TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 232, 1977

CLOSED CONVEX INVARIANT SUBSETS OF $L_p(G)$

BY

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ABSTRACT. Let G be a locally compact group. We characterize in this paper closed convex subsets K of $L_p(G)$, $1 < p < \infty$, that are invariant under all left or all right translations. We prove, among other things, that $K = \{0\}$ is the only nonempty compact (weakly compact) convex invariant subset of $L_p(G)$ ($L_1(G)$). We also characterize affine continuous mappings from $P_1(G)$ into a bounded closed invariant subset of $L_p(G)$ which commute with translations, where $P_1(G)$ denotes the set of nonnegative functions in $L_1(G)$ of norm one. Our results have a number of applications to multipliers from $L_q(G)$ into $L_p(G)$.

1. **Introduction.** Let G be a locally compact group with a fixed left Haar measure λ and modular function Δ defined by the identity

$$\Delta(a)\int k(xa)\,dx = \int k(x)\,dx$$

for continuous functions k vanishing off compact subsets of G. Left and right translations in $L_p(G)$, $1 \leq p \leq \infty$, by an element g in G are defined respectively by

$$(l_g f)(x) = f(gx) \quad \text{and} \quad (r_g f)(x) = \Delta^{1/p}(g) f(xg)$$

for each $x \in G$ (here $1/\infty = 0$). In this case, $l_a l_b = l_{ba}$ and $r_a r_b = r_{ab}$ for all $a, b \in G$. Furthermore, each l_a and r_a is a linear isometry on $L_p(G)$.

A subset K of $L_p(G)$ is called left (right) invariant if $l_g(K) \subseteq K$ ($r_g(K) \subseteq K$) for each $g \in G$.

In this paper we shall be concerned with closed convex left or right invariant subsets of $L_p(G)$. We prove, in §4, that if G is locally compact noncompact, and $1 < p < \infty$, then each closed convex left or right invariant subset K of $L_p(G)$ must contain the origin. Furthermore if K is assumed to be compact, then K must be trivial, i.e. K contains only the origin. We also show that $K = \{0\}$ is the only nonempty weakly compact convex left or right invariant subset of $L_1(G)$.

Received by the editors September 16, 1975 and, in revised form, December 25, 1975.

AMS (MOS) subject classifications (1970). Primary 43A15; Secondary 43A22.

Key words and phrases. Locally compact group, left translations, right translations, convex sets, multipliers, operators commuting with translations, L_p-spaces, measure algebra, convolution, multipliers, affine mappings.

(1) This research is supported by NRC Grant A-7679.

In §5 we study affine continuous mappings T from a closed convex left (right) invariant subset A of $L_q(G)$ into a closed convex invariant subset B of $L_p(G)$, where $1 < q < \infty$ and $1 < p < \infty$. We characterize all such maps when A is the set of probability measures contained in $L_1(G)$, and B is any closed bounded convex left (right) invariant subset of $L_p(G)$, $1 < p < \infty$.

Our results have a number of natural applications to various properties of multipliers from $L_q(G)$ into $L_p(G)$ as contained in Akemann [1], Brainerd and Edwards [2], Gaudry [4], Kitchen [10], Hörmander [8], Sakai [13] and Wendel [14].

2. **Some notations.** If K is a subset of a normed linear space E, then co K will denote the convex hull of K. If η is a topology on E, then the η-closure of K in E will be denoted by $K^{-\eta}$. In the event that η is the topology induced by the norm of E, then $K^{-\eta}$ will often be abbreviated as K^{-}; the η-closed convex hull of K will be denoted by $\overline{\text{co}}^{\eta}K$ (or by $\overline{\text{co}}\, K$ in case η is the norm topology).

Let G be a locally compact group with fixed left Haar measure λ, symbols like $\int \ldots dx$ will always denote integration with respect to λ. The spaces $L_p(G)$, $1 < p < \infty$, and $M(G)$ are defined exactly as in [6]. Let $C(G)$ be the space of bounded continuous complex-valued functions on G, and let $C_0(G)$ be the subspace of $C(G)$ consisting of all those functions that vanish at infinity. Given any function f on G, the function $\tilde{f}$ on G will be defined by $\tilde{f}(x) = f(x^{-1})$ for each $x \in G$. Convolutions of two functions f and g on G will be defined by

$$(f * g)(x) = \int f(y)\, g(y^{-1}x)\, dx$$

whenever it makes sense. If $\mu \in M(G)$, and $f \in L_p(G)$, $1 < p < \infty$, then the function $\mu * f$ in $L_p(G)$ is defined by

$$(\mu * f)(x) = \int f(y^{-1}x)\, d\mu(y).$$

Convolution of two measures in $M(G)$ is defined exactly as in [6, p. 266]. If $\mu \in M(G)$, and $f \in L_1(G)$, then

$$(f * \mu)(x) = \int \Delta(y^{-1}) f(xy^{-1})\, d\mu(y)$$

(see [6, Theorem 20.9]).

