

ON ANALYTICALLY INVARIANT SUBSPACES AND SPECTRA

BY

DOMINGO A. HERRERO

ABSTRACT. Let T be a bounded linear operator from a complex Banach space \mathcal{X} into itself. Let \mathcal{E}_T and \mathcal{E}_T^a denote the weak closure of the polynomials and the rational functions (with poles outside the spectrum $\sigma(T)$ of T) in T , respectively. The lattice $\text{Lat } \mathcal{E}_T^a$ of (closed) invariant subspaces of \mathcal{E}_T^a is a very particular subset of the invariant subspace lattice $\text{Lat } \mathcal{E}_T = \text{Lat } T$ of T . It is shown that: (1) If the resolvent set of T has finitely many components, then $\text{Lat } \mathcal{E}_T^a$ is a clopen (i.e., closed and open) sublattice of $\text{Lat } T$, with respect to the "gap topology" between subspaces. (2) If $\mathfrak{M}_1, \mathfrak{M}_2 \in \text{Lat } T$, $\mathfrak{M}_1 \cap \mathfrak{M}_2 \in \text{Lat } \mathcal{E}_T^a$ and $\mathfrak{M}_1 + \mathfrak{M}_2$ is closed in \mathcal{X} and belongs to $\text{Lat } \mathcal{E}_T^a$, then \mathfrak{M}_1 and \mathfrak{M}_2 also belong to $\text{Lat } \mathcal{E}_T^a$. (3) If $\mathfrak{M} \in \text{Lat } T$, R is the restriction of T to \mathfrak{M} and \bar{T} is the operator induced by T on the quotient space \mathcal{X}/\mathfrak{M} , then $\sigma(T) \subset \sigma(R) \cup \sigma(\bar{T})$. Moreover, $\sigma(T) = \sigma(R) \cup \sigma(\bar{T})$ if and only if $\mathfrak{M} \in \text{Lat } \mathcal{E}_T^a$. The results also include an analysis of the semi-Fredholm index of R and \bar{T} at a point $\lambda \in \sigma(R) \cup \sigma(\bar{T}) \setminus \sigma(T)$ and extensions of the results to algebras between \mathcal{E}_T and \mathcal{E}_T^a .

1. Properties of the lattice $\text{Lat } \mathcal{E}_T^a$. The study of the lattice $\text{Lat } \mathcal{E}_T^a$ (the *analytically invariant* subspaces of T) began in [5]. This article is based on and complements the results contained there. In what follows, \mathcal{X} will denote a Banach space over the complex field \mathbb{C} ; *operator* and *subspace* will mean *bounded linear map* from a Banach space into itself and *closed linear manifold*, respectively. We shall consider invariant subspace lattices under the topology induced by the "gap between subspaces", i.e., the metric in the family of all subspaces of \mathcal{X} defined by $\hat{d}(\mathcal{X}_1, \mathcal{X}_2) =$ Hausdorff distance between the closed unit ball of the subspace \mathcal{X}_1 and the closed unit ball of the subspace \mathcal{X}_2 . The Banach algebra of all operators in \mathcal{X} will be denoted by $\mathcal{L}(\mathcal{X})$.

Let Σ be a subset of $\mathcal{L}(\mathcal{X})$. It is well known (see [1]; [5]; [6, Chapter IV]) that $(\text{Lat } \Sigma, \hat{d})$ is a complete metric space; therefore, $\text{Lat } \mathcal{E}_T^a$ is always a closed subset of $\text{Lat } T$. We shall show that, under suitable restrictions on the spectrum of T , $\text{Lat } \mathcal{E}_T^a$ is also open in $\text{Lat } T$.

THEOREM 1. *Let $T \in \mathcal{L}(\mathcal{X})$. For each λ in the resolvent set $\rho(T) = \mathbb{C} \setminus \sigma(T)$ of T , there exists a constant $r(T, \lambda) > 0$ such that $\hat{d}(\mathfrak{M}, \mathfrak{N}) > r(T, \lambda)$ for all*

Received by the editors July 30, 1973.

AMS (MOS) subject classifications (1970). Primary 47A10, 47A15.

Key words and phrases. Invariant subspace, analytically invariant subspace, bi-invariant subspace, lattice, spectrum, gap topology, restriction of an operator, operator induced on the quotient space, semi-Fredholm operator, index.

© American Mathematical Society 1977

$\mathfrak{N} \in \text{Lat } T \cap \text{Lat}(T - \lambda)^{-1}$ and all $\mathfrak{U} \in \text{Lat } T \setminus \text{Lat}(T - \lambda)^{-1}$. In particular, $\text{Lat } T \cap \text{Lat}(T - \lambda)^{-1}$ is a clopen sublattice of $\text{Lat } T$.

