARK 1. When ξ_n has $p (> 3)$ components the proof is on similar lines. We describe below the limit set $K_p(\alpha)$ in p dimensions.

Write

$$\xi_n(m) = (|T_{1,n}|^{\theta_n}, |T_{2,n}|^{\theta_n} \dots |T_{m,n}|^{\theta_n}),$$

$m = 1, 2, 3, \dots$, where $T_{j,n} = n^{-1/\alpha}(S_{j,n} - a_n)$ and $S_{j,n}$, $j = 1, 2, \dots, m$, are independent copies of S_n . Let $C_m(\alpha)$ be the m -dimensional cube $[0, e^{1/\alpha}]^m$.

For an arbitrary point $X_p = (x_1, \dots, x_p)$ in $C_p(\alpha)$ let $q = q(X_p)$ denote the number of components with values in $[0, 1)$; $r = r(X_p)$ be the minimum of these components if $q > 1$ and be equal to 1 if $q = 0$; $s = s(X_p)$ be the maximum of these components if $q > 1$ and be equal to 1 if $q = 0$.

Define the functions

$$A(x) = \begin{cases} 1 & \text{if } 0 < x < 1, \\ x^\alpha & \text{if } 1 \leq x < e^{1/\alpha}; \end{cases}$$

$$B(x) = \begin{cases} x^{-1} & \text{if } 0 < x < 1, \\ 1 & \text{if } 1 \leq x < e^{1/\alpha}. \end{cases}$$

Let $u = e^{-1} \prod_{j=1}^p A(x_j)$ and $v = \prod_{j=1}^p B(x_j)$.

For $p > 2$, a point $X_{p+1} = (x_1, x_2, \dots, x_{p+1})$ is a limit point of $\xi_n(p+1)$, i.e. $X_{p+1} \in K_{p+1}(\alpha)$, if and only if

- (i) $X_{p+1} \in C_{p+1}(\alpha)$,
- (ii) $X_p \in K_p(\alpha)$, $x_{p+1} \in K_1(\alpha)$, and
- (iii) either $0 < \alpha < 1$ and

$$0 < \min\{uvr^\alpha, (uv)^{1/(1-\alpha)}\} < x_{p+1} < 1, \text{ or}$$

$$0 < \alpha < 1 \text{ and } \max\{e^{-1}, uvr^\alpha\} \leq x_{p+1}^{-\alpha} \leq 1, \text{ or}$$

$$1 \leq \alpha < 2, q = 0 \text{ and } 0 \leq x_{p+1} < 1, \text{ or}$$

$$1 < \alpha < 2, q \geq 1, 0 \leq x_{p+1} < 1 \text{ and } \min(r, x_{p+1}) > u^{1/(q+1-\alpha)}, \text{ or}$$

$$1 < \alpha < 2 \text{ and } \max\{e^{-1}, us^{-(q-\alpha)}\} < x_{p+1}^{-\alpha} < 1.$$

REMARK 2. When the summands are standard normal variables, minor modifications of our steps establish that every point in the unit square $[0, 1]^2$ is a limit point of (ξ_n) . That no point outside this square is a limit point of (ξ_n) is a consequence of the classical law of the iterated logarithm [5]. When ξ_n has p (≥ 3) components, the set $K_p(2)$ of the limit points can, on similar lines, be proved to be $[0, 1]^p - [0, e^{-1/(p-2)}]^p$. Thus for $p > 3$, the limit set is not the cube. For a different sequence (ξ_n) , LePage [6] has obtained the limit set to be the unit ball in space of p -dimensions.

REMARK 3. One may consider component sequences $(\text{sign } T_n)|T_n|^{\theta_n}$ and arrive at all the limit points following closely our steps for the multidimensional situation. In case $\alpha > 1$ and $\beta = \pm 1$ use will have to be made of the fact that the appropriate tail of the distribution tends to zero exponentially fast.

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REFERENCES

1. L. Breiman, *Probability*, Addison-Wesley, Reading, Mass., 1968. MR 37 #4841.
2. J. Chover, *A law of the iterated logarithm for stable summands*, Proc. Amer. Math. Soc. 17 (1966), 441-443. MR 32 #6523.
3. K. L. Chung and W. H. J. Fuchs, *On the distribution of values of sums of random variables*, Mem. Amer. Math. Soc. No. 6 (1951), 12 pp. MR 12, 722.
4. W. Feller, *An introduction to probability theory and its applications*, Vol. 2, Wiley, New York, 1966. MR 35 #1048.
5. P. Hartman and A. Wintner, *On the law of the iterated logarithm*, Amer. J. Math. 63 (1941), 169-176. MR 2, 228.
6. R. D. LePage, *Log log law for gaussian processes*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 25 (1972/73), 103-108. MR 48 #9789.
7. J. L. Mijneer, *Sample path properties of stable processes*, Math. Centre Tracts, no. 59, Mathematische Centrum, Amsterdam, 1975. MR 51 #7008.
8. F. Spitzer, *Principles of random walk*, Van Nostrand, Princeton, N.J., 1964. MR 30 #1521.

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