For each $a \in G$, let ε_a denote the measure in $M(G)$ such that $\varepsilon_a(A) = 1_A(a)$, where 1_A is the characteristic function on a subset A of G.

We shall frequently be concerned with the following subsets of $M(G)$:

$$P(G) = \{\mu \in M(G);\ \|\mu\| = 1 \text{ and } \mu > 0\},$$
$$P_1(G) = \{\phi \in L_1(G);\ \|\phi\|_1 = 1 \text{ and } \phi > 0\},$$
$$E(G) = \{\varepsilon_a;\ a \in G\}.$$

3. **Technical lemmas.** Given any locally compact group G, let τ denote the separated locally convex topology on $M(G)$ determined by the family of seminorms $Q = \{p_f;\ f \in C(G)\}$ where $|p_f(\mu)| = |\int f\, d\mu|$ for each $\mu \in M(G)$. Then τ is stronger than the weak*-topology on $M(G)$, i.e. the $\sigma(M(G), C_0(G))$ topology, but weaker than the weak topology on $M(G)$.

LEMMA 3.1. *For any locally compact group G, we have*

$$P(G) = P_1(G)^{-\tau} = \overline{\text{co}}^{\tau} E(G).$$

PROOF. It is clear that $P_1(G)^{-\tau} \subseteq P(G)$ and $\overline{\text{co}}^{\tau} E(G) \subseteq P(G)$. The other direction follows from the fact that $P_1(G)$ and co $E(G)$ are both weak*-dense in the set of positive linear functionals ϕ in $C(G)^*$ with norm one (see [5, p. 2] and [9, p. 92]).

LEMMA 3.2. *Let G be a locally compact group.*

(a) *For each $f \in L_p(G)$, $1 < p < \infty$, the map $\mu \to \mu * f$ from $M(G)$ into $L_p(G)$ is continuous when $M(G)$ has the weak*-topology and $L_p(G)$ has the weak topology.*

(b) *For each $f \in L_1(G)$, the maps $\mu \to \mu * f$ and $\mu \to f * \mu$ from $M(G)$ into $L_1(G)$ are continuous when $M(G)$ has the τ-topology and when $L_1(G)$ has the weak topology.*

(c) *For each $f \in L_\infty(G)$, the map $\mu \to \mu * f$ from $M(G)$ into $L_\infty(G)$ is continuous when $M(G)$ has the τ-topology and $L_\infty(G)$ has the weak*-topology.*

(d) *For each $\gamma \in M(G)$, the maps $\mu \to \mu * \gamma$ and $\mu \to \gamma * \mu$ from $(M(G)$, weak*) into $(M(G)$, weak*) are continuous.*

PROOF. (a) Let $\{\mu_\alpha\}$ be a net in $M(G)$ converging to some μ_0 in $M(G)$ in the weak*-topology. Let $h \in L_{p'}(G)$ where $1/p + 1/p' = 1$. Then $\langle \mu_\alpha * f, h\rangle = \langle h * \tilde{f}, \mu_\alpha\rangle$ which converges to $\langle h * \tilde{f}, \mu_0\rangle = \langle \mu_0 * f, h\rangle$ since $h * \tilde{f} \in C_0(G)$ (see [6, p. 215]).

The proof of (b) and (c) are very similar.

(d) That $\mu \to \mu * \gamma$ is continuous from $(M(G)$, weak*) into $(M(G)$, weak*) follows immediately from the definition of convolution of two measures as defined in [6, p. 266].

To see that $\mu \to \gamma * \mu$ is continuous from $(M(G)$, weak*) into $(M(G)$, weak*), it is sufficient to show that the map is continuous from $(M(G), \mathfrak{T})$ into $(M(G)$, weak*) where $\mathfrak{T}$ is the Mackey topology on $M(G)$ for the pair $(M(G), C_0(G))$. Let $\{\mu_\alpha\}$ be a net in $M(G)$ converging to some μ_0 in the $\mathfrak{T}$-topology, then for each $f \in C_0(G)$, $\sup\{|\mu_\alpha(l_a f) - \mu_0(l_a f)|;\ a \in G\} \to 0$, since $\{l_a f;\ a \in G\}$ is relatively compact in the weak topology of $C_0(G)$ (see [5, p. 38]). Consequently, $\gamma * \mu_\alpha(f) \to \gamma * \mu_0(f)$.

For each $1 \leq p \leq \infty$, let Π_p denote the linear isometry from $L_p(G)$ onto $L_p(G)$ defined by

$$\Pi_p(f)(t) = \left(1/\Delta^{1/p}(t)\right)\tilde{f}(t)$$

for each $t \in G$, and each $f \in L_p(G)$. We gather in the next lemma a few basic properties of Π_p that we shall use. Since their proofs are routine, we omit the details.

LEMMA 3.3. *Let* $1 \leq p \leq \infty$.

(a) *Each* Π_p *is a linear isometry from* $L_p(G)$ *onto* $L_p(G)$. *Furthermore,* Π_∞ *is also weak*-weak* continuous.*

(b) *For each* $f \in L_p(G)$, *and each* $x \in G$, *we have*

(i) $l_x\Pi_p(f) = \Pi_p(r_{x^{-1}}f)$,

(ii) $r_x\Pi_p(f) = \Pi_p(l_{x^{-1}}f)$,

(iii) $l_x\Pi_p^{-1}(f) = \Pi_p^{-1}(r_{x^{-1}}f)$,

(iv) $r_x\Pi_p^{-1}(f) = \Pi_p^{-1}(l_{x^{-1}}f)$.

(c) *For each* $h \in L_1(G)$ *and each* $f \in L_p(G)$, *we have*

(i) $h * \Pi_p(f) = \Pi_p(f * \tilde{h}/\Delta^{1/p})$,

(ii) $h * \Pi_p^{-1}(f) = \Pi_p^{-1}(f * \tilde{h}/\Delta^{1/p})$.

(d) *For each* $x \in G$ *and each* $f \in L_p(G)$, *we have*

$$\varepsilon_a * \Pi_p(f) = \Pi_p(r_a f).$$

4. **Invariant subsets of $L_p(G)$.** It is well known that a closed linear subset I of $L_1(G)$ is a left (resp. right) ideal of $L_1(G)$ if and only if I is left (resp. right) invariant (see [12, p. 125]). Our first result is a generalization of this fact to closed convex subsets of $L_p(G)$.

THEOREM 4.1. *Let G be a locally compact group.*

(a) *If K is a closed convex subset of* $L_p(G)$, $1 \leq p < \infty$, *then K is left (resp. right) invariant if and only if* $\phi * K \subseteq K$ (*resp.* $K * \tilde{\phi}/\Delta^{1/p} \subseteq K$) *for each* $\phi \in P_1(G)$.

(b) *If K is a weak*-closed convex subset of* $L_\infty(G)$, *then K is left (resp. right) invariant if and only if* $\phi * K \subseteq K$ (*resp.* $K * \tilde{\phi} \subseteq K$) *for each* $\phi \in P_1(G)$.

PROOF. (a) Assume that K is left invariant, and $\phi \in P_1(G)$. Let

$$\phi_\alpha = \sum_{i=1}^{n} \lambda_i \varepsilon_{g_i}$$

be a net in co $E(G)$ such that ϕ_α converges to ϕ in the τ-topology. Since $\phi_\alpha * f = \sum_{i=1}^n \lambda_i l_{g_i^{-1}} f$, and K is closed convex and left invariant, it follows immediately from Lemma 3.2 that $\phi * f \in K$.

Conversely, if $\phi * f \in K$ for each $\phi \in P_1(G)$, and $x \in G$, let $\{\phi_\alpha\}$ be a net in $P_1(G)$ such that ϕ_α converges to $\varepsilon_{x^{-1}}$ in the τ-topology (Lemma 3.1). Since $\phi_\alpha * f \in K$ for each α, and K is closed, by Lemma 3.2 again, $\varepsilon_{x^{-1}} * f = l_x f \in K$.

Assume that K is right invariant. By Lemma 3.3(b), $\Pi_p(K)$ is left invariant. Hence $\phi * \Pi_p(K) \subseteq \Pi_p(K)$ for each $\phi \in P_1(G)$. Using Lemma 3.3(c), we have

$$\Pi_p^{-1}[\phi * \Pi_p(f)] = f * \tilde{\phi}/\Delta^{1/p},$$

which is in K for each $\phi \in P_1(G)$. Proof of the other direction is similar.

Assertion (b) can be proved in exactly the same way.

Corollary 4.2. (a) *Let* $f \in L_p(G)$, $1 \leqslant p < \infty$; *then*

(i) $\overline{\text{co}}\{l_x f;\ x \in G\} = \{\phi * f;\ \phi \in P_1(G)\}^-$ *and*

(ii) $\overline{\text{co}}\{r_x f;\ x \in G\} = \{f * \tilde{\phi}/\Delta^{1/p};\ \phi \in P_1(G)\}^-$.

(b) (*Wong* [15, *p*. 42] *and* [16, *Lemma* 6.3]). *Let* $f \in L_\infty(G)$; *then*

(i) $\overline{\text{co}}^{w^*}\{l_x f;\ x \in G\} = \{\phi * f;\ \phi \in P_1(G)\}^{-w^*}$,

(ii) $\overline{\text{co}}^{w^*}\{r_x f;\ x \in G\} = \{f * \tilde{\phi};\ \phi \in P_1(G)\}^{-w^*}$,

where w^* *denotes the weak*-topology on* $L_\infty(G)$.

Proof. (a) Let $K_1 = \overline{\text{co}}\{l_x f;\ x \in G\}$ and $K_2 = \overline{\text{co}}\{\phi * f;\ \phi \in P_1(G)\}$. Then $l_x(K_1) \subseteq K_1$ for each $x \in G$. Hence by Theorem 4.1, $\phi * K_1 \subseteq K_1$ for each $\phi \in P_1(G)$. In particular, $K_2 \subseteq K_1$. Conversely, the set K_2 also contains f. Indeed, if ϕ_α is a net in $P_1(G)$ converging to ε_e, where e is the identity of G, then $\phi_\alpha * f$ converges to f in the weak topology (Lemma 3.2). Hence $f \in K_2$. Since $\phi * K_2 \subseteq K_2$ for each $\phi \in P_1(G)$, it follows that $l_x(K_2) \subseteq K_2$ for each $x \in G$. Hence $l_x f \in K_2$ for each $x \in G$. This proves part (i) of (a). To prove (ii), consider the identities

$$\Pi_p \overline{\text{co}}\,\{r_x f;\ x \in G\} = \overline{\text{co}}\,\{l_x \Pi_p f;\ x \in G\} = \{\phi * \Pi_p f;\ \phi \in P_1(G)\}^-$$

which is valid by Lemma 3.3 and part (i). Consequently using Lemma 3.3(c), we have

$$\begin{aligned}\overline{\text{co}}\,\{r_x f;\ x \in G\} &= \Pi_p^{-1}\left(\{\phi * \Pi_p f;\ \phi \in P_1(G)\}^-\right)\\ &= \{f * \tilde{\phi}/\Delta^{1/p};\ \phi \in P_1(G)\}^-.\end{aligned}$$

The proof of (b) is similar.

For $p = 1$, the following result is proved by Kitchen [10] for compact abelian groups, and by Akemann [1] and Gaudry [4] by arbitrary compact groups.

Corollary 4.3. *If G is a compact group, then for each* $f \in L_p(G)$, $1 \leqslant p < \infty$, *the operator* $h \to h * f$ ($h \to f * \tilde{h}$) *from* $L_1(G)$ *into* $L_p(G)$ *is compact.*

Proof. If $h \in L_1(G)$ and $\|h\|_1 \leqslant 1$, then $h = (h_1 - h_2) + i(h_3 - h_4)$ where each h_i is positive, and $\|h_i\| \leq 1$. Hence if $K = \{\lambda k;\ 0 \leq \lambda \leq 1$, and $k \in K_2\}$ where $K_2 = \{\phi * f;\ \phi \in P_1(G)\}$, then $h * f \in (K - K) + i(K - K)$ whenever $\|h\|_1 \leq 1$. If G is compact, then K is compact. Hence the map $h \to h * f$ is compact. That $h \to f * \tilde{h}$ is compact can be proved similarly.

Corollary 4.3 is false for $p = \infty$ unless we assume $f \in C(G)$. But we have

COROLLARY 4.4. *Let G be any locally compact group. If $f \in L_\infty(G)$ such that $\{l_x f;\ x \in G\}$ is relatively compact in the norm topology (weak topology) of $L_\infty(G)$, then the maps $h \to h * f$ and $h \to f * \tilde{h}$ from $L_1(G)$ into $L_\infty(G)$ are compact (weakly compact) linear operators.*

PROOF. If $\{l_x f;\ x \in G\}$ is relatively compact in the weak topology of $L_\infty(G)$, then the set $K_1 = \overline{\text{co}}\{l_x f;\ x \in G\}$ is a weakly compact subset of $L_\infty(G)$. Since the weak*-topology is Hausdorff on K_1, it follows that the weak*- and weak topologies agree on K_1. Consequently,

$$K_1 = \overline{\text{co}}^{w^*}\{l_x f;\, x \in G\} = \{\phi * f;\, \phi \in P_1(G)\}^{-w^*} = \{\phi * f;\, \phi \in P_1(G)\}^{-}.$$

An argument similar to the proof of Corollary 4.3 shows that the map $h \to h * f$ from $L_1(G)$ into $L_\infty(G)$ is weakly compact. Also, if $\{l_x f;\ x \in G\}$ is relatively compact in the weak topology of $L_\infty(G)$, then $\{r_x f;\ x \in G\}$ is relatively compact in the weak topology of $L_\infty(G)$. Hence the map $h \leq f * \tilde{h}$ from $L_1(G)$ into $L_\infty(G)$ is also weakly compact.

The proof for the norm compact case is similar.

THEOREM 4.5. *Let G be any locally compact group and $1 < p < \infty$. Then G is noncompact if and only if each closed convex left or right invariant nonempty subset K of $L_p(G)$ contains the origin.*

PROOF. If G is compact, then the set $K = \{1\}$, where 1 is the one function on G, is a closed, convex and invariant subset of $L_p(G)$. Conversely, assume that G is not compact and K is left invariant. For each compact subset $\alpha \subset G$, choose a_α such that $a_\alpha \not\subseteq \alpha$. Then the net $\{\varepsilon_{a_\alpha}\}$ converges to 0 in the weak*-topology of $M(G)$. Hence $\varepsilon_{a_\alpha} * f$ converges to 0 in the weak topology of $L_p(G)$ for each $f \in K$ by Lemma 3.1. Consequently, $0 \in K$.

