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ON ANALYTICALLY INVARIANT SUBSPACES AND SPECTRA

BY

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ABSTRACT. Let T be a bounded linear operator from a complex Banach space $\mathfrak{X}$ into itself. Let $\mathcal{A}_T$ and $\mathcal{A}_T^a$ denote the weak closure of the polynomials and the rational functions (with poles outside the spectrum $\sigma(T)$ of T) in T, respectively. The lattice Lat $\mathcal{A}_T^a$ of (closed) invariant subspaces of $\mathcal{A}_T^a$ is a very particular subset of the invariant subspace lattice Lat $\mathcal{A}_T$ = Lat T of T. It is shown that: (1) If the resolvent set of T has finitely many components, then Lat $\mathcal{A}_T^a$ is a clopen (i.e., closed and open) sublattice of Lat T, with respect to the "gap topology" between subspaces. (2) If $\mathfrak{M}_1, \mathfrak{M}_2 \in$ Lat T, $\mathfrak{M}_1 \cap \mathfrak{M}_2 \in$ Lat $\mathcal{A}_T^a$ and $\mathfrak{M}_1 + \mathfrak{M}_2$ is closed in $\mathfrak{X}$ and belongs to Lat $\mathcal{A}_T^a$, then $\mathfrak{M}_1$ and $\mathfrak{M}_2$ also belong to Lat $\mathcal{A}_T^a$. (3) If $\mathfrak{M} \in$ Lat T, R is the restriction of T to $\mathfrak{M}$ and $\bar{T}$ is the operator induced by T on the quotient space $\mathfrak{X}/\mathfrak{M}$, then $\sigma(T) \subset \sigma(R) \cup \sigma(\bar{T})$. Moreover, $\sigma(T) = \sigma(R) \cup \sigma(\bar{T})$ if and only if $\mathfrak{M} \in$ Lat $\mathcal{A}_T^a$. The results also include an analysis of the semi-Fredholm index of R and $\bar{T}$ at a point $\lambda \in \sigma(R) \cup \sigma(\bar{T}) \setminus \sigma(T)$ and extensions of the results to algebras between $\mathcal{A}_T$ and $\mathcal{A}_T^a$.

1. **Properties of the lattice** Lat $\mathcal{A}_T^a$. The study of the lattice Lat $\mathcal{A}_T^a$ (the *analytically invariant* subspaces of T) began in [5]. This article is based on and complements the results contained there. In what follows, $\mathfrak{X}$ will denote a Banach space over the complex field $\mathbf{C}$; *operator* and *subspace* will mean *bounded linear map* from a Banach space into itself and *closed linear manifold*, respectively. We shall consider invariant subspace lattices under the topology induced by the "gap between subspaces", i.e., the metric in the family of all subspaces of $\mathfrak{X}$ defined by $\hat{d}(\mathfrak{X}_1, \mathfrak{X}_2)$ = Hausdorff distance between the closed unit ball of the subspace $\mathfrak{X}_1$ and the closed unit ball of the subspace $\mathfrak{X}_2$. The Banach algebra of all operators in $\mathfrak{X}$ will be denoted by $\mathcal{L}(\mathfrak{X})$.

Let Σ be a subset of $\mathcal{L}(\mathfrak{X})$. It is well known (see [1]; [5]; [6, Chapter IV]) that (Lat Σ, $\hat{d}$) is a complete metric space; therefore, Lat $\mathcal{A}_T^a$ is always a closed subset of Lat T. We shall show that, under suitable restrictions on the spectrum of T, Lat $\mathcal{A}_T^a$ is also open in Lat T.

THEOREM 1. *Let $T \in \mathcal{L}(\mathfrak{X})$. For each λ in the resolvent set $\rho(T) = \mathbf{C} \setminus \sigma(T)$ of T, there exists a constant $r(T, \lambda) > 0$ such that $\hat{d}(\mathfrak{M}, \mathfrak{N}) \geq r(T, \lambda)$ for all*

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$\mathfrak{M} \in \operatorname{Lat} T \cap \operatorname{Lat}(T-\lambda)^{-1}$ *and all* $\mathfrak{N} \in \operatorname{Lat} T \setminus \operatorname{Lat}(T-\lambda)^{-1}$. *In particular*, $\operatorname{Lat} T \cap \operatorname{Lat}(T-\lambda)^{-1}$ *is a clopen sublattice of* $\operatorname{Lat} T$.

PROOF. For each pair of subspaces $\mathfrak{X}_1, \mathfrak{X}_2$ define

$$\delta(\mathfrak{X}_1, \mathfrak{X}_2) = \begin{cases} 0 & \text{if } \mathfrak{X}_1 = \{0\}, \\ \sup\{\operatorname{distance}(x_1, \mathfrak{X}_2) : x_1 \in \mathfrak{X}_1, \|x_1\| = 1\} & \text{if } \mathfrak{X}_1 \neq \{0\}, \end{cases}$$

and

$$\hat{\delta}(\mathfrak{X}_1, \mathfrak{X}_2) = \max\{\delta(\mathfrak{X}_1, \mathfrak{X}_2), \delta(\mathfrak{X}_2, \mathfrak{X}_1)\}.$$

Then (see [6, p. 198])

$$\hat{\delta}(\mathfrak{X}_1, \mathfrak{X}_2) \leq \hat{d}(\mathfrak{X}_1, \mathfrak{X}_2) \leq 2\hat{\delta}(\mathfrak{X}_1, \mathfrak{X}_2);$$

if $\mathfrak{X}_1 \supset \mathfrak{X}_2$ and $\mathfrak{X}_1 \neq \mathfrak{X}_2$, then it follows from Riesz' lemma that $\delta(\mathfrak{X}_1, \mathfrak{X}_2) = \hat{d}(\mathfrak{X}_1, \mathfrak{X}_2) = 1$.

Without loss of generality, we can assume that $\lambda = 0$, i.e., that T is invertible. Let $\mathfrak{M} \in \operatorname{Lat} T \cap \operatorname{Lat} T^{-1}$ and $\mathfrak{N} \in \operatorname{Lat} T \setminus \operatorname{Lat} T^{-1}$; then (see [5]) $\mathfrak{M} = T\mathfrak{M}$, while $T\mathfrak{N} \subset \mathfrak{N}$ but $\mathfrak{N} \neq T\mathfrak{N}$. We have

$$\begin{aligned} 1 = \hat{d}(\mathfrak{N}, T\mathfrak{N}) &\leq \hat{d}(\mathfrak{N}, \mathfrak{M}) + \hat{d}(\mathfrak{M}, T\mathfrak{N}) \\ &\leq \hat{d}(\mathfrak{N}, \mathfrak{N})(1 + 2\|T\| \cdot \|T^{-1}\|), \end{aligned}$$

where the first equality follows from Riesz' lemma, the first inequality is just the triangular inequality for the metric $\hat{d}$, and the second one follows from the relations between $\hat{\delta}$ and $\hat{d}$ and Lemma 4.2 of [5], which implies that

$$\hat{\delta}(\mathfrak{M}, T\mathfrak{N}) = \hat{\delta}(T\mathfrak{M}, T\mathfrak{N}) \leq \|T\| \cdot \|T^{-1}\| \hat{\delta}(\mathfrak{M}, \mathfrak{N}).$$

Therefore,

$$\hat{d}(\mathfrak{M}, \mathfrak{N}) \geq r(T, 0) = (1 + 2\|T\| \cdot \|T^{-1}\|)^{-1}.$$

The general case and the second statement follow immediately from this result. $\square$

COROLLARY 2. *Let* $\sigma(T; \mathcal{A}_T)$ *denote the spectrum of* T *in the Banach algebra* $\mathcal{A}_T$. *If* $\sigma(T; \mathcal{A}_T) \setminus \sigma(T)$ *has finitely many components, then* $\operatorname{Lat} \mathcal{A}_T^a$ *is a clopen sublattice of* $\operatorname{Lat} T$. *In the general case*, $\operatorname{Lat} \mathcal{A}_T^a$ *is a countable intersection of clopen sublattices of* $\operatorname{Lat} T$.

PROOF. It is enough to recall that $\mathcal{A}_T^a$ is generated by T and $\{(T-\lambda_n)^{-1}\}$, where the (possibly empty) countable set $\{\lambda_n\}$ has exactly one point in common with each bounded component of $\mathbf{C} \setminus \sigma(T; \mathcal{A}_T)$ [5]. Now the result follows immediately from Theorem 1. $\square$

REMARK. Since $\sigma(T) \subset \sigma(T; \mathcal{Q}_T)$ and $\sigma(T; \mathcal{Q}_T) \setminus \sigma(T)$ is the union of a (possibly empty) subfamily of bounded components of $\rho(T)$, it easily follows that Lat $\mathcal{Q}_T^a$ is clopen in Lat T whenever $\rho(T)$ has finitely many components (see [5] for details).

EXAMPLE A. Let T be the bilateral shift "multiplication by e^{ix}" acting on $L^2(\partial D, dm)$, where D denotes the unit disc of the complex plane, ∂D and D^- are the boundary and the closure of D, respectively, and $dm = dx/2\pi$ is the normalized Lebesgue measure on ∂D. Then (see [2], [3])

$$\operatorname{Lat} \mathcal{Q}_T^a = \{L^2(M, dm) \colon M \text{ is a measurable subset of } \partial D\}$$

($L^2(M_1, dm) = L^2(M_2, dm)$ if and only if $m(M_1 \triangle M_2) = 0$), and

$$\operatorname{Lat} T = \operatorname{Lat} \mathcal{Q}_T^a \cup (\operatorname{Lat} T)',$$

where

$$(\operatorname{Lat} T)' = \{\{0\}, L^2(\partial D, dm)\} \cup \{uH^2 \colon u \in L^\infty(\partial D, dm), |u(e^{ix})| = 1 \text{ (a.e., } dm)\}$$

(H^2 is a subspace of L^2 spanned by the orthonormal set $\{e^{inx}\}_{n=0}^{\infty}$; $uH^2 = vH^2$ if and only if $u\bar{v}$ is constant a.e.). We have:

(i) By Theorem 1, Lat $\mathcal{Q}_T^a$ and (Lat T)$'$ are clopen subsets of Lat T; Lat $\mathcal{Q}_T^a$ is actually a boolean algebra and $\hat{d}(L^2(M_1), L^2(M_2)) = 1$ whenever $L^2(M_1) \neq L^2(M_2)$. The spectrum of T is equal to ∂D and the constant $r(T, 0)$ can be chosen as being equal to 1.

(ii) (Lat T)$'$ is another sublattice of Lat T. The topological properties of (Lat T)$'$ are very far from those of Lat $\mathcal{Q}_T^a$. Indeed, $\{uH^2 \colon u \in L^\infty, |u(e^{ix})| = 1 \text{ (a.e.)}\}$ is an *arcwise connected* subset of Lat T.

(iii) The operator theoretical properties of these two lattices are also very different. In fact, Lat $\mathcal{Q}_T^a$ is a *reflexive* lattice in the sense of H. Radjavi and P. Rosenthal [8]. On the contrary, if $A \in \mathcal{L}(L^2)$ leaves invariant every subspace in (Lat T)$'$, then (see [4, §3])$A \in \mathcal{Q}_T$ and, therefore, Lat $A \subset$ Lat T $\neq$ (Lat T)$'$; i.e., (Lat T)$'$ is not reflexive.

(iv) Let χ be the characteristic function of the upper half part of ∂D and let $\mathfrak{M}_1 = H^2$, $\mathfrak{M}_2 = (1 - 2\chi)H^2$. Then [2], [3] $\mathfrak{M}_1, \mathfrak{M}_2 \in \operatorname{Lat} T \setminus \operatorname{Lat} \mathcal{Q}_T^a$, $\mathfrak{M}_1 \cap \mathfrak{M}_2 = \{0\}$ and closure $(\mathfrak{M}_1 + \mathfrak{M}_2) = L^2$ (this property implies that the hypothesis "$\mathfrak{M}_1 + \mathfrak{M}_2$ is closed" of Theorem 4 below cannot be relaxed).

2. **Analytically invariant subspaces and the spectrum of** T. Our next two theorems are consequences of the results contained in [5, §§2 and 6]. Recall that $T \in \mathcal{L}(\mathfrak{X})$ is a *semi-Fredholm operator* if it has closed range and either dim ker T or codim ran $T = \dim \mathfrak{X}/T\mathfrak{X}$ is finite; in that case, the *index* of T is defined by ind T = dim ker T-codim ran T. The reader is referred to [6, Chapter IV] for the properties of the semi-Fredholm operators.

THEOREM 3. *Let* $T \in \mathcal{L}(\mathfrak{X})$ *and let* $\mathfrak{M} \in \operatorname{Lat} T$. *If* $\pi\colon \mathfrak{X} \to \mathfrak{X}/\mathfrak{M}$ *is the canonical projection of* $\mathfrak{X}$ *onto the quotient space,* $\bar{T} \in \mathcal{L}(\mathfrak{X}/\mathfrak{M})$ *is defined by* $\bar{T}(\pi x) = \pi T x$ *and* $R = T|\mathfrak{M}$ *is the restriction of* T *to* $\mathfrak{M}$, *then*

$$\sigma(T) