DISINTEGRATION OF MEASURES ON COMPACT
TRANSFORMATION GROUPS

BY
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Abstract. Let $G$ be a compact metrizable group which acts freely on a
locally compact Hausdorff space $X$. Let $\mu$ be a measure on $X$, $\pi: X \rightarrow X/G$
$\equiv Y$ the projection, $\nu = \pi(\mu)$. We show that there is a $\nu$-Lusin-measurable
disintegration of $\mu$ with respect to $\pi$. We use this result to prove a structure
theorem concerning $T$-ergodic measures on bitransformation groups $(G, X, T)$ with $G$ metric and $X$ compact. We finish with some remarks
concerning the case when $G$ is not metric.

Introduction. This paper falls naturally into two parts. The first deals with
the following situation: $G$, a compact metric group, acts freely on a locally
compact space $X$ (thus, if $g \cdot x = x$ for any $x \in X$ and $g \in G$, then $g =$
identity in $G$). The quotient $Y = X/G$ is locally compact; let $\pi: X \rightarrow Y$ be
the canonical projection. We show that each measure $\mu$ on $X$ has a
$\nu(\mu) = \nu$-Lusin-measurable disintegration with respect to $\pi$ (see §0 for defini-
tions; see [6] for a detailed discussion of disintegrations and their relationship
to liftings). No theorem known to the author yields this result, although it is
similar to theorems on the disintegration of a measure on a product space (see
[2] and [6]).

The second part considers a special case: $\mu$ is a $T$-ergodic measure on a
compact Hausdorff space $X$ which is the phase space of a bitransformation
group $(G, X, T)$ with $G$ metric. Let

$$G_0 = \left\{ g \in G \mid \int_X f(gx) \, d\mu(x) = \int_X f(x) \, d\mu(x) \text{ for all } f \in C(X) \right\},$$

and let $\gamma_0$ be Haar measure on $G_0$. We show that, if $y \rightarrow \lambda_y$ is the
disintegration of $\mu$, then each $\lambda_y$ "looks like" $\gamma_0$ in a certain sense. The
following result is crucial: If $Z$ is a Hausdorff space and $f: X \rightarrow Z$ a
$\mu$-Lusin-measurable, $T$-invariant map, then $f(x) = \text{const } \mu$-a.e. Finally, in §6,
we remove the metrizability assumption on $G$; we assume the existence of a
strong lifting on $(Y, \nu)$ (the only place in the paper where this is done).

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0. Preliminaries. We quote definitions and results from [1], [3], [6], and [12]; see these references for more details.

0.1. Let $W$ be a locally compact Hausdorff space, $K(W)$ the set of continuous complex functions on $W$ with compact support, with the topology of uniform convergence on compact sets. A (Radon) measure on $W$ is a continuous complex linear functional on $K(W)$; we denote the set of all such functionals by $C^*(W)$. We will always assume $C^*(W)$ is given the topology of pointwise convergence (the vague topology). Let $M_+(W)$ consist of those positive elements $\eta$ of $C^*(W)$ for which $\|\eta\| = \eta(W) < \infty$. See [1, Chapter III, §1; Chapter IV, §4, \eta^97, Proposition 12].

0.2. Let $W$ be as above, $\eta$ a positive measure on $W$, $Z$ a topological space, $\pi: W \to Z$ a map. Say $\pi$ is $\eta$-Lusin-measurable [12] if, for each compact $K \subset W$, there is a sequence $(K_i)$ of pairwise disjoint compact sets such that (i) $\eta(K \sim \bigcup K_i) = 0$; (ii) $\eta|K_i$ is continuous $(i > 1)$. If $\pi$ is $\eta$-Lusin-measurable, then $\pi^{-1}(V)$ is $\eta$-measurable for each open (closed) $V \subset Z$. See [1, Chapter IV, §5].

0.3. Let $W, \eta$ be as above. Let $M^1(W, \eta)$ be the set of $\eta$-integrable complex functions on $W$, $L^1(W, \eta)$ the (usual) set of equivalence classes modulo null sets. Let $N_1$ be the norm on $L^1(W, \eta)$. Similarly, define $M^{\infty}(W, \eta)$ and $L^{\infty}(W, \eta)$.

0.4. Let $W, \eta, \pi$ be as above, and let $Z$ be locally compact Hausdorff. Call $\pi$ $\eta$-proper if, for each compact $K \subset Z$, $\pi^{-1}(K)$ is $\eta$-integrable. If $\pi$ is $\eta$-proper, the map $f \mapsto \eta(f \circ \pi): K(Z) \to C$ defines a Radon measure $\gamma$ on $Z$; we write $\gamma = \pi(\eta)$. If $f \in L^1(Z, \gamma)$, then $\int_W (f \circ \pi) \, d\mu = \int_Z f \, d\gamma$. See [1, Chapter V, §6].

0.5. Let $\lambda: W \to M_+(Z): w \mapsto \lambda_w$ be a map. Say $\lambda$ is weakly $\eta$-measurable if $\gamma \mapsto \langle \lambda_w, f \rangle$ is $\eta$-measurable for each $f \in K(Z)$. The map $\lambda$ is weakly essentially $\eta$-integrable if $f \in K(Z) \Rightarrow w \mapsto \langle \lambda_w, f \rangle$ is essentially $\eta$-integrable. In this case, the formula $f \mapsto \int_W \langle \lambda_w, f \rangle \, d\eta(w)$ defines a measure $\gamma$ on $Z$. If $f \in L^1(Z, \gamma)$, then the map $w \mapsto \langle \lambda_w, f \rangle$ is defined $\eta$-a.e., is $\eta$-integrable, and $\gamma(f) = \int_W \langle \lambda_w, f \rangle \, d\eta(w)$. We write $\gamma = \int_W \lambda_w \, d\eta(w)$. Finally, if $\lambda': W \to M_+(Z)$ is another map, say $\lambda' = \lambda$ weakly $\nu$-a.e. if $\langle \lambda', f \rangle = \langle \lambda_w, f \rangle$ $\nu$-a.e. for each $f \in K(Z)$. See [1, Chapter V, §3].

0.6. Definition. Let $\pi: W \to Z$ be $\eta$-proper, $\gamma = \pi(\eta)$. A weakly essentially $\gamma$-integrable map $\lambda: Z \to M_+(W)$: $z \mapsto \lambda_z$ is a disintegration of $\eta$ with respect to $\pi$ if:

(a) $\lambda$ is $\gamma$-adequate [1, Chapter V, §3, \eta^91, Definition 1];
(b) $\|\lambda_z\| = 1$ for all $z$;
(c) Support($\lambda_e$) $\subset \pi^{-1}(z)$ ($z \in Z$);
(d) $\eta = \int x \lambda_\gamma d\gamma(z)$.

If $\lambda$ is $\gamma$-Lusin-measurable (with respect to the vague topology) and satisfies $||\lambda_\gamma|| \leq \text{const} < \infty \gamma$-a.e., then $\lambda$ is $\nu$-adequate. See [1, Chapter VI, §3, $\eta^01$; Chapter V, §3, $\eta^01$, Proposition 2].

0.7. Theorem (Dunford and Pettis). Let $E$ be a separable Banach space, $E'$ its dual with the norm topology. Let $\xi: L^1(\eta) \rightarrow E'$ be a bounded linear map. There exists a map $\lambda: W \rightarrow E'$: $w \rightarrow \lambda_w$ such that (i) $w \rightarrow \langle e, \lambda_w \rangle$ is $\eta$-measurable, and (ii) if $f \in L^1(\eta)$, then

$$\langle \xi(f), e \rangle = \int_w f(w) \langle \lambda_w, e \rangle \, d\eta(w) \quad (e \in E).$$

One has $||\xi|| = \text{ess sup}_{w \in W} ||\lambda_w||$. Further, if $\lambda': W \rightarrow E'$ is another map satisfying (i) and (ii), then $\lambda' = \lambda$ locally $\eta$-a.e.