If K is right invariant, then the set $\Pi_p(K)$ is left invariant by Lemma 3.3. Hence $0 \in \Pi_p(K)$, which implies $0 \in K$.

THEOREM 4.6. *Let G be any locally compact group and $1 < p < \infty$. Then G is noncompact if and only if any compact convex left or right invariant nonempty subset K of $L_p(G)$ consists only of the origin.*

PROOF. If G is compact, let $K = \{1\}$. Conversely, if G is not compact, K is left invariant and $f \in K$ such that $f \neq 0$, let $k \in L_{p'}(G)$, where $1/p + 1/p' = 1$, and consider the function h on G defined by $h(g) = \langle l_g f, k\rangle$. Then h is a bounded continuous almost periodic function on G (use Lemma 4.8 and Theorem 5.1 in [3]). Furthermore, $h = \tilde{f} * k$, which is in $C_0(G)$. Consequently, $h \equiv 0$ (see [3, p. 82]). Hence $\tilde{f} * k = 0$ for each $k \in L_{p'}(G)$. Let $\{V_\alpha\}$ be a decreasing set of compact neighbourhoods of e and let $k_\alpha = 1_{\tilde{V}_\alpha}/|V_\alpha|$; then $\|\tilde{f} * k_\alpha - \tilde{f}\|_p \to 0$. Since $k_\alpha \in L_{p'}(G)$ for each α, it follows that $\tilde{f} = 0$. Hence $f = 0$.

If K is right invariant, then $\Pi_p(K)$ is left invariant weakly compact and convex by Lemma 3.3. Hence $\Pi_p(K) = \{0\}$. So $K = \{0\}$.

Sakai proved implicitly [13, Theorem 1] that if T is any weakly compact linear operator from $L_1(G)$ into $L_1(G)$ commuting with right translations, and G is noncompact, then $T = 0$. The following is an analogue of Sakai's result for $L_p(G)$, $1 < p < \infty$:

COROLLARY 4.7. *Let G be any locally compact noncompact group, $1 < q < \infty$ and $1 < p < \infty$. If T is any compact linear operator from $L_q(G)$ into $L_p(G)$ commuting with either all left translations or all right translations, then $T = 0$.*

PROOF. Let $K = \{T(f);\ \|f\|_q \leq 1\}^-$. If T commutes with all left translations, then K is a compact convex left invariant subset of $L_p(G)$. By Theorem 4.6, $K = \{0\}$. Hence $T = 0$.

When $p = 1$, we have an even stronger result:

THEOREM 4.8. *Let G be any locally compact group. Then G is noncompact, if and only if every weakly compact convex left or right invariant nonempty subset K of $L_1(G)$ consists of the origin only.*

PROOF. As before, if G is compact, let $K = \{1\}$. Conversely if G is not compact, K is right invariant, and $f \in K$, then a similar proof as that of Corollary 4.3 shows that the map T: $h \to h * \tilde{f}$ is a weakly compact operator from $L_1(G)$ into $L_1(G)$. Furthermore T commutes with all right translations. Hence by the proof of Sakai [13, Theorem 1], $T = 0$. Hence $\tilde{f} = 0$, which implies $f = 0$.

If K is left invariant, then $\Pi_1(K)$ is also weakly compact and right invariant. Hence $\Pi_1(K) = \{0\}$ i.e. $K = \{0\}$.

Let $m \in L_\infty(G)^*$. Following Wong [16, p. 355], let m_L: $L_\infty(G) \to L_\infty(G)$ be the topological left introversion operator of m defined by $m_L(f)(\phi) = m(\tilde{\phi}/\Delta * f)$ for each $f \in L_\infty(G)$ and each $\phi \in L_1(G)$. Similarly, the topological right introversion m_R: $L_\infty(G) \to L_\infty(G)$ is defined by $m_R(f)(\phi) = m(f * \tilde{\phi})$ for each $f \in L_\infty(G)$ and each $\phi \in L_1(G)$. A mean on $L_\infty(G)$ is a positive linear functional on $L_\infty(G)$ of norm one. The following proposition shows that a left (right) introverted subspace X of $L_\infty(G)$, i.e. $m_L(X) \subseteq X$ ($m_R(X) \subseteq X$) for each mean m on $L_\infty(G)$, as considered by Wong [16, p. 356], can be characterized in terms of certain invariance by right (left) translation operators.