PROOF. For each pair of subspaces $\mathfrak{X}_1, \mathfrak{X}_2$ define

$$\delta(\mathfrak{X}_1, \mathfrak{X}_2) = \begin{cases} 0 & \text{if } \mathfrak{X}_1 = \{0\}, \\ \sup\{\text{distance}(x_1, \mathfrak{X}_2) : x_1 \in \mathfrak{X}_1, \|x_1\| = 1\} & \text{if } \mathfrak{X}_1 \neq \{0\}, \end{cases}$$

and

$$\hat{\delta}(\mathfrak{X}_1, \mathfrak{X}_2) = \max\{\delta(\mathfrak{X}_1, \mathfrak{X}_2), \delta(\mathfrak{X}_2, \mathfrak{X}_1)\}.$$

Then (see [6, p. 198])

$$\hat{\delta}(\mathfrak{X}_1, \mathfrak{X}_2) \leq \hat{d}(\mathfrak{X}_1, \mathfrak{X}_2) \leq 2\hat{\delta}(\mathfrak{X}_1, \mathfrak{X}_2);$$

if $\mathfrak{X}_1 \supset \mathfrak{X}_2$ and $\mathfrak{X}_1 \neq \mathfrak{X}_2$, then it follows from Riesz' lemma that $\delta(\mathfrak{X}_1, \mathfrak{X}_2) = \hat{d}(\mathfrak{X}_1, \mathfrak{X}_2) = 1$.

Without loss of generality, we can assume that $\lambda = 0$, i.e., that T is invertible. Let $\mathfrak{N} \in \text{Lat } T \cap \text{Lat } T^{-1}$ and $\mathfrak{U} \in \text{Lat } T \setminus \text{Lat } T^{-1}$; then (see [5]) $\mathfrak{N} = T\mathfrak{N}$, while $T\mathfrak{U} \subset \mathfrak{U}$ but $\mathfrak{U} \neq T\mathfrak{U}$. We have

$$\begin{aligned} 1 = \hat{d}(\mathfrak{U}, T\mathfrak{U}) &\leq \hat{d}(\mathfrak{U}, \mathfrak{N}) + \hat{d}(\mathfrak{N}, T\mathfrak{U}) \\ &\leq \hat{d}(\mathfrak{U}, \mathfrak{U})(1 + 2\|T\| \cdot \|T^{-1}\|), \end{aligned}$$

where the first equality follows from Riesz' lemma, the first inequality is just the triangular inequality for the metric \hat{d} , and the second one follows from the relations between $\hat{\delta}$ and \hat{d} and Lemma 4.2 of [5], which implies that

$$\hat{\delta}(\mathfrak{N}, T\mathfrak{U}) = \hat{\delta}(T\mathfrak{N}, T\mathfrak{U}) \leq \|T\| \cdot \|T^{-1}\| \hat{\delta}(\mathfrak{N}, \mathfrak{U}).$$

Therefore,

$$\hat{d}(\mathfrak{N}, \mathfrak{U}) \geq r(T, 0) = (1 + 2\|T\| \cdot \|T^{-1}\|)^{-1}.$$

The general case and the second statement follow immediately from this result. \square

COROLLARY 2. Let $\sigma(T; \mathcal{A}_T)$ denote the spectrum of T in the Banach algebra \mathcal{A}_T . If $\sigma(T; \mathcal{A}_T) \setminus \sigma(T)$ has finitely many components, then $\text{Lat } \mathcal{A}_T^a$ is a clopen sublattice of $\text{Lat } T$. In the general case, $\text{Lat } \mathcal{A}_T^a$ is a countable intersection of clopen sublattices of $\text{Lat } T$.

PROOF. It is enough to recall that \mathcal{A}_T^a is generated by T and $\{(T - \lambda_n)^{-1}\}$, where the (possibly empty) countable set $\{\lambda_n\}$ has exactly one point in common with each bounded component of $\mathbb{C} \setminus \sigma(T; \mathcal{A}_T)$ [5]. Now the result follows immediately from Theorem 1. \square

REMARK. Since $\sigma(T) \subset \sigma(T; \mathcal{Q}_T)$ and $\sigma(T; \mathcal{Q}_T) \setminus \sigma(T)$ is the union of a (possibly empty) subfamily of bounded components of $\rho(T)$, it easily follows that $\text{Lat } \mathcal{Q}_T^a$ is clopen in $\text{Lat } T$ whenever $\rho(T)$ has finitely many components (see [5] for details).

EXAMPLE A. Let T be the bilateral shift "multiplication by e^{ix} " acting on $L^2(\partial D, dm)$, where D denotes the unit disc of the complex plane, ∂D and D^- are the boundary and the closure of D , respectively, and $dm = dx/2\pi$ is the normalized Lebesgue measure on ∂D . Then (see [2], [3])

$$\text{Lat } \mathcal{Q}_T^a = \{L^2(M, dm) : M \text{ is a measurable subset of } \partial D\}$$

$(L^2(M_1, dm) = L^2(M_2, dm)$ if and only if $m(M_1 \triangle M_2) = 0$), and

$$\text{Lat } T = \text{Lat } \mathcal{Q}_T^a \cup (\text{Lat } T)'$$

where

$$(\text{Lat } T)' = \{\{0\}, L^2(\partial D, dm)\}$$

$$\cup \{uH^2 : u \in L^\infty(\partial D, dm), |u(e^{ix})| = 1 \text{ (a.e., } dm)\}$$

(H^2 is a subspace of L^2 spanned by the orthonormal set $\{e^{inx}\}_{n=0}^\infty$; $uH^2 = vH^2$ if and only if $u\bar{v}$ is constant a.e.). We have:

(i) By Theorem 1, $\text{Lat } \mathcal{Q}_T^a$ and $(\text{Lat } T)'$ are clopen subsets of $\text{Lat } T$; $\text{Lat } \mathcal{Q}_T^a$ is actually a boolean algebra and $\hat{d}(L^2(M_1), L^2(M_2)) = 1$ whenever $L^2(M_1) \neq L^2(M_2)$. The spectrum of T is equal to ∂D and the constant $r(T, 0)$ can be chosen as being equal to 1.

(ii) $(\text{Lat } T)'$ is another sublattice of $\text{Lat } T$. The topological properties of $(\text{Lat } T)'$ are very far from those of $\text{Lat } \mathcal{Q}_T^a$. Indeed, $\{uH^2 : u \in L^\infty, |u(e^{ix})| = 1 \text{ (a.e.)}\}$ is an *arcwise connected* subset of $\text{Lat } T$.

(iii) The operator theoretical properties of these two lattices are also very different. In fact, $\text{Lat } \mathcal{Q}_T^a$ is a *reflexive* lattice in the sense of H. Radjavi and P. Rosenthal [8]. On the contrary, if $A \in \mathcal{L}(L^2)$ leaves invariant every subspace in $(\text{Lat } T)'$, then (see [4, §3]) $A \in \mathcal{Q}_T$ and, therefore, $\text{Lat } A \subset \text{Lat } T \neq (\text{Lat } T)'$; i.e., $(\text{Lat } T)'$ is not reflexive.

(iv) Let χ be the characteristic function of the upper half part of ∂D and let $\mathfrak{N}_1 = H^2$, $\mathfrak{N}_2 = (1 - 2\chi)H^2$. Then [2], [3] $\mathfrak{N}_1, \mathfrak{N}_2 \in \text{Lat } T \setminus \text{Lat } \mathcal{Q}_T^a$, $\mathfrak{N}_1 \cap \mathfrak{N}_2 = \{0\}$ and closure $(\mathfrak{N}_1 + \mathfrak{N}_2) = L^2$ (this property implies that the hypothesis " $\mathfrak{N}_1 + \mathfrak{N}_2$ is closed" of Theorem 4 below cannot be relaxed).

2. Analytically invariant subspaces and the spectrum of T . Our next two theorems are consequences of the results contained in [5, §§2 and 6]. Recall that $T \in \mathcal{L}(\mathcal{X})$ is a *semi-Fredholm operator* if it has closed range and either $\dim \ker T$ or $\text{codim } \text{ran } T = \dim \mathcal{X} / T\mathcal{X}$ is finite; in that case, the *index* of T is defined by $\text{ind } T = \dim \ker T - \text{codim } \text{ran } T$. The reader is referred to [6, Chapter IV] for the properties of the semi-Fredholm operators.

THEOREM 3. *Let $T \in \mathcal{L}(\mathcal{X})$ and let $\mathfrak{N} \in \text{Lat } T$. If $\pi: \mathcal{X} \rightarrow \mathcal{X}/\mathfrak{N}$ is the canonical projection of \mathcal{X} onto the quotient space, $\bar{T} \in \mathcal{L}(\mathcal{X}/\mathfrak{N})$ is defined by $\bar{T}(\pi x) = \pi Tx$ and $R = T|_{\mathfrak{N}}$ is the restriction of T to \mathfrak{N} , then*

$$\sigma(T) \cup \sigma(R) = \sigma(T) \cup \sigma(\bar{T}) = \sigma(R) \cup \sigma(\bar{T}).$$

If $\lambda \in \sigma(R) \setminus \sigma(T)$ or $\lambda \in \sigma(\bar{T}) \setminus \sigma(T)$, then $\lambda \in \sigma(R) \cap \sigma(\bar{T})$, $R - \lambda$ and $\bar{T} - \lambda$ are semi-Fredholm operators and $\text{ind}(\bar{T} - \lambda) = -\text{ind}(R - \lambda) > 0$. In particular, $\sigma(T) = \sigma(R) \cup \sigma(\bar{T})$ if and only if $\mathfrak{N} \in \text{Lat } \mathcal{A}_T^a$.

PROOF. Assume that $0 \notin \sigma(R) \cup \sigma(\bar{T})$. Since \bar{T} is invertible, given $x \in \mathcal{X}$ there exists $y \in \mathcal{X}$ such that $\pi x = \bar{T}\pi y = \pi Ty$; hence, $z = Ty - x \in \mathfrak{N}$. Since R is invertible, $z = Rw$ for some $w \in \mathfrak{N}$. It follows that $x = Ty - Rw = T(y - w)$; therefore, T maps \mathcal{X} onto \mathcal{X} .

On the other hand, if $Tx = 0$, then $\pi Tx = \bar{T}\pi x = 0$, and the invertibility of \bar{T} implies that $\pi x = 0$, i.e., $x \in \mathfrak{N}$. Finally, since R is also invertible, $Rx = Tx = 0$ implies that $x = 0$. We conclude that T is invertible. Replacing T by $T - \lambda$ for each $\lambda \notin \sigma(R) \cup \sigma(\bar{T})$, it follows that $\sigma(T) \subset \sigma(R) \cup \sigma(\bar{T})$. A fortiori $\sigma(T) \cup \sigma(R)$ and $\sigma(T) \cup \sigma(\bar{T})$ are also contained in $\sigma(R) \cup \sigma(\bar{T})$.