For a proof, see [6, Corollary 1, p. 89].

0.8. The following notation will be fixed from now on. $G$ will be a compact Hausdorff topological group, $X$ a locally compact Hausdorff space (often compact) with Radon measure $\mu$. We assume $(G, X)$ is a left-transformation group [3] such that $G$ acts freely. Define $\pi: X \rightarrow X/G \equiv Y$ to be the canonical projection, and let $\nu = \pi(\mu)$. If $A \subset X$, let $G \cdot A = \{ g \cdot x | g \in G, x \in A \}$. If $f \in C(X)$, let $(f \cdot g)(x) = f(gx)$; if $\mu \in C^*(X)$, let $(g\mu)(f) = \mu(fg)$. We will sometimes write $dg$ for Haar measure on $G$. Finally, in §§4–6, $T$ will denote an arbitrary group such that $(X, T)$ is a right transformation group; in §§5–6, $(G, X, T)$ will be a bitransformation group (the actions of $G$ and $T$ commute). See [3].

I. COMPACT TRANSFORMATION GROUPS

In §1, we assume $G$ is a Lie group and $X$ is compact. In §2, $G$ is allowed to be metric; in §3, $X$ becomes locally compact.

1. $X$ compact, $G$ Lie. Let $\mu$ be a measure on $X$, $\nu = \pi(\mu)$. To disintegrate $\mu$ with respect to $\pi$, we first express $X$ as a “measurable product” $Y \times G$. We then apply the Dunford-Pettis theorem (0.7) to the map $\xi: L^1(\nu) \rightarrow C^*(G)$ given by $\xi(f) = \pi_2[(f \circ \pi) \cdot \mu]$; here $\pi_2: X \rightarrow G$ is the projection. The map $\omega: Y \rightarrow C^*(G)$ that results will then be used to construct a disintegration of $\mu$.

The key to this section is the following result; it is an immediate corollary of [8, §5.4, Theorem 1].

1.1. Theorem. For each $x \in X$, there is a compact neighborhood $U$ of $x$ and a compact $F \subset U$ such that $\pi^{-1}(y) \cap F$ is a single point whenever $y \in \pi(U)$.

1.2. Notation. For each $x \in X$, pick sets $U_x, F_x$ satisfying the conditions of 1.1. It is clear that we can replace each $U_x$ by its saturation $G \cdot U_x$. Assume
this done, and choose sets $U_i$ $(1 < i < r)$ which cover $X$. Let $U_i \equiv U_{x_i}$, $F_i \equiv F_{x_i}$, $V_i \equiv \pi(U_i)$. Define maps $\tau_i: V_i \to U_i$ by $\{\tau_i(y)\} = F_i \cap \pi^{-1}(y)$. Let $A_1 = V_1$, $A_i = V_1 \cup \bigcup_{j=1}^{i-1} V_j$ $(2 < i < r)$. Then the $A_i$ are Borel, and $\bigcup_{i=1}^{r} A_i = Y$. Let $B_i = \pi^{-1}(A_i)$. Define $\tau: Y \to X$ by $\tau|_{A_i} = \tau_i$.

The following lemma is a consequence of the definitions.

1.3. **Lemma.** The maps $(g, x) \to g \cdot x: G \times F_i \to U_i$ and $\psi_i: (g, y) \to g \cdot \tau_i(y): G \times V_i \to U_i$ are homeomorphisms $(1 < i < r)$. The map $\tau$ is a section of $X$ over $Y$ (i.e., $\tau(y)$ is an element of $\pi^{-1}(y)$ for each $y \in Y$), and $\tau$ is $\nu$-Lusin-measurable.

1.4. **Definition.** For $x \in X$, let $\pi_2(x) \in G$ be determined by $\pi_2(x) \cdot \tau(x) = x$ ($y = \pi(x)$); thus $\pi_2: X \to G$.

In other words, $\pi_2(X)$ is that $g$ such that $g \cdot \tau(y) = x$. Using 1.3, one sees that $\pi_2$ is continuous on each set $B_i$. By 0.2, $\pi_2$ is $\eta$-Lusin measurable for every $\eta \in C^*(X)$. Also, if $K \subset G$ is compact, then $\pi_2^{-1}(K)$ is $\eta$-integrable $(\eta \in C^*(G))$. Therefore,

1.5. **Lemma.** The map $\pi_2$ is $\eta$-proper (0.4) for every $\eta \in C^*(G)$.

1.6. **Theorem.** There is a $\nu$-Lusin-measurable map $\omega: Y \to M_+(G): y \to \omega_y$ such that:

(a) $\|\omega_y\| = 1$ ($y \in Y$).

(b) If $f \in M^1(Y, \nu)$ and $h \in C(G)$, then

$$\int_X f(\pi(x)) h(\pi_2(x)) \, d\mu(x) = \int_Y f(y) \omega_y(h) \, d\nu(y).$$

If $\omega': Y \to C^*(G): y \to \omega'_y$ is another $\omega^*-\nu$-measurable map satisfying (b) such that $\|\omega'_y\| \leq M \nu$-a.e. for some $M$, then $\omega'_y = \omega_y \nu$-a.e.

**Proof.** Let $f \in M^1(Y, \nu)$. Since $\pi_2$ is $(f \circ \pi) \cdot \mu$-proper (0.4), $\pi_2[(f \circ \pi) \cdot \mu] \equiv \xi(f)$ is a measure on $G$. We estimate the norm of $\xi(f)$:

$$\|\xi(f)\| = \sup_{\|h\| \leq 1} |\xi(f) \cdot h|,$$

$$= \sup_{\|h\| \leq 1} \left| \int_X f(\pi(x)) h(\pi_2(x)) \, d\mu(x) \right|,$$

$$\leq \int_X |f \circ \pi_2(x)| \, d\mu(x) = N_1(f).$$

It follows immediately that $\xi$ induces a linear map of $L^1(Y, \nu)$ into $C^*(G)$ such that $\|\xi(f)\| < N_1(f)$. By the Dunford-Pettis theorem (0.7), there is a $\nu$-Lusin-measurable map $\omega: Y \to C^*(G): y \to \omega_y$ such that (b) holds; further, if $\omega$ satisfies the description in 1.6, then $\omega' = \omega \nu$-a.e.

It must still be shown that (perhaps after modification on a set of measure
zero) one has \( \omega_y \in M_+(X) \) and \( \|\omega_y\| = 1 \) for all \( y \). Let \( h \in C_+(G) = \{ f \in C(G) | f(g) > 0 \text{ for all } g \} \), and let \( Y \subset B_h = \{ y | \omega_y(h) < 0 \} \). Then \( B_h \) is \( \nu \)-measurable (since \( \omega \) is); let \( \theta \) be the characteristic function of \( B_h \). Then

\[
0 < \int_X \theta(\pi(x))h(\pi(x)) \, d\mu(x) = \int_Y \theta(y)\omega_y(h) \, d\nu(y) < 0.
\]

Hence \( \nu(B_h) = 0 \). Let \( (h_i)_{i=1}^\infty \) be a countable dense subset of \( C_+(G) \); it is easily seen that \( h \in C_+(G) \Rightarrow B_h \subset \bigcup_{i=1}^\infty B_{h_i} \). It follows that \( \omega_y \geq 0 \) \( \nu \)-a.e. To check that \( \|\omega_y\| = 1 \) \( \nu \)-a.e., let \( h_0(g) \equiv 1 \) (\( g \in G \)). Then \( \|\omega_y\| = \omega_y(h_0) \), but by (b),

\[
\int_Y f(y) \, d\nu(y) = \int_X f(\pi(x)) \, d\mu(x)
\]

for all \( f \in M_1(\nu) \). It follows that \( \omega_y(h_0) = 1 \) \( \nu \)-a.e.

We are still using the notation of 1.2.

1.7. Lemma. Let \( \varepsilon > 0 \) and \( f \in C(X) \) be given. For each \( i, 1 < i < r \), there exist \( h_i \in C(G) \) and a bounded Borel function \( \psi_i : Y \to \mathbb{C} \) such that, on \( B_i \), \( |f(x) - \psi_i(\pi(x))h_i(\pi(x))| < \varepsilon \).

**Proof.** Let \( \eta_i : G \times V_i \to U_i \) be the homeomorphism of 1.3, and let \( f_i = f \circ \eta_i \). There are functions \( \psi'_i \in C(V_i) \), \( h_i \in C(G) \) such that \( |f_i(g, y) - \psi'_i(y)h_i(g)| < \varepsilon \) (\( y \in V_i, g \in G \)). Let

\[
\psi_i(y) = \begin{cases} 
\psi'_i(y), & y \in A_i, \\
0, & y \notin A_i.
\end{cases}
\]

Then \( \psi_i \) is bounded Borel. Now on \( B_i \), \( \pi_2(x) = g \) and \( \pi(x) = y \), where \( \eta_i(g, y) = x \). Thus, on \( B_i \), \( |f(x) - \psi_i(\pi(x))h_i(\pi(x))| < \varepsilon \).

1.8. Definition. If \( y \in Y \), let \( \phi_y : G \to X : g \to g \cdot \tau(y) \) (\( \tau \) is defined in 1.3).

Observe that \( \phi_y \) is a homeomorphism onto \( \pi^{-1}(y) \).

1.9. Theorem. There exists a \( \nu \)-Lusin-measurable disintegration \( \lambda : Y \to M_+(X) : y \to \lambda_y \) of \( \mu \) with respect to \( \pi \). If \( \lambda' : y \to \lambda' \) is another \( \nu \)-Lusin-measurable map satisfying 0.6 (c), (d) and such that \( \|\lambda'\| < M < \infty \) \( \nu \)-a.e., then \( \lambda' = \lambda \) \( \nu \)-a.e.

**Proof.** Let \( \omega \) be the map of 1.6, and define a measure \( \lambda_y \) on \( X \) by

\[
\langle \lambda_y, f \rangle = \langle \omega_y, f \circ \phi_y \rangle (f \in C(X)).
\]

It follows immediately from 1.6 and this definition of \( \lambda_y \) that \( \lambda_y > 0, \|\lambda_y\| = 1, \) and \( \text{Supp}(\lambda_y) \subset \pi^{-1}(y) \). We must show
that \( \lambda: y \to \lambda_y \) is \( \nu \)-Lusin-measurable, and that \( \mu(f) = \int_Y \lambda_y(f) \, d\nu(y) \) for all \( f \in C(X) \).

For measurability, fix \( \varepsilon > 0 \), and choose a compact \( F \subset Y \) such that (i) \( \nu(\sim F) < \varepsilon \), and (ii) both \( \omega|_F \) and \( \tau|_F \) are continuous. Let \( f \in C(X) \), and suppose \( y_n \to y \) in \( F \). Then

\[
|\lambda_{y_n}(f) - \lambda_y(f)| = |\lambda_{y_n}(f \circ \phi y_n) - \lambda_y(f \circ \phi y)|
\]
\[
< |\lambda_{y_n}(f \circ \phi y_n) - \lambda_y(f \circ \phi y)| + |\lambda_{y_n}(f \circ \phi y) - \lambda_y(f \circ \phi y)|
\]
\[
< \|f \circ \phi y_n - f \circ \phi y\|_{C(G)} + |\lambda_{y_n}(f \circ \phi y) - \lambda_y(f \circ \phi y)|.
\]

By (ii) above, both terms tend to zero. Thus \( \lambda_{y_n} \to \lambda_y \) (vaguely) \( \Rightarrow \lambda|_F \) is continuous \( \Rightarrow \lambda \) is \( \nu \)-Lusin-measurable.

By \( \nu \)-measurability and the fact that \( \|\lambda_y\| = 1 \) for all \( y \), the formula

\[
\mu'(f) = \int_Y \lambda_y(f) \, d\nu(y) \quad (f \in C(X))
\]
defines a measure on \( X \) (0.5). We will show that \( \mu' = \mu \).

Observe first that

\[(\ast) \quad \langle \lambda_y, h \circ \pi_2 \rangle = \langle \omega_y, h \rangle \quad (y \in Y, h \in C(G)).\]

Let \( \theta_i \) be the characteristic function of \( A_i \subset Y \). Recalling that \( B_i = \pi^{-1}(A_i) \), we have

\[
\mu(f) = \sum_{i=1}^r \int_{B_i} f(x) \, d\mu(x) = \sum_{i=1}^r \int_X \theta_i(\pi(x)) f(x) \, d\mu(x) \quad (f \in C(X)).
\]

Fix \( f \), and let \( \varepsilon > 0 \) be given. By 1.7, there are functions \( h_i \in C(G) \) and bounded Borel functions \( \psi_i \) on \( Y \) such that \( |f(x) - \psi_i(\pi(x)) h_i(\pi_2(x))| < \varepsilon \) \( (x \in B_i) \). Let \( f' \) be defined by

\[
f'(x) = \psi_i(\pi(x)) h_i(\pi_2(x)) \quad (x \in B_i, 1 < i < r).
\]

Then \( f' \) is \( \mu' \)-integrable, so

\[
\mu'(f') = \int_Y \lambda_y(f') \, d\nu(y) \quad (\text{by 0.5})
\]
\[
= \sum_{i=1}^r \int_Y \theta_i(y) \psi_i(y) \lambda_y(h \circ \pi_2) \, d\nu(y)
\]
\[
= \sum_{i=1}^r \int_Y \theta_i(y) \psi_i(y) \omega_y(h) \, d\nu(y) \quad (\text{by (\ast) above})
\]
\[
= \sum_{i=1}^r \int_Y \theta_i(\pi(x)) \psi_i(\pi(x)) h(\pi_2(x)) \, d\mu(x) \quad (\text{by 1.6})
\]
\[
= \mu(f').
\]
It now follows easily from the uniform bound $|f(x) - f'(x)| < \varepsilon$ ($x \in X$) that $\mu(f) = \mu'(f) = \int_x \lambda_y f \, dv(y)$.

Uniqueness remains to be shown. Let $\lambda'$ be as in the statement of 1.8, and let $\omega'_y = \pi_2(\lambda'_y)$. It is straightforward to check that $y \rightarrow \omega'_y$ is $\nu$-Lusin-measurable. Let $f \in M^1(\nu)$, $h \in C(G)$; then $(f \circ \pi) \cdot (h \circ \pi_2)$ is $\mu$-integrable, hence (0.5)

$$((f \circ \pi) \cdot (h \circ \pi_2)) = \int_y \lambda'_y ((f \circ \pi) \cdot (h \circ \pi_2)) \, dv(y)$$

$$= \int_y f(y) \lambda'_y (h \circ \pi_2) \, dv(y)$$

$$= \int_y f(y) \omega'_y (h) \, dv(y).$$

By uniqueness in 1.6, $\omega'_y = \omega_y \nu$-a.e., and it follows that $\lambda'_y = \lambda_y \nu$-a.e.

2. $X$ compact, $G$ metric. The following result ([8, Chapter IV, §7] or [5, p. 67]) is basic.

2.1. Theorem. Let $H$ be a compact topological group. Then every neighborhood of the identity element contains a closed normal subgroup $L$ such that $H/L$ is a Lie group.

2.2. Notation. Consider a transformation group $(G, X)$ where $G$ is not necessarily metric. Let $\{G_i\}$ be a decreasing net of closed normal subgroups such that $G/G_i$ is a Lie group; this exists by 2.1. If $G$ is metric, $\{G_i\}$ may be taken to be a sequence. Let $X_i = X/G_i$, $\pi_i: X \rightarrow X_i$; observe that $(G/G_i, X_i)$ is a transformation group with $G/G_i$ Lie. Each space $C(X_i)$ may be embedded in $C(X)$; one then has that $\bigcup_i C(X_i)$ is dense in $C(X)$.