PROPOSITION 4.9. *Let X be a nonempty subset of $L_\infty(G)$. Then*

(a) *$m_L(X) \subseteq X$ for each mean m on $L_\infty(G)$ if and only if $\overline{\text{co}}^{w^*}\{r_x f; x \in G\} \subseteq X$ for each $f \in X$.*

(b) *$m_R(X) \subseteq X$ for each mean m on $L_\infty(G)$ if and only if $\overline{\text{co}}^{w^*}\{l_x f; x \in G\} \subseteq X$ for each $f \in X$.*

In particular, if X is weak-closed and convex, then $m_L(X) \subseteq X$ ($m_R(X) \subseteq X$) for each mean m if and only if X is right (left) invariant.*

PROOF. (a) Assume that $m_L(X) \subseteq X$ for each mean m. Let $f \in X$. By [16, Lemma 6.3], it is sufficient to show that $\overline{\text{co}}^{w^*}\{f * \tilde{\phi}; \phi \in P_1(G)\} \subseteq X$ for each $f \in X$. Indeed, let $f \in X$, and let $\{\phi_\alpha\}$ be a net in $P_1(G)$ such that the net $\{f * \tilde{\phi}_\alpha\}$ converges to some h in $L_\infty(G)$ in the weak*-topology. By passing to a subnet if necessary, we may assume that the net $\{\phi_\alpha\}$ converges in the weak*-topology of $L_\infty(G)^*$ to some mean m. Then for each $\phi \in L_1(G)$, we have

$$m_L(f)(\phi) = m(\tilde{\phi}/\Delta * f) = \lim_\alpha \langle \tilde{\phi}/\Delta * f, \phi_\alpha \rangle$$
$$= \lim_\alpha \langle f * \tilde{\phi}_\alpha, \phi \rangle = \langle h, \phi \rangle$$

using Lemma 3.1(c) in [16]. Hence $m_L(f) = h$ is in X. The converse can be proved by approximating each mean m by a net $\{\phi_\alpha\}$ in $P_1(G)$ in the weak*-topology [9, p. 92].

The proof of (b) is similar.

5. **Affine mappings which commute with translations.** Bounded linear operators T from $L_q(G)$ into $L_p(G)$ that commute with left (or right) translations have been studied extensively by various authors (see [11] and [7, §35] for references). In this section we shall study affine mappings T from a left (right) invariant closed convex subset A of $L_q(G)$ into a closed convex left (right) invariant subset of $L_p(G)$ commuting with left (right) translations, i.e. $l_x(Tf) = T(l_x f)$ ($r_x(Tf) = T(r_x f)$) for each $x \in G$ and each $f \in A$.

THEOREM 5.1. *Let A and B be closed left (right) invariant convex subsets of $L_q(G)$ and $L_p(G)$, respectively, where $1 \leq q < \infty$, and $1 \leq p < \infty$. Let T be an affine continuous mapping from A into B. The following are equivalent:*

(a) *T commutes with left (right) translations.*

(b)

$$T(\phi * f) = \phi * T(f) \qquad \left(T(f * \tilde{\phi}/\Delta^{1/q}) = T(f) * \tilde{\phi}/\Delta^{1/p}\right)$$

for each $\phi \in P_1(G)$ and $f \in A$.

PROOF. (a) $\Rightarrow$ (b): Assume that T commutes with all left translations and $\phi \in P_1(G)$. Let $\phi_\alpha = \sum_{i=1}^{n_\alpha} \lambda_i^\alpha \varepsilon_{a_i^\alpha}$ be a net in co $E(G)$ converging to ϕ in the τ-topology. Then $\phi_\alpha * f$ converges to $\phi * f$ in the weak topology for each

$f \in A$. Since A is closed and convex and T is affine, T is also continuous when A and B have the respective weak topologies. Let b_i^α be the inverse of a_i^α. Then

$$T(\phi * f) = \lim_\alpha T(\phi_\alpha * f) = \lim_\alpha T\left(\sum_{i=1}^{n_\alpha} \lambda_i^\alpha l_{b_i^\alpha} f\right)$$

$$= \lim_\alpha \sum_{i=1}^{n_\alpha} \lambda_i^\alpha l_{b_i^\alpha} T(f) = \lim_\alpha \phi_\alpha * T(f) = \phi * T(f),$$

using Lemma 3.2.

Now assume that A is right invariant, and T commutes with all right translations. With ϕ, ϕ_α and b_i^α chosen as before, then repeated application of Lemmas 3.2 and 3.3 yields

$$T(f * \tilde{\phi}/\Delta^{1/q}) = T(\Pi_q^{-1}[\phi * \Pi_q(f)]) = \lim_\alpha T(\Pi_q^{-1}[\phi_\alpha * \Pi_q(f)])$$

$$= \lim_\alpha T\left(\sum_{i=1}^{n_\alpha} \lambda_i^\alpha r_{b_i^\alpha} f\right) = \lim_\alpha \sum_{i=1}^{n_\alpha} \lambda_i r_{b_i^\alpha} T(f) = \lim_\alpha \Pi_p^{-1}[\phi_\alpha * \Pi_p(T(f))]$$

$$= \Pi_p^{-1}[\phi * \Pi_p(T(f))] = T(f) * \frac{1}{\Delta^{1/p}} \tilde{\phi}$$

for each $\phi \in P_1(G), f \in A$.