Assume that $\lambda \in \sigma(R) \cup \sigma(\bar{T}) \setminus \sigma(T)$; then either $\lambda \in \sigma(R) \setminus \sigma(T)$ or $\lambda \in \sigma(\bar{T}) \setminus \sigma(T)$. In both cases the conclusion is the same: $\mathfrak{N} \notin \text{Lat } \mathcal{A}_T^a$ and $\lambda \in \sigma(R) \cap \sigma(\bar{T})$ (see [5, Lemmas 2.2 and 6.3]). Therefore $\sigma(T) \cup \sigma(R) \supset \sigma(R) \cup \sigma(\bar{T})$ and $\sigma(T) \cup \sigma(\bar{T}) \supset \sigma(R) \cup \sigma(\bar{T})$, whence we obtain the equalities of the first statement; moreover, the same arguments show that $\sigma(T) = \sigma(R) \cup \sigma(\bar{T})$ if and only if $\mathfrak{N} \in \text{Lat } \mathcal{A}_T^a$.

Since $\lambda \notin \sigma(T)$, $(T - \lambda)\mathfrak{N}$ is closed; in fact, it is a *proper* subspace of \mathfrak{N} , and therefore $(R - \lambda)$ is a semi-Fredholm operator of negative index because $\ker(R - \lambda) \subset \ker(T - \lambda) = \{0\}$. On the other hand, $(T - \lambda)$ maps \mathcal{X} onto \mathcal{X} and, therefore, $(\bar{T} - \lambda)$ maps \mathcal{X}/\mathfrak{N} onto \mathcal{X}/\mathfrak{N} , i.e., $(\bar{T} - \lambda)$ is a semi-Fredholm operator of positive index. Finally, observe that $\ker(\bar{T} - \lambda) = (T - \lambda)^{-1}\mathfrak{N}/\mathfrak{N}$ is isomorphic to $\mathfrak{N}/(T - \lambda)\mathfrak{N} = \mathfrak{N}/(R - \lambda)\mathfrak{N}$ and, therefore, $\text{ind}(\bar{T} - \lambda) = -\text{ind}(R - \lambda) > 0$. \square

THEOREM 4. *Let $T \in \mathcal{L}(\mathcal{X})$ and let $\mathfrak{N}_1, \mathfrak{N}_2 \in \text{Lat } T$. Assume that $\mathfrak{N}_3 = \mathfrak{N}_1 + \mathfrak{N}_2$ is closed in \mathcal{X} and let $\mathfrak{N}_0 = \mathfrak{N}_1 \cap \mathfrak{N}_2$. Then*

$$\begin{aligned} \sigma(T_0) \cup \sigma(T_3) &= \sigma(T_0) \cup \sigma(T_1) \cup \sigma(T_2) \\ &= \sigma(T_1) \cup \sigma(T_2) \cup \sigma(T_3), \end{aligned}$$

where $T_j = T|_{\mathfrak{N}_j}$, $j = 0, 1, 2, 3$. Moreover, if $\lambda \in [\sigma(T_0) \cup \sigma(T_3)] \setminus [\sigma(T_1) \cup \sigma(T_2)]$, then $\lambda \in \sigma(T_0) \cap \sigma(T_3)$, $T_0 - \lambda$ and $T_3 - \lambda$ are semi-Fredholm operators and $\text{ind}(T_3 - \lambda) = -\text{ind}(T_0 - \lambda) > 0$.

PROOF. Let $R = T|_{\mathfrak{N}_3}$, let \bar{T} be the operator induced by T on $\mathcal{X}/\mathfrak{N}_0$ (as in Theorem 3 above) and let $\bar{R} = \bar{T}|_{\mathfrak{N}_3/\mathfrak{N}_0}$. Then the fact that \mathfrak{N}_3 is

closed implies that $\mathfrak{N}_3/\mathfrak{N}_0 = \mathfrak{N}_1/\mathfrak{N}_0 \oplus \mathfrak{N}_2/\mathfrak{N}_0$ (algebraic direct sum).

According to [5, Theorems. 6.1 and 6.2], $\mathfrak{N}_1, \mathfrak{N}_2 \in \text{Lat } R$ and $\mathfrak{N}_1/\mathfrak{N}_0, \mathfrak{N}_2/\mathfrak{N}_0 \in \text{Lat } \bar{R}$; moreover, it is clear that \bar{R} commutes with the projection of $\mathfrak{N}_3/\mathfrak{N}_0$ onto $\mathfrak{N}_1/\mathfrak{N}_0$ along $\mathfrak{N}_2/\mathfrak{N}_0$ and, therefore, $\mathfrak{N}_1/\mathfrak{N}_0, \mathfrak{N}_2/\mathfrak{N}_0 \in \text{Lat } \mathcal{Q}_T'' \subset \text{Lat } \mathcal{Q}_T^a$ (where $\mathcal{Q}_T'' = \{S \in \mathcal{L}(\mathcal{X}): SV = VS \text{ for all } V \in \mathcal{L}(\mathcal{X}) \text{ commuting with } L\}$ denotes the double commutant of $L \in \mathcal{L}(\mathcal{X})$; see [5, Lemma 2.3]). Hence

$$\sigma(\bar{R}) = \sigma(\bar{R}|\mathfrak{N}_1/\mathfrak{N}_0) \cup \sigma(\bar{R}|\mathfrak{N}_2/\mathfrak{N}_0).$$

By applying Theorem 3 to T_1, T_2 and T_3 , we obtain

$$\begin{aligned} \sigma(T_0) \cup \sigma(T_3) &= \sigma(T_0) \cup \sigma(\bar{R}) \\ &= \sigma(T_0) \cup \sigma(\bar{R}|\mathfrak{N}_1/\mathfrak{N}_0) \cup \sigma(\bar{R}|\mathfrak{N}_2/\mathfrak{N}_0) \\ &= \bigcup_{j=0}^2 \sigma(T_j) = \bigcup_{j=0}^3 \sigma(T_j). \end{aligned}$$

Let $\lambda \in [\sigma(T_0) \cup \sigma(T_3)] \setminus [\sigma(T_1) \cup \sigma(T_2)]$. Since $\bigcup_{j=0}^2 \sigma(T_j) = \bigcup_{j=0}^3 \sigma(T_j)$, it follows that $\lambda \in \sigma(T_0)$. On the other hand, $(T - \lambda)\mathfrak{N}_0$ is closed and $\ker(T_0 - \lambda) = \{0\}$, because $T_1 - \lambda$ is invertible and \mathfrak{N}_0 is a subspace of \mathfrak{N}_1 ; therefore $T_0 - \lambda$ is a semi-Fredholm operator of negative index.

Consider the map $W: \mathfrak{N}_1 \oplus \mathfrak{N}_2 \rightarrow \mathfrak{N}_3$ defined by $W(x_1, x_2) = x_1 - x_2$. Clearly, $\lambda \in \sigma(T_1 \oplus T_2) = \sigma(T_1) \cup \sigma(T_2)$, $\ker W = \{(x_0, x_0): x_0 \in \mathfrak{N}_0\}$ is an "isometrically isomorphic copy" of \mathfrak{N}_0 and $\text{ind}(T_1 \oplus T_2|_{\ker W} - \lambda) = \text{ind}(T_0 - \lambda)$. By using the canonical isomorphism between $\mathfrak{N}_3 = \text{ran } W$ and $\mathfrak{N}_1 \oplus \mathfrak{N}_2/\ker W$ and applying Theorem 3, we conclude that $\lambda \in \sigma(T_3)$ and, moreover, that $T_3 - \lambda$ is a semi-Fredholm operator with index $\text{ind}(T_3 - \lambda) = -\text{ind}(T_0 - \lambda) > 0$. This proves, in particular, that $\bigcup_{j=0}^2 \sigma(T_j) = \bigcup_{j=1}^3 \sigma(T_j)$. \square

COROLLARY 5. *Let \mathfrak{N}_1 and \mathfrak{N}_2 be two invariant subspaces of T satisfying the hypotheses of Theorem 4 and assume, moreover, that $\mathfrak{N}_0, \mathfrak{N}_3 \in \text{Lat } \mathcal{Q}_T^a$. Then $\mathfrak{N}_1, \mathfrak{N}_2 \in \text{Lat } \mathcal{Q}_T^a$.*

PROOF. Our hypotheses on $\mathfrak{N}_0, \mathfrak{N}_3$, Theorem 4 and Lemma 2.2 of [5] imply that $\sigma(T_1) \cup \sigma(T_2) \subset \sigma(T_0) \cup \sigma(T_3) \subset \sigma(T)$. Hence $\sigma(T_j) \subset \sigma(T)$, $j = 1, 2$, and (according to [5, Lemma 2.2]) this is equivalent to $\mathfrak{N}_1, \mathfrak{N}_2 \in \text{Lat } \mathcal{Q}_T^a$. \square

The arguments of the proof of Theorem 4 can be applied to other situations; namely, we have

THEOREM 6. *Let $\{\mathfrak{N}_\nu: \nu \in \Phi\}$ be an arbitrary family of invariant subspaces of $T \in \mathcal{L}(\mathcal{X})$ and assume that $\mathcal{X} = \sum_{\nu \in \Phi} \mathfrak{N}_\nu$, the algebraic sum of the \mathfrak{N}_ν 's. Then every $\lambda \in \sigma(T) \setminus \bigcup_{\nu} \sigma(T|_{\mathfrak{N}_\nu})$ is an interior point of $\sigma(T)$ such that*

$T - \lambda$ is a semi-Fredholm operator of positive index.

PROOF. Let $\lambda \in \sigma(T) \setminus \bigcup \sigma(T|_{\mathfrak{M}_\nu})$. Then $(T - \lambda)\mathfrak{M}_\nu = \mathfrak{M}_\nu$ for all $\nu \in \Phi$ and, therefore,

$$(T - \lambda)\mathfrak{X} = (T - \lambda) \sum_{\nu} \mathfrak{M}_\nu = \sum_{\nu} (T - \lambda)\mathfrak{M}_\nu = \sum_{\nu} \mathfrak{M}_\nu = \mathfrak{X},$$

i.e. $T - \lambda$ maps \mathfrak{X} onto \mathfrak{X} . Hence, $T - \lambda$ is a semi-Fredholm operator of positive index and, therefore (see [6, Chapter IV]), λ is an interior point of $\sigma(T)$. \square

EXAMPLE B. Let T be as in Example A, let $S = T|_{H^2}$ (the unilateral shift) and set $L = T^* \oplus S^*$ (where L^* denotes the adjoint of the operator L) acting in the usual fashion on the orthogonal direct sum $\mathfrak{X} = L^2 \oplus H^2$. Then we can decompose

$$\begin{aligned} \mathfrak{X} &= [(H^2)^\perp \oplus H^2] \oplus H^2 \\ &= (H^2)^\perp \oplus \{(f, f): f \in H^2\} \oplus \{(f, -f): f \in H^2\} \\ &= \mathfrak{M}_1 + \mathfrak{M}_2, \end{aligned}$$

where $\mathfrak{M}_1 = (H^2)^\perp \oplus \{(f, f): f \in H^2\}$ and $\mathfrak{M}_2 = (H^2)^\perp \oplus \{(f, -f): f \in H^2\}$. Straightforward computations show that \mathfrak{M}_1 and \mathfrak{M}_2 are invariant under L ; $L|_{\mathfrak{M}_1}$ and $L|_{\mathfrak{M}_2}$ are similar to T and, therefore, $\sigma(T|_{\mathfrak{M}_1}) = \sigma(T|_{\mathfrak{M}_2}) = \partial D$. However, $\mathfrak{M}_0 = \mathfrak{M}_1 \cap \mathfrak{M}_2 = (H^2)^\perp \oplus \{0\}$ and $L|_{\mathfrak{M}_0}$ is unitarily equivalent to S ; hence $\sigma(L|_{\mathfrak{M}_0}) = \sigma(L|_{\mathfrak{M}_3}) = D^-$ ($\mathfrak{M}_3 = \mathfrak{X}$).

EXAMPLE C. Now set $B = T \oplus T$ acting on $L^2 \oplus L^2$. Then $\sigma(B) = \partial D$; $\mathfrak{M}_1 = L^2 \oplus H^2$ and $\mathfrak{M}_2 = H^2 \oplus L^2$ belong to $\text{Lat } B \setminus \text{Lat } \mathcal{A}_B^a$, because $\sigma(B|_{\mathfrak{M}_1}) = \sigma(B|_{\mathfrak{M}_2}) = D^-$ is not included in ∂D . This example shows that, in general, from $\mathfrak{X} = \mathfrak{M}_1 + \mathfrak{M}_2$, $\mathfrak{M}_1 \neq \mathfrak{X} \neq \mathfrak{M}_2$, we cannot conclude that $\sigma(B) \supset \sigma(B|_{\mathfrak{M}_1})$.

3. Algebras between \mathcal{A}_T and \mathcal{A}_T^a . Let $T \in \mathcal{L}(\mathfrak{X})$, let Λ be a subset of \mathbf{C} containing at most one point of each bounded component of $\rho(T)$, and let $\mathcal{A}_T(\Lambda)$ denote the weakly closed algebra generated by T and $\{(T - \lambda)^{-1}: \lambda \in \Lambda\}$ (for instance, $\mathcal{A}_T = \mathcal{A}_T(\emptyset)$). Then part of the results of [5, §6] and the above theorems can be extended to the algebras $\mathcal{A}_T(\Lambda)$ by using the same kind of arguments. Thus, we shall establish without proof the following:

THEOREM 7. (i) If $\mathfrak{M} \in \text{Lat } T$, $R = T|_{\mathfrak{M}}$ and \bar{T} is the operator induced by T on $\mathfrak{X}/\mathfrak{M}$, then the following are equivalent: (a) $\mathfrak{M} \in \text{Lat } \mathcal{A}_T(\Lambda)$; (b) $\Lambda \subset \rho(R)$; (c) $\Lambda \subset \rho(\bar{T})$; (d) $\Lambda \subset \rho(R) \cup \rho(\bar{T})$; (e) $\sigma(T; \mathcal{A}_T(\Lambda)) = \sigma(R) \cup \sigma(\bar{T})$.

(ii) If $\mathfrak{M} \in \text{Lat } \mathcal{A}_T(\Lambda)$ and \mathfrak{N} is a subspace of \mathfrak{M} , then $\mathfrak{N} \in \text{Lat } \mathcal{A}_R(\Lambda)$ implies that $\mathfrak{N} \in \text{Lat } \mathcal{A}_T(\Lambda)$.

(iii) If $\mathfrak{M} \in \text{Lat } \mathcal{Q}_T(\Lambda)$ and $\overline{\mathfrak{N}}$ is a subspace of $\mathfrak{X}/\mathfrak{M}$, then $\overline{\mathfrak{N}} \in \text{Lat } \mathcal{Q}_T(\Lambda)$ implies that $\mathfrak{N} = \pi^{-1}(\overline{\mathfrak{N}}) \in \text{Lat } \mathcal{Q}_T(\Lambda)$.

(iv) If Λ is finite, there exists a constant $s(T, \Lambda) > 0$, such that $\hat{d}(\mathfrak{M}, \mathfrak{N}) > s(T, \Lambda)$ for all $\mathfrak{M} \in \text{Lat } \mathcal{Q}_T(\Lambda)$ and all $\mathfrak{N} \in \text{Lat } T \setminus \text{Lat } \mathcal{Q}_T(\Lambda)$. In particular, $\text{Lat } \mathcal{Q}_T(\Lambda)$ is a clopen sublattice of $\text{Lat } T$. Moreover, both results remain true under the weaker assumption: Λ only intersects finitely many components of $\mathbb{C} \setminus \sigma(T; \mathcal{Q}_T(\Lambda))$.

(v) If $\mathfrak{M}_1, \mathfrak{M}_2 \in \text{Lat } T$, $\mathfrak{M}_0 = \mathfrak{M}_1 \cap \mathfrak{M}_2 \in \text{Lat } \mathcal{Q}_T(\Lambda)$ and $\mathfrak{M}_3 = \mathfrak{M}_1 + \mathfrak{M}_2$ is closed and belongs to $\text{Lat } \mathcal{Q}_T(\Lambda)$, then $\mathfrak{M}_1, \mathfrak{M}_2 \in \text{Lat } \mathcal{Q}_T(\Lambda)$.

Our last result says that each of the lattices $\text{Lat } \mathcal{Q}_T(\Lambda)$ is invariant under “small perturbations of the dimension”. In fact, we have

THEOREM 8. *If $\mathfrak{M}, \mathfrak{N} \in \text{Lat } T$, $\mathfrak{M} \subset \mathfrak{N}$ and $\dim \mathfrak{N}/\mathfrak{M} < \infty$, then $\mathfrak{M} \in \text{Lat } \mathcal{Q}_T(\Lambda)$ if and only if $\mathfrak{N} \in \text{Lat } \mathcal{Q}_T(\Lambda)$. Moreover, $\sigma(T|\mathfrak{N}) = \sigma(T|\mathfrak{M}) \cup \{\lambda_1, \dots, \lambda_n\}$, where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of the finite dimensional operator induced by $T|\mathfrak{N}$ on $\mathfrak{N}/\mathfrak{M}$ and $\sigma(\overline{T}_{\mathfrak{M}}) = \sigma(\overline{T}_{\mathfrak{N}}) \cup \{\lambda_1, \dots, \lambda_n\}$, where $\overline{T}_{\mathfrak{M}}$ and $\overline{T}_{\mathfrak{N}}$ denote the operators induced by T on $\mathfrak{X}/\mathfrak{M}$ and $\mathfrak{X}/\mathfrak{N}$, respectively. In particular, if $\mathfrak{M} \in \text{Lat } T$ and $\dim \mathfrak{N} < \infty$ or $\text{codim } \mathfrak{N} < \infty$, then $\mathfrak{N} \in \text{Lat } \mathcal{Q}_T^a$.*

PROOF. By applying Theorem 3 to $T|\mathfrak{N}$, we obtain

$$\sigma(T|\mathfrak{N}) \cup \sigma(T|\mathfrak{M}) = \sigma(T|\mathfrak{M}) \cup \{\lambda_1, \dots, \lambda_n\} = \sigma(T|\mathfrak{N}) \cup \{\lambda_1, \dots, \lambda_n\}$$

Since $\{\lambda_1, \dots, \lambda_n\}$, the spectrum of the operator $(T|\mathfrak{N})^-$ induced by $T|\mathfrak{N}$ on $\mathfrak{N}/\mathfrak{M}$, is a finite set, we have that $\sigma((T|\mathfrak{N})^-) = \partial\sigma((T|\mathfrak{N})^-) \subset \sigma(T|\mathfrak{N})$ (see [5, §2]), we conclude that $\sigma(T|\mathfrak{N}) = \sigma(T|\mathfrak{M}) \cup \{\lambda_1, \dots, \lambda_n\}$.

Similarly, by applying Theorem 3 to $\overline{T}_{\mathfrak{M}}$, we obtain the equality $\sigma(\overline{T}_{\mathfrak{M}}) = \sigma(\overline{T}_{\mathfrak{N}}) \cup \{\lambda_1, \dots, \lambda_n\}$ (to see this, observe that $(T|\mathfrak{N})^-$ coincides with $\overline{T}_{\mathfrak{M}}|(\mathfrak{N}/\mathfrak{M})$). We have proved that the symmetric difference $\sigma(T|\mathfrak{N}) \Delta \sigma(T|\mathfrak{M})$ is contained in the finite set $\{\lambda_1, \dots, \lambda_n\}$; thus, since $\sigma(T|\mathfrak{N}) \Delta \sigma(T|\mathfrak{M})$ cannot contain a component of $\rho(T)$, \mathfrak{M} and \mathfrak{N} belong to exactly the same lattices $\text{Lat } \mathcal{Q}_T(\Lambda)$ (where Λ runs over all possible sets of the above described type).

If $\dim \mathfrak{M} < \infty$ ($\text{codim } \mathfrak{M} < \infty$, resp.), then $\mathfrak{M} \in \text{Lat } \mathcal{Q}_T^a$ because $\dim \mathfrak{M}/\{0\} < \infty$ and $\{0\} \in \text{Lat } \mathcal{Q}_T^a$ ($\dim \mathfrak{X}/\mathfrak{M} < \infty$ and $\mathfrak{X} \in \text{Lat } \mathcal{Q}_T^a$, resp.) \square

It is interesting to observe that not every finite dimensional (or finite codimensional) invariant subspace of T is *bi-invariant*, i.e., invariant under the double commutant \mathcal{Q}_T'' of T . Indeed, we have the following counterexample, inspired in a paper of A. L. Lambert and T. R. Turner:

EXAMPLE D. Let S_α and S_β be injective unilateral weighted shifts in l^2 such that the operator $L = S_\alpha^* \oplus S_\beta^* \in \mathcal{L}(l^2 \oplus l^2)$ satisfies the relations $\mathcal{Q}_L = \mathcal{Q}_L^\alpha \neq \mathcal{Q}_L'' = \mathcal{Q}_L'$ (for a concrete numerical example, see [7]). Assume that S_α (S_β , resp.) is defined in the usual way with respect to the orthonormal basis $\{e_n\}_{n=0}^\infty$ of $l^2 \oplus \{0\}$ ($\{f_n\}_{n=0}^\infty$ of $\{0\} \oplus l^2$, resp.). Then $\ker L = \{\lambda e_0 + \mu f_0 : \lambda, \mu \in \mathbb{C}\} \in \text{Lat } \mathcal{Q}_L' = \text{Lat } \mathcal{Q}_L''$ (i.e. $\ker L$ is actually a *hyperinvariant* subspace), and the orthogonal projection P of $l^2 \oplus l^2$ onto $l^2 \oplus \{0\}$ belongs to \mathcal{Q}_L'' . Clearly, every one-dimensional subspace of $\ker L$ belongs to $\text{Lat } \mathcal{Q}_L'$. On the other hand, since $P \in \mathcal{Q}_L''$, $\text{Lat } \mathcal{Q}_L''$ splits with respect to the above direct sum decomposition, i.e., if $\mathfrak{N} \in \text{Lat } \mathcal{Q}_L''$, then $\mathfrak{N} = P\mathfrak{N} \oplus (I - P)\mathfrak{N}$ (see [5, §5]). It is not hard to conclude that L has exactly two one-dimensional bi-invariant subspaces: the ones generated by e_0 and by f_0 .

Furthermore, this example answers strongly in the negative a question raised in [5, §6]:

(i) $\ker L \in \text{Lat } \mathcal{Q}_L'$ and $\{\lambda(e_0 + f_0) : \lambda \in \mathbb{C}\} \in \mathcal{Q}_L''_{|\ker L}$. However, $\{\lambda(e_0 + f_0)\} \notin \text{Lat } \mathcal{Q}_L''$.

(ii) Let S_α and S_β be as above and set $Q = S_\alpha \oplus S_\beta \in \mathcal{L}(l^2 \oplus l^2)$. Then the subspace spanned by $\{e_n, f_n\}_{n=1}^\infty$ is equal to $\mathfrak{N} = \overline{\text{ran } Q} \in \text{Lat } \mathcal{Q}_Q'$ and $\overline{\mathfrak{N}} = \mathfrak{N} \oplus \{\lambda(e_0 + f_0) : \lambda \in \mathbb{C}\} / \mathfrak{N} \in \text{Lat } \mathcal{Q}_Q''$, where $\overline{Q} = 0$ is the operator induced by Q on $l^2 \oplus l^2 / \mathfrak{N}$. However, $\overline{\mathfrak{N}} = \pi^{-1}(\overline{\mathfrak{N}}) = \mathfrak{N} \oplus \{\lambda(e_0 + f_0) : \lambda \in \mathbb{C}\} \notin \text{Lat } \mathcal{Q}_Q''$.

Let $T \in \mathcal{L}(\mathfrak{X})$ and let $\mathfrak{M}, \mathfrak{N} \in \text{Lat } \mathcal{Q}_T''$, $\mathfrak{M} \subset \mathfrak{N}$. Whether or not $\mathfrak{M} \in \text{Lat } \mathcal{Q}_{T|_{\mathfrak{M}}}''$ (and, similarly, $\mathfrak{N}/\mathfrak{M} \in \text{Lat } \mathcal{Q}_{\overline{T}}''$, where \overline{T} is the operator induced by T on $\mathfrak{X}/\mathfrak{M}$) is an open problem, even under the stronger assumption $\mathfrak{M} \in \text{Lat } \mathcal{Q}_T'$.

REFERENCES

1. R. G. Douglas and C. Pearcy, *On a topology for invariant subspaces*, J. Functional Analysis 2 (1968), 323–341.
2. P. R. Halmos, *A Hilbert space problem book*, Van Nostrand, Princeton, N. J., 1967.
3. H. Helson, *Lectures on invariant subspaces*, Academic Press, New York, 1964.
4. D. A. Herrero and A. L. Lambert, *On strictly cyclic algebras, \mathfrak{P} -algebras and reflexive operators*, Trans. Amer. Math. Soc. 185 (1975), 229–235.
5. D. A. Herrero and N. Salinas, *Analytically invariant and bi-invariant subspaces*, Trans. Amer. Math. Soc. 173 (1972), 117–136.
6. T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, New York, 1966.
7. A. L. Lambert and T. R. Turner, *The double commutants of invertible shifts*, Duke Math. J. 39 (1972), 385–389.
8. H. Radjavi and P. Rosenthal, *On invariant subspaces and reflexive algebras*, Amer. J. Math. 91 (1969), 683–692.

UNIVERSIDADE ESTADUAL DE CAMPINAS, IMECC, CAMPINAS, SP, BRAZIL

UNIVERSIDAD NACIONAL DE RÍO IV⁰, CORDOBA, ARGENTINA

Current address: Department of Mathematics, IVIC, AP 1827, Caracas 101, Venezuela