2.3. Discussion. Let $(G, X)$ be a transformation group with $G$ metric, $\{G_i\}$ a decreasing sequence of normal subgroups as in 2.2. Let $\mu$ be a measure on $X$, $\mu_i = \pi_i(\mu)$. Apply 1.9 to a fixed $\mu_i$ to obtain a $\nu$-Lusin-measurable disintegration $\tilde{\lambda}_i^f: Y \rightarrow M_+(X_i)$; $y \rightarrow \tilde{\lambda}_i^f$. Define a map $\lambda^f: Y \rightarrow M_+(X)$ by Haar-lifting elements $\tilde{\lambda}_i^f$ by $G_i$; thus

$$\lambda^f(x) = \int_{X_i} \left( \int_{G_i} f(g_i x) \, dg_i \right) \, d\tilde{\lambda}_i^f,$$

where $dg_i$ refers to Haar measure on $G_i$ (observe that the quantity in parentheses defines a continuous function on $X_i$ when $f \in C(X_i)$). It is easily checked that $\lambda^f$ is $\nu$-Lusin-measurable.

2.4. Theorem. There is a $\nu$-Lusin-measurable disintegration $\lambda$ of $\mu$ with respect to $\pi$. The uniqueness statement of 1.9 holds here, also.

Proof. Let $l$, $m$ be positive integers, $l > m$, and let $A_{l,m} = \{y \in Y|\lambda^f(y)
= \lambda^m_y(f) \text{ for all } f \in C(X_m). \text{ Since } \lambda^l_y|_{C(X_m)} \text{ is clearly a } \nu\text{-Lusin-measurable disintegration of } \mu_m, \text{ uniqueness in 1.9 implies that } \nu(A_m) = 1. \text{ Let } A = \cap_{i>m} A_i; \text{ also has } \nu\text{-measure 1. Now if } f \in Q = \bigcup_{i=1}^\infty C(X_i) \subset C(X), \text{ and if } y \in A, \text{ then there is an } l_0 \text{ so that } l > l_0 \Rightarrow \langle f, \lambda^l_y \rangle \text{ is constant. That is, } \lim_{l \to \infty} \langle f, \lambda^l_y \rangle \text{ exists for each } f \in Q. \text{ Since } Q \text{ is dense in } C(X), \text{ we conclude that } \lambda_y = \lim_{l \to \infty} \lambda^l_y \in M_+(X) \text{ exists for each } y \in A. \text{ Define } \lambda: Y \to M_+(X) \text{ to be } \lambda_y \text{ if } y \in A, \text{ and some point-mass supported on } \pi^{-1}(y) \text{ if } y \not\in A.

We show that } \lambda \text{ is } \nu\text{-Lusin-measurable (this is not immediate, since } M_+(X) \text{ is not separable). Let } 0 < \varepsilon < 1, \text{ and choose compact sets } K_l \subset A \subset Y, \nu(K_l) > 1 - \varepsilon \cdot 2^{-(l+1)}, \text{ such that } \lambda^l|_{K_l} \text{ is continuous. Let } K = \cap_{l=1}^\infty K_l; \text{ then } \nu(K) > 1 - \varepsilon. \text{ If } y_n \to y \text{ in } K, \text{ and if } f \in C(X_i), \text{ then }

\lambda_y(f) = \lambda^l_y(f) \to \lambda_y(f). \text{ It follows that } \lambda \text{ is } \nu\text{-Lusin-measurable.}

That } \text{Supp}(\lambda_y) \subset \pi^{-1}(y) \text{ is a consequence of } \text{Supp}(\lambda^l_y) \subset \pi^{-1}(y) \text{ (} l > 1). \text{ The other conditions of 0.6 obviously hold. The uniqueness is obtained as follows. Let } \sigma \text{ be another } \nu\text{-Lusin-measurable disintegration of } \mu, \text{ and let } \sigma_l = \sigma|_{C(X)} \text{ Then } \sigma_l \text{ defines a disintegration of } \mu_l, \text{ hence equals } \lambda^l|_{C(X)} \text{ on a set } B_l \text{ of } \nu\text{-measure 1. Let } B = \cap_{l=1}^\infty B_l; \sigma = \lambda \text{ on } B.

3. \text{X locally compact, } G \text{ metric. For the material in 3.1–3.5, see [1, Chapter IV, §§5, N°s 9, 10].}

3.1. Definition. A family } (A_i)_{i \in I} \text{ of subsets of } Y \text{ is locally countable if, for each } y \in Y, \text{ there is a neighborhood } V \text{ of } y \text{ such that } V \cap A_i \text{ is nonempty for at most countably many } i.

3.2. Theorem. There is a locally countable family } (K_i)_{i \in I} \text{ of compact subsets of } Y, \text{ pairwise disjoint, such that } Y \sim \bigcup_{i \in I} K_i \text{ is locally } \nu\text{-null.}

3.3. Definition. Let } A \subset Y \text{ be } \nu\text{-measurable, and let } f \text{ map } A \text{ to a topological space } Z. \text{ Say that } f \text{ is } \nu\text{-measurable if every extension of } f \text{ to } Y, \text{ constant on } Y \sim A, \text{ is } \nu\text{-Lusin-measurable.}

3.4. Theorem. Let } (A_i)_{i \in I} \text{ be a locally countable family of } \nu\text{-measurable subsets of } Y, \text{ } A = \bigcup_{i \in I} A_i. \text{ A map } f: A \to Z \text{ is } \nu\text{-measurable iff } f|A_i \text{ is measurable for each } i.

3.5. Theorem. Let } K \subset Y \text{ be compact, } f: K \to Z \text{ a map. Then } f \text{ is } \nu\text{-measurable iff } f|_K \text{ is } \nu|_K\text{-Lusin-measurable. Here } \nu|_K \text{ is the restriction of } \nu \text{ to } K.

Choose a locally countable collection } (K_i) \text{ of disjoint compact subsets of } Y \text{ as in 4.2, and let } L_i = \pi^{-1}(K_i) \text{ for each } i. \text{ Note } L_i \subset X \text{ is compact. Let } \tau_i: L_i \to X \text{ be the injection for fixed } i; \text{ the induced map } \tau_i: M_+(L_i) \to M_+(X) \text{ is vaguely continuous. Define a map } \lambda^l \text{ as follows: if } \tilde{\lambda}^l: K_i \to M_+(L_i) \text{ is a disintegration of } \mu_{|L_i} \text{ (as in 2.4) with respect to } \tau_i: L_i \to K_i, \text{ let } \lambda^l = \tau_i \circ \tilde{\lambda}^l.
Then $\lambda^i$ is $\nu|_{K^i}$-measurable. Define $\lambda: Y \to M_+(X)$ by: $\lambda_y = \lambda^i_y$ if $y \in K^i$; $\lambda_y = \delta_x$ for some $x \in \pi^{-1}(y)$ if $y \not\in \bigcup_{i \in I} K^i$.

3.6. Theorem. The map $\lambda$ is a $\nu$-Lusin-measurable disintegration of $\mu$ with respect to $\pi$. The uniqueness statement of 1.9 holds with "$\nu$-a.e." replaced by "locally $\nu$-a.e.".

Proof. Let $N = Y \sim \bigcup_{i \in I} K^i$. Any function defined on $N$ is $\nu$-measurable since $N$ is locally-$\nu$-null. Combining 3.5, 3.4, and 3.3 shows that $\lambda$ is $\nu$-Lusin-measurable. Clearly $\lambda_y$ is supported on $\pi^{-1}(y)$, and $\|\lambda_y\| = 1$ ($y \in Y$).

Let $f \in K(X)$ be nonnegative. Observe that, since $S = \text{Support}(f)$ is compact, the function $r(y) = \lambda_y(f)$ is $\nu$-integrable. Also, the set $J = \{i \in I: S \cap L^i \neq \emptyset\}$ is countable. So:

$$
\int_Y \lambda_y(f) \, d\nu(y) = \int_Y \sum_{j \in J} r(y) \cdot \phi_{K^j} \, d\nu(y) = \sum_{j \in J} \int_{K^j} r(y) \, d(\nu|_{K^j})(y)
$$

$$
= \sum_{j \in J} \int_{K^j} \lambda^i_y(f|_{K^j}) \, d(\nu|_{K^j})(y) = \sum_{j \in J} \int_{L^j} (f \circ \tau_j) \, d(\mu|_{L^j}) = \int_X f \, d\mu.
$$

This shows that $\mu$ is a $\nu$-Lusin-measurable disintegration of $\mu$ with respect to $\pi$.

To show uniqueness, let $\lambda'$ be another $\nu$-Lusin-measurable disintegration. Restricting $\lambda'$ to $K^i$ for each $i$ and applying uniqueness in 2.4 shows that $\lambda' = \lambda$ locally $\nu$-a.e.

II. ERGODIC MEASURES ON BITRANSFORMATION GROUPS

4. Generalities on ergodic measures. We give some basic material, then prove a lemma (4.4) which is of importance in §5.

Let $(X, T)$ be a transformation group with $X$ compact Hausdorff and $T$ an arbitrary group. If $t \in T$ and $A \subset X$, define $A \cdot t = \{xt | x \in X\}$. If $f \in C(X)$ and $t \in T$, let $(ft)(x) = f(xt)$ ($x \in X$); if $\mu \in C^*(X)$, let $(\tau_t)(f) = \mu(ft)$ ($f \in C(X)$).

4.1. Definition. A measure $\mu$ on $X$ is $T$-invariant if $t\mu = \mu$ for all $t \in T$. Then (0.5) $\mu(A \cdot t^{-1}) = \mu(A)$ for each $\mu$-measurable $A \subset X$ and each $t \in T$.

4.2. Definition. A measure $\mu$ on $X$ is $T$-ergodic if: (i) it is a positive probability measure (i.e., $\|\mu\| = 1$); (ii) whenever $A \subset X$ is $\mu$-measurable and $\mu(A \triangle A t^{-1}) = 0$ for all $t \in T$, one has $\mu(A) = 0$ or $\mu(A) = 1$. Here $\triangle = \text{symmetric difference}$.

It is easily seen that this definition is equivalent to the one obtained by replacing "$A$ is $\mu$-measurable" by "$A$ is Borel".

We will later (§6) use the following well-known result; see Phelps [11] for a proof.
4.3. Theorem. A measure $\mu$ on $X$ is ergodic iff $\mu$ is extreme in the compact convex set of $T$-invariant probabilities on $X$.

4.4. Lemma. A measure $\mu$ on $X$ is ergodic $\iff$ the following holds: if $Z$ is a Hausdorff space and $f: X \to Z$ a $\mu$-measurable map satisfying $f(x_t) = f(x)$ $\mu$-a.e. for each $t \in T$, then $f = \text{constant}$ $\mu$-a.e.

Proof. $\Leftarrow$: Let $A \subset X$ be a $\mu$-measurable with $\mu(A \triangle A_t^{-1}) = 0$ ($t \in T$). If $\phi_A$ is the characteristic function of $A$, then $\phi_A(x_t) = \phi_A(x)$ $\mu$-a.e. for each $t$. By the hypothesis, one now obtains $\phi_A(x) = 0$ or 1 for $\mu$-almost all $x$.

$\Rightarrow$: Let $Q_1 = \{z \in Z \mid \text{there exists an open set } V \text{ containing } z \text{ such that } \mu(f^{-1}(V)) = 0\}$, and let $Q = \sim Q_1$. There are three steps: (i) $Q$ contains at most one point; (ii) $Q$ contains at least one point; (iii) if $Q = \{b\}$, then $f(x) = b$ $\mu$-a.e.

(i) If $a, b \in Q$, $a \neq b$, choose disjoint open sets $V_a, V_b$ containing $a, b$ respectively, and let $A = f^{-1}(V_a), B = f^{-1}(V_b)$. Then $A \cap B = \emptyset$, and $\mu(A) > 0, \mu(B) > 0$. Since $\mu(X) = 1$, one has $0 < \mu(A) < 1, 0 < \mu(B) < 1$. However, $f(x_t) = f(x)$ $\mu$-a.e. ($t \in T$) $\Rightarrow$ $\mu(A \triangle A_t^{-1}) = 0 = \mu(B \triangle B_t^{-1}) = 0$. Therefore ergodicity of $\mu$ is violated.

(ii) Suppose $Q = \emptyset$. Let $K$ be a compact subset of $X$, $\mu(K) > 0$, such that $f|_K$ is continuous. Then $K_1 = f(K)$ is compact, Each $z \in K_1$ has a neighborhood $V_z$ in $Z$ such that $\mu(f^{-1}(V_z)) = 0$. Let $K_1 \subset \bigcup_{i=1}^n V_{z_i}$; then

$$0 < \mu(K) < \mu(f^{-1}(K_1)) \leq \mu\left(\bigcup_{i=1}^n V_{z_i}\right) \leq \sum_{i=1}^n \mu(f^{-1}(V_{z_i})) = 0.$$ 

Thus $Q \neq \emptyset$, and there is a $b \in Z$ such that $Q = \{b\}$.

(iii) We show that, given $\epsilon > 0$, there is a set $K$, $\mu(K) > 1 - \epsilon$, such that $f(K) = \{b\}$. Let $L$ be a compact set such that $\mu(L) > 1 - \epsilon$ and $f|_L$ is continuous. Let $\mu_L$ be the restriction of $\mu$ to $L$, and let the support of $\mu_L$ be $K \subset L$. If $\mu_K$ is $\mu$ restricted to $K$, then $\text{Supp } \mu_K = K$, and if $V$ is open in $K$, then $0 < \mu_K(V) = \mu(V)$ [1]. Further, $\mu(K) = \mu(L) > 1 - \epsilon$ [1]. We claim that $f(K) = \{b\}$. For let $c \in f(K), c \neq b$. Let $V_c$ be a $Z$-open set containing $b'$ such that $\mu(f^{-1}(V_c)) = 0$. Then $(f|_K)^{-1}(V_c) = K \cap f^{-1}(V_c)$ is open in $K$ and has $\mu$-measure zero. This contradiction establishes that $f(K) = \{b\}$.

5. The disintegration of an ergodic measure. Let $(G, X, T)$ be a bitransformation group with $G$ compact metric, and let $\mu$ be a $T$-ergodic measure on $X$. Let $\nu = \pi(\mu)$, and let $G_0 = \{g \in G \mid g\mu = \mu\}$. Then $G_0$ is a closed subgroup of $G$. By 4.3, $\mu$ has a $\nu$-Lusin-measurable disintegration $\lambda$ with respect to $\pi$. If
5.1. Definition. Let $H: X \to C^*(G)$ be given by $\langle H(x), \hat{h} \rangle = \langle \lambda_y, h \circ \phi_y^{-1} \rangle$ where $\hat{h}(g) = h(g^{-1})$ $(g \in G, x \in X, y = \pi(x), h \in C(G))$.

Here $h \circ \phi_y^{-1}$ is assumed to be continuously extended to all of $X$; the choice of the extension does not matter because $\lambda_y$ is supported on $\pi^{-1}(y)$.

5.2. Proposition. (a) $H(gx) = g \cdot H(x)$ $(g \in G, x \in X)$.
(b) $H(xt) = H(x) \mu$-a.e. ($t \in T$).
(c) $H$ is $\mu$-Lusin-measurable.