(b) $\Rightarrow$ (a). Let $a \in G$, and $\{\phi_\alpha\}$ be a net in $P_1(G)$ such that ϕ_α converges to $\varepsilon_{a^{-1}}$ in the τ-topology. If $T(f * \phi) = T(f) * \phi$ for each $\phi \in P_1(G)$ and each $f \in A$, then

$$T(l_a f) = T(\varepsilon_{a^{-1}} * f) = \lim_\alpha T(\phi_\alpha * f) = \lim_\alpha \phi_\alpha * T(f)$$

$$= \varepsilon_{a^{-1}} * T(f) = l_a T(f).$$

The remaining case can be proved similarly using the operators Π_p and Π_q as in (a) $\Rightarrow$ (b).

A simple modification of the proof of Theorem 5.1 proves

THEOREM 5.2. *Let A be a closed left (right) invariant convex subset of $L_q(G)$, $1 \leq q < \infty$ and B be a weak*-closed left (right) invariant convex subset of $L_\infty(G)$. Let T be an affine continuous map from A into B. The following are equivalent*:

(a) *T commutes with left (right) translations,*

(b)

$$T(\phi * f) = \phi * T(f) \qquad \left(T(f * \tilde{\phi}/\Delta^{1/q}) = T(f) * \tilde{\phi}\right)$$

for each $\phi \in P_1(G)$ and $f \in A$.

THEOREM 5.3. *Let G be a locally compact noncompact group. Let B be a nonempty closed convex left (right) invariant subset of $L_1(G)$ and let A be a* *nonempty bounded closed convex left (right) invariant subset of* $L_q(G)$, $1 < q < \infty$. *If* T *is a continuous affine mapping from* A *into* B *commuting with left (right) translations, then* $T(f) = 0$ *for all* $f \in A$.

PROOF. Since A and B are closed and convex, and T is affine, T is also continuous when A and B have their respective weak topologies. Since A is weakly compact and convex, $T(A)$ is a weakly compact convex left (resp. right) invariant subset of $L_1(G)$. By Theorem 4.8, $T(A) = \{0\}$.

Hörmander proved [8] that if T is a bounded linear operator from $L_q(\mathbf{R}^n)$ into $L_p(\mathbf{R}^n)$ with $1 \leq p < q < \infty$, and T commutes with translations, then $T = 0$. His proof can be easily adapted to prove (see also [11, p. 149])

THEOREM 5.4. *Let* G *be a locally compact noncompact group and let* T *be a bounded linear operator from* $L_q(G)$ *into* $L_p(G)$ *with* $1 \leq p < q < \infty$. *Let* A, B *be closed left (right) invariant convex subsets of* $L_q(G)$ *and* $L_p(G)$, *respectively, such that* $T(A) \subseteq B$. *If* T *commutes with left (right) translations when restricted to* A, *then* $T(f) = 0$ *for each* $f \in A$. (*Note that the set* B *must contain* 0 *by Theorem* 4.5.)

REMARK. We do not know whether Theorem 5.4 still remains true when T is just an affine continuous map from A into B.

THEOREM 5.5. *Let* G *be any locally compact group and let* B *be a closed bounded left (right) invariant subset of* $L_p(G)$, $1 \leq p < \infty$. *Let* T *be a continuous affine mapping from* $P_1(G)$ *into* B.

(a) *If* $p = 1$, *then* T *commutes with all left (right) translations if and only if there exists a regular Borel measure* μ *in* $M(G)$ *such that* $T(\phi) = \phi * \mu$ ($T(\phi) = \mu * \phi$) *for each* $\phi \in P_1(G)$.

(b) *If* $1 < p \leq \infty$, *or if* $p = 1$ *and* B *is a weakly compact subset of* $L_1(G)$, *then* T *commutes with all left (right) translations if and only if there exists* $h \in B$ *such that* $T(\phi) = \phi * h$ ($T(\phi) = h * \Delta^{1/p'}\phi$) *for all* $\phi \in P_1(G)$.

PROOF. (a) It is easy to check that if there exist $\mu \in M(G)$ such that $T(\phi) = \phi * \mu$ ($T(\phi) = \mu * \phi$), then T commutes with left (right) translations. Conversely, if T commutes with all right translations, then $T(\phi * \tilde{\psi}/\Delta) = T(\phi) * \tilde{\psi}/\Delta$ for $\phi, \psi \in P_1(G)$, by Theorem 5.1. Since $\psi \to \tilde{\psi}/\Delta$ maps $P_1(G)$ one-one onto $P_1(G)$, it follows that $T(\phi * \psi) = T(\phi) * \psi$. Let $\{\phi_\alpha\}$ be a net in $P_1(G)$ such that $\{\phi_\alpha\}$ converges to ε_e in the τ-topology of $M(G)$. Since $\{T(\phi_\alpha)\}$ is bounded, we may assume (by passing to a subnet if necessary) that $T(\phi_\alpha)$ converges to $\mu \in M(G)$ in the weak*-topology. It follows from Lemma 3.2(b) that the net $T(\phi_\alpha) * \psi = T(\phi_\alpha * \psi)$ converges to $T(\psi)$ in the weak topology of $L_1(G)$. On the other hand, the net $\{T(\phi_\alpha) * \psi\}$ also converges to $\mu * \psi$ in the weak*-topology of $M(G)$ (Lemma 3.2(d)). Hence $T(\psi) = \mu * \psi$

for each $\psi \in P_1(G)$. The proof for mappings commuting with left translation is similar.

(b) It is routine to check that the map $T(\phi) = \phi * h$ $(T(\phi) = h * \Delta^{1/p'}\phi)$ commutes with left (right) translations. To see the converse, we first assume that $1 < p < \infty$. If T commutes with right translation, then $T(\phi * \tilde{\psi}/\Delta) = T(\phi) * \tilde{\psi}/\Delta^{1/p}$ for each $\phi, \psi \in P_1(G)$. Let $\{\phi_\alpha\}$ be a net in $P_1(G)$ converging to ε_e in the τ-topology. Since $T(\phi_\alpha) \in B$, and B is weakly compact, it follows that $T(\phi_\alpha)$ has a weak cluster point $h \in B$. Consequently $T(\tilde{\psi}/\Delta) = h * \tilde{\psi}/\Delta^{1/p}$ for each $\psi \in P_1(G)$, or $T(\phi) = h * \Delta^{1/p'}\phi$ for each $\phi \in P_1(G)$.

If T commutes left translation, then $\Pi_p \circ T \circ \Pi_1^{-1}$ is an affine map from $P_1(G)$ into $\Pi_p(B)$ commuting with right translations. By above, we can find $k \in \Pi_p(B)$ such that

$$\Pi_p \circ T \circ \Pi_1^{-1}(\tilde{\phi}/\Delta) = k * \tilde{\phi}/\Delta^{1/p}$$

for each $\phi \in P_1(G)$. Hence

$$T(\phi) = \Pi_p^{-1}\left(k * \tilde{\phi}/\Delta^{1/p}\right) = \phi * \Pi_p^{-1}(k)$$

for each $\phi \in P_1(G)$ by Lemma 3.3(c). Now let $h = \Pi_p^{-1}(k)$.

If $p = \infty$, B is weak*-compact. Hence if T commutes with right translation, let g be a weak*-cluster point of the net $\{T(\phi_\alpha)\}$, where ϕ_α is chosen as above. By passing to a subnet if necessary, we may assume that $T(\phi_\alpha)$ converges to g in the weak*-topology of $L_\infty(G)$. Then the net $T(\phi_\alpha) * \tilde{\psi}$ also converges to $g * \tilde{\psi}$ in the weak*-topology of $L_\infty(G)$. Hence

$$T(\tilde{\psi}/\Delta) = \lim_\alpha T\left(\phi_\alpha * \tilde{\psi}/\Delta\right) = \lim_\alpha T(\phi_\alpha) * \tilde{\psi} = g * \tilde{\psi}.$$

The proof for maps commuting with left translation is similar to that of the case $1 < p < \infty$.

REMARK. Let T be a bounded linear operator from $L_1(G)$ into $L_p(G)$, $1 < p < \infty$. Let $B =$ closure $\{T(\phi); \phi \in P_1(G)\}$; then (a) yields the classical result of Wendel [14] (see also [7, p. 376]) for multipliers from $L_1(G)$ into $L_1(G)$. Part (b) yields the Akemann [1], Gaudry [4] and Kitchen [10] characterization of weakly compact and compact multipliers from $L_1(G)$ into $L_1(G)$. Also, if $1 < p \leq \infty$, Theorem 5.5 yields Brainerd and Edwards' characterization of left and right multipliers from $L_1(G)$ into $L_p(G)$ in [2, Corollary 2.6.2]. Note that Brainerd and Edwards' definition or right translation by $a \in G$, denoted by ρ_a, on $L_p(G)$ corresponds with our r_a only when $p = 1$ or when G is unimodular. (In fact, $\rho_a = r_{a^{-1}}$ in this case, and note that Brainerd and Edwards' modular function $\Delta(a)$ is our $\Delta(a^{-1})$.) Moreover, ρ_a on $L_p(G)$, when $p > 1$ and G is not unimodular, fails to be an isometry. This is exactly why in this case there is no bounded linear operator from $L_1(G)$ into $L_p(G)$, when $p > 1$, commuting with all ρ_a, $a \in G$ (see [2, p. 304]). However, it follows from Theorem 5.5(b) that linear operators T_h: $L_1(G) \to L_p(G)$, where

$T_h(f) = h * \Delta^{1/p'}f$, and $h \in L_p(G)$, commute with all right translations in our sense. Furthermore any such bounded linear operators from $L_1(G)$ into $L_p(G)$ must be of this form.

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