Proof. (a) This follows immediately from the definition.
(b) Fix $t \in T$, and define $\omega: Y \to C^*(X)$: $\omega_y = t^{-1}(\lambda_y)$. Since $\nu$ is $\nu$-invariant, it is easily seen that $\omega$ is $\nu$-Lusin measurable; by 0.5, the formula $\eta(f) = \int_Y \omega_y(f) \, d\nu(y)$ ($f \in C(X)$) defines a measure $\eta$ on $X$. Now

$$\eta(f) = \int_Y \langle \lambda_y, ft^{-1} \rangle \, d\nu(y) = \int_Y \langle \lambda_y, ft^{-1} \rangle \cdot td\nu(y)$$

(by 0.4)

$$= \nu(f).$$

By uniqueness in 3.3, $\omega = \lambda \nu$-a.e. One now checks that $\langle H(xt), \hat{h} \rangle = \langle \omega_y, h \circ \phi_y^{-1} \rangle$ ($h \in C(G)$); the conclusion follows.

(c) It suffices to prove (c) when $G$ is a Lie group. For, let $G_l, X_l, X'$ $(l \geq 1)$ be as in §2, and define $H_l$ as in 5.1 by replacing $(G, X, T)$ by $(G/G_l, X_l, T)$ and $\lambda$ and $\lambda_l$. Since $\lambda_y = \lim_{l \to \infty} \lambda_y^l$, one has $H(x) = \lim_{l \to \infty} H_l(x)$ $(x \in X, y = \pi(x))$. Since $C^*(G)$ is separable, $H$ is $\mu$-Lusin-measurable if the $H_l$ are.

Assume $G$ is Lie. By 1.3, there is a $\nu$-measurable section $\tau: Y \to X$. Fix $0 < \varepsilon < 1$, and let $B \subset Y$ be a compact set such that $\nu(B) > 1 - \varepsilon$ and such that both $\tau|_B$ and $\lambda|_B$ are continuous. It is enough to show that $H$ is continuous when restricted to $\pi^{-1}(B) = A$, since $\mu(A) = \nu(B) > 1 - \varepsilon$. So, let $x_n \to x_0$ in $A$, and let $h \in C(G)$. It must be verified that $\langle \lambda_y, \phi_y^{-1}h \rangle \to \langle \lambda_y, \phi_y^{-1}h \rangle$ if $y = \pi(x_n), y = \pi(x_0)$.

Observe that the map $\zeta: G \times B \to A$ $(g, y) \to g \cdot \tau(y)$ is continuous and bijective, hence a homeomorphism. Let $x_n = \zeta(g_n, y_n), x_0 = \zeta(g_0, y_0)$. On $A$, define functions $f_n, f$ by $f_n \circ \zeta(g, y) = h(g g_n^{-1}), f \circ \zeta(g, y) = h(g g_0^{-1})$. Then $f_n \to f$ uniformly on $A$. Extend $f_n, f$ continuously to $X$, calling the extensions $f_n, f$ also $(f_n$ may not converge uniformly to $f$ on $X$, but this will not matter).

It may be checked that $\phi_y^{-1}h = f_n|\pi^{-1}(y_n), \phi_y^{-1}h = f|\pi^{-1}(y).$ Thus the proof will be completed if it is shown that $\langle \lambda_y, f_n \rangle \to \langle \lambda_y, f \rangle$. But

$$|\langle \lambda_y, f_n \rangle - \langle \lambda_y, f \rangle| \leq |\langle \lambda_y, f_n \rangle - \langle \lambda_y, f \rangle| + |\langle \lambda_y, f \rangle - \langle \lambda_y, f \rangle|.$$
Since $||\lambda_\nu|| = 1$ and $f_n \to f$ uniformly on $\pi^{-1}(y_n)$, the first term tends to zero. The second term goes to zero because $\lambda|_\mu$ is continuous.

From 4.4, 5.2(b), and 5.2(c), we see that $H(x) = \text{const} \mu$-a.e. To identify the constant we use the following result; it is a corollary of the proofs of 1.5.4 and 1.5.5(1) in [9]. Let $\xi(\nu) = \{\tau \in M_+(X)|\tau\text{ is } T\text{-ergodic, } \pi(\tau) = \nu\}$.

5.3. THEOREM. The map $g \to g\mu: G \to \xi(\nu)$ induces a homeomorphism of $G/G_0$ (left coset space) onto $\xi(\nu)$. If $f \in C(X)$, then $\mu(f) = \int_{G/G_0}(g'\mu)(f) \, d\delta_{G_0}(g'G_0)$ gives an integral representation of $\mu$ over $\xi(\nu) \cong G/G_0$. Here $\delta_{G_0}$ is the Dirac measure at the coset $[G_0]$. This representation is unique: if $\mu(f) = \int_{G/G_0}(g'\mu)(f) \, d\eta(g'G_0)$ ($f \in C(X)$), then $\eta = \delta_{G_0}$.

5.4. PROPOSITION. $H(x) = \gamma_0 \mu$-a.e., where $\gamma_0$ is Haar measure on $G_0$.

PROOF. We know that $F(x) = \beta \mu$-a.e. for some $\beta \in C^*(G)$. By mimicking the proof of 5.2(b), it may be seen that $F(g_0x) = F(x)$ ($g_0 \in G_0$); combining this with 5.2(a), one has $g_0 \cdot \beta = \beta$. If we show that $\text{Supp}(\beta) \subset G_0$, it will follow that $\beta = \gamma_0$.

Define $\bar{\beta} \in M_+(G)$ by $\bar{\beta}(h) = \int_G h(g^{-1}) \, d\beta(g)$. The measure $\bar{\beta}$ induces a measure $\bar{\beta}$ on $G/G_0$. Fix $f \in C(X)$. On the set $A_\mu$ of $\mu$-measure 1 where $H(x) = \beta$, one has $\langle \lambda_\nu, f \rangle = \langle \bar{\beta}, f \circ \phi_x \rangle$ (5.1). Therefore

$$
\mu(f) = \int_Y \langle \lambda_\nu, f \rangle \, d\nu(y) = \int_Y \langle \bar{\beta}, f \circ \phi_x \rangle \, d\mu(x) = \int_{G/G_0} \int_X f(gx) \, d\mu(x) \, d\bar{\beta}(g) = \int_{G/G_0} (g\mu)(f) \, d\bar{\beta}(g).
$$

By uniqueness in 5.3, $\bar{\beta} = \delta_{[G_0]}$; it follows that $\beta$, and hence $\beta$, is supported on $G_0$.

5.5. REMARK. Proposition 5.4 gives a precise formulation of the fact that "each $\lambda_\nu$ looks like $\gamma_0$"; in fact, for $\nu$-a.a. $y \in Y$ and $x \in \pi^{-1}(y)$, one sees that $\phi_x^{-1}(\lambda_\nu) = \gamma_0 \cdot g_x$ for some $g_x \in G$. Here $(\gamma_0 \cdot g)(h) = \int_G h(gg) \, d\gamma_0(g)$. If $H(x) = \gamma_0$ then $\phi_x^{-1}(\lambda_\nu) = \gamma_0$.

We indicate some corollaries of 5.4.

5.6. DEFINITION. An ergodic decomposition of $\xi(\nu)$ is a collection $\{A_\eta|\eta \in \xi(\nu)\}$ of pairwise disjoint Borel sets such that $\eta(A_\eta) = 1$ and $\eta(A_\eta \Delta A_{\eta^t}) = 0$ ($t \in T$).

This relaxes the usual definition, according to which the $A_\eta$ would be strictly $T$-invariant.
We will use the set \( \mathcal{A}_\mu = \{ x | H(x) = \gamma_0 \} \) to obtain an ergodic decomposition of \( \xi(\nu) \) which "splits up" fibers \( \pi^{-1}(y) \) in a nice way.

5.7. **Lemma.** (a) If \( x \in \mathcal{A}_\mu \), then \( gx \in \mathcal{A}_\mu \) iff \( g \in G_0 \); i.e., \( \mathcal{A}_\mu \) is \( G_0 \)-saturated. If \( y = \pi(x) \), then \( \mathcal{A}_\mu \cap \pi^{-1}(y) \) is homeomorphic to \( G_0 \).

(b) \( \mu(A_\mu \triangle A_\mu \cdot t^{-1}) = 0 \) (\( t \in T \)).

**Proof.** (a) Note \( H(gx) = gy_0 \), which equals \( \gamma_0 \) iff \( g \in G_0 \). This implies that \( gx \in \mathcal{A}_\mu \) iff \( g \in G_0 \), which in turn implies that \( \phi_x(G_0) = \mathcal{A}_\mu \cap \pi^{-1}(y) \).

(b) This follows from 5.2(b).

5.8. **Remark.** There is a set \( \mathcal{A}'_\mu \subset \mathcal{A}_\mu \), \( \mu(\mathcal{A}'_\mu) = 1 \), such that \( \mathcal{A}'_\mu \) is an \( F_\sigma \) in \( X \) and 5.7 holds with \( \mathcal{A}'_\mu \) in place of \( \mathcal{A}_\mu \). To see this, write \( \mathcal{A}_\mu = \bigcup_{i=1}^\infty K_i \cup N \) where \( K_i \) is compact \( (i > 1) \) and \( \mu(N) = 0 \). Let \( \mathcal{A}'_\mu = G_0 \cdot \bigcup_{i=1}^\infty K_i \subset \mathcal{A}_\mu \) (since \( \mathcal{A}_\mu \) is \( G_0 \)-saturated). Then each \( G_0 \cdot K_i \) is compact, so \( \mathcal{A}'_\mu \) is an \( F_\sigma \); it is easily shown that 5.7 remains valid with \( \mathcal{A}'_\mu \) replacing \( \mathcal{A}_\mu \).

Replace \( \mathcal{A}_\mu \) by \( \mathcal{A}'_\mu \), retaining the notation \( \mathcal{A}_\mu \).

5.9. **Proposition.** \( \{ gA_\mu | g \in G \} \) is an ergodic decomposition of \( \xi(\nu) \); if \( \eta = g\mu \), then \( \mathcal{A}_\eta = g \cdot \mathcal{A}_\mu \).

The notation is meant to indicate the class of distinct sets \( gA_\mu \).

**Proof.** By 5.7(a), \( g \cdot \mathcal{A}_\mu = \mathcal{A}_\mu \) if \( g \in G_0 \), \( gA_\mu \cap \mathcal{A}_\mu = \emptyset \) if \( g \notin G_0 \), i.e., if \( g\mu \neq \mu \). Thus \( \{ gA_\mu | g \in G \} \) is a pairwise disjoint collection of Borel sets. Also \( \mu(g\mu) = \mu(g^{-1}gA_\mu) = 1 \), and \( \mu(g\mu)(gA_\mu \triangle A_\mu \cdot t^{-1}) = \mu(A_\mu \triangle A_\mu \cdot t^{-1}) = 0 \) (5.7). By 5.3, \( \xi(\nu) \) is exactly \( \{ g\mu | g \in G \} \). Thus all conditions of 5.6 are satisfied.

We state without proof a theorem which depends on 5.9. Let \( \tilde{\mu} \) be the Haar lift of \( \nu \); i.e., \( \mu(f) = \int_Y \int_G f(gx) \, dv(y) \, dv(g) \) \((f \in C(X))\). Observe that \( \tilde{\mu} \) is \( G \)-invariant, i.e., \( g\mu = \mu \) for all \( g \). Hence there is defined a natural unitary representation \( (G, L^2(X, \tilde{\mu})) \) of \( G \) on \( L^2(X, \mu) \) via the formula \( (g \cdot f)(x) = f(gx) \). Similarly, there is a unitary representation \( (G_0, L^2(X, \mu)) \).

5.10. **Theorem.** \( (G, L^2(X, \tilde{\mu})) \) is the representation induced (see [4]) by \( (G_0, L^2(X, \mu)) \).

The proof is contained in [7]. Using this result, one may define and discuss a generalization of "functions of type \( \gamma \)" [10].

5.11. **Question.** If \( T = \) integers or reals, ergodic sets have an interpretation (indeed, may be defined) in terms of regular points. May the ergodic sets of 5.6 be interpreted in some analogous way?

6. **G nonmetrizable, \( Y \) has strong lifting.**

6.1. We retain the assumptions and notation of \( \S 5 \), except that \( G \) need not be metric. We suppose that \( \text{Support}(\nu) = Y \) and that \( \rho \) is a strong lifting of \( L^\infty(Y, \mu) \). Thus \( \rho \) is a map from \( M^\infty(Y, \nu) \) to \( M^\infty(Y, \nu) \) such that: (i) \( \rho \) is
linear; (ii) \( \rho(f) = f \) \( \nu \)-a.e. for all \( f \); (iii) \( f_1 = f_2 \) \( \nu \)-a.e. \( \Rightarrow \rho(f_1) = \rho(f_2) \); (iv) \( f > 0 \Rightarrow \rho(f) > 0 \); (v) \( \rho(f_1 \cdot f_2) = \rho(f_1) \cdot \rho(f_2) \); (vi) \( \rho(1) = 1 \); (vii) \( \rho(f) = f \) for each \( f \in C(Y) \) (see [6] for a complete discussion). It is the last property which is crucial; a function \( \rho \) satisfying (i)-(vi) always exists on \( M'^\infty(Y, \nu) \) [6, Chapter IV, Theorem 3].

Our goal is 6.9, which is an analogue of 5.4 (see also 5.5). We note that if a strong lifting of \( L^\infty(Y, \nu) \) exists, then every extension \( \tilde{\nu} \) of \( \nu \) has a weakly \( \nu \)-measurable (0.5) disintegration with respect to \( \tau \); see [6].

**6.2. Theorem.** Let \( \lambda: Y \to M^+(X) \) be weakly \( \nu \)-measurable and satisfy 
\[ ||\lambda_y|| < \text{const} < \infty \] \( \nu \)-a.e. There is a map \( \lambda': Y \to M^+(X) \), satisfying the conditions just stated, such that \( \lambda' = \lambda \) weakly \( \nu \)-a.e. and \( \rho(\lambda') = \lambda' \).

The last condition means that, if \( f \in C(Y) \), the functions \( \langle \lambda'_y, f \rangle \) and \( \rho(\langle \lambda'_y, f \rangle) \) are equal for all \( y \in Y \). For details and a proof, see [6, Chapter VI, §4]. We observe here that the conditions \( ||\lambda_y|| < \text{const} \) \( \nu \)-a.e. and \( \rho(\lambda') = \lambda' \), together with [6, Chapter IX, Proposition 5], show that \( \lambda' \) is \( \nu \)-adequate (0.6(a)). Moreover, if \( ||\lambda_y|| = 1 \) for all \( y \), then \( 1 = \rho(1) = \rho(\langle \lambda_y, 1 \rangle) = \langle \lambda'_y, 1 \rangle \Rightarrow ||\lambda'_y|| = 1 \) for all \( y \).

Recall that \( G_0 = \{ g \in G | \mu g = \mu \} \); let \( \xi_0 = \{ \eta \in M^+(X) | ||\eta|| = 1, \eta g = \eta \text{ for all } g \in G_0 \} \).

**6.3. Lemma.** Let \( Z = X/G_0 \), \( \sigma: X \to Z \), the projection. Then \( \sigma \) induces an affine isomorphism of \( \xi_0 \) onto \( M^+(Z) \cap \{ \tau \in M^+(Z) | ||\tau|| = 1 \} \).

**Proof.** We need only show that the induced map is bijective. To show injectivity, suppose \( \sigma(\eta_1) = \sigma(\eta_2) \). Then \( \eta_1(f) = \eta_2(f) \) for all \( f \in C(X) \) satisfying \( f g_0 = f(g_0 \in G_0) \). Pick \( f \in C(X) \), and let \( f_0(x) = f(g_0 x) \ dg_0 \). Then \( f_0 \) is \( G_0 \)-adequate; further
\[
\eta_1(f_0) = \int_X f(g_0 x) \ dg_0 \ d\eta_1 = \int_{G_0} \int_X f(g_0 x) \ dg_1 \ dg_0 = \eta_1(f),
\]
and similarly \( \eta_2(f_0) = \eta_2(f) \). One concludes that \( \eta_1(f) = \eta_2(f) \). For surjectivity, pick \( \tau \in M^+(Z) \), and let \( \eta \) be its \( G_0 \)-Haar lift: \( \eta(f) = \int_Y (f g_0(x) \ dg_0) \ d\tau(y) \). Clearly \( g_0 \eta = \eta \) (\( g \in G_0 \)) and \( \sigma(\eta) = \tau \).

**6.4. Proposition.** \( \eta \) is extreme in \( \xi_0 \) \( \Leftrightarrow \eta(f \cdot h) = \eta(f) \cdot \eta(h) \) for all \( f, h \in C(Z) \subset C(X) \).

**Proof.** If \( \eta \) is extreme, 6.3 implies that \( \sigma(\eta) \) is also. Hence \( \sigma(\eta) \) is a Dirac measure placed at some \( z \in Z \), so \( \eta \) is multiplicative on \( C(Z) \). On the other hand, if \( \eta \) is multiplicative on \( C(Z) \), so is \( \sigma(\eta) \Rightarrow \sigma(\eta) \) is extreme \( \Rightarrow \) (by 6.3 again) \( \eta \) is extreme in \( \xi_0 \).

Combining 6.3 and 6.4 shows that those measures on \( X \) ergodic with respect to \( G_0 \) are the \( G_0 \)-Haar lifts of Dirac measures on \( Z \).
6.5. Corollary. Let $\rho: M^\infty(Y, \nu) \rightarrow M^\infty(Y, \nu)$ satisfy (i)-(vi) of 6.1 (thus $\rho$ is a lifting of $M^\infty(Y, \nu)$). Let $\lambda: Y \rightarrow M_+(X)$ be weakly $\nu$-measurable, and suppose $\lambda_y$ is extreme in $\xi_0$ for all $y$. Then $\rho(\lambda)_y$ is also extreme in $\xi_0$ for all $y$.

Proof. Let $f, h \in C(Z), y \in Y$. Then

$$
\rho(y)_{\tilde{y}}(f \cdot h) = [\rho(\lambda_y, f \cdot h)](y) = [\rho(\lambda_y, f) \cdot \rho(\lambda_y, h)](y)
$$

by 6.1(vi) and 6.4. Thus $\rho(\lambda)_y$ is multiplicative on $C(Z)$ for each $\tilde{y}$, hence extreme in $\xi_0$.

The next lemma is used in the proof of 6.9; it will also allow us to tie 5.4 and 6.7 together. Observe that if $\eta$ is extreme in $\xi_0$, then $\eta$ is supported on $\pi^{-1}(y)$ for some $\tilde{y} \in Y$.

6.6. Lemma. Let $\eta \in M_+(X)$. Then $\eta$ is extreme in $\xi_0$ iff $\phi^{-1}(\eta) = \gamma_0 \cdot g$ for some $g \in G (x \in \pi^{-1}(\tilde{y}))$.

Proof. Follows easily from the fact that $\{\gamma_0 \cdot g | g \in G\}$ is precisely the set of measures on $G$ ergodic with respect to the left action (defined by the group multiplication) of $G_0$ on $G$.

6.7. In what follows, we will use the notation of §2. Thus $(G_l)$ is a decreasing net of closed normal subgroups of $G$ such that $G/G_l$ is Lie, and $\lambda^l: Y \rightarrow M_+(X)$ is obtained as in 2.3 (metrizability of $G$ is not necessary there). Let $G_{0,l}$ consist of those $g \in G$ such that $g$ projects to a point $\tilde{g} \in G/G_l$ satisfying $\tilde{g} \cdot \mu_l = \mu_l$. It is not hard to show that (i) each $G_{0,l}$ is a closed subgroup of $G$; (ii) $\cap_l G_{0,l} = G_0$; (iii) if $\gamma_l$ is Haar measure on $G_{0,l}$, then $\gamma_l \rightarrow \gamma_0$ (vague convergence) in $M_+(G)$ (for the arguments here, see [7, Appendix A]). By use of 5.4 and 5.5, it is easily seen that (for fixed $l$ and $\nu$-a.a. $y$) $\phi^{-1}(\lambda^l_y) = \gamma_l \cdot g_x$ for some $g_x \in G (x \in \pi^{-1}(\tilde{y}))$. Lemma 6.6 applies equally well if $\xi_0$ is replaced by $\xi_0 = \{\eta \in M_+(X) \| \eta \| = 1, g \cdot \eta = \eta \text{ for all } g \in G_{0,l}\}$; hence $\lambda^l_y$ is extreme in $\xi_0$ for $\nu$-a.a. $y$. Modifying on a set of measure zero if necessary, we assume this is so for all $y$.

6.8. Let $\rho$ be a strong lifting on $(Y, \nu)$. Applying 6.2 to replace each $\lambda^l$ by a new map (again called $\lambda^l$) which is equal to the old one weakly $\nu$-a.e., is weakly $\nu$-measurable, and which satisfies $\rho(\lambda^l) = \lambda^l$. By 6.5, $\lambda^l_y$ is extreme in $\xi_0$ for all $y$. Since $\rho$ is strong, $\text{Supp}(\lambda^l_y) \subset \pi^{-1}(\tilde{y})$ for all $y$.

6.9. Theorem. There is a weakly $\nu$-measurable disintegration $\lambda$ of $\mu$ with respect to $\pi$ such that $\lambda_y \in \xi_0$ for all $y \in Y$.

Proof. We show first that $\lim \lambda^l_y$ exists for all $y \in Y$, and defines a weakly measurable map $\lambda: Y \rightarrow \lambda_y$. Let $f \in C(X)$. Since $\cup_l C(X_l)$ is dense in $C(X)$, there is a sequence $(f_{(n)})$ in this union which converges to $f$ uniformly. Fix $n$; there is an $l_n$ such that $l > l_n \Rightarrow f \in C(X_l)$. We claim that, if $l > l_n$, then
\[ \langle \lambda'_y, f_n \rangle = \langle \lambda'_y, f_n \rangle = h_n(y) \text{ for all } y. \] To see this, note uniqueness in 1.9 and the definitions of the \( \lambda'_f \) imply that \( \langle \lambda'_y, f_n \rangle = h_n(y) \) \( \nu \)-a.e. But then \( \langle \lambda'_y, f_n \rangle = \rho \langle \lambda'_y, f_n \rangle = \rho(h_n)(y) = h_n(y) \) for all \( y \), proving the assertion. Simple estimates now show that \( (\lambda'_f) \) is Cauchy in \( l \) (\( f \) is arbitrary), and that, if \( \lambda_y = \lim \lambda'_y \), then

\[ \langle \lambda_y, f \rangle = \lim_{n \to \infty} h_n(y) \] (the limit is actually uniform). Thus \( \lambda \) exists and is weakly measurable.

To see that \( \lambda_y \in \xi_0 \) for fixed \( y \), let \( x \in \pi^{-1}(y) \). By 6.7, \( \phi_x^{-1}(\lambda'_y) = \gamma_{t} \cdot g_t \) for some \( g_t \in G \). Using (iii) of 6.7 and choosing a convergent subsequence, we obtain \( \phi_x^{-1}(\lambda_y) = \gamma_0 \cdot g \). By 6.6, \( \lambda_y \in \xi_0 \).

Now replace \( \lambda \) by \( \rho(\lambda) \). The remarks after 6.2 and the fact that \( \rho \) is strong show that 0.6(a), (b), and (c) hold. To obtain 0.6(d), use (\( \star \)), the uniformity of the convergence, and the fact that \( \mu_y(f_n) = \mu(f_n) \) for all \( n \). By 6.5, we still have \( \lambda_y \in \xi_0 \) for all \( y \). The proof is completed.

**References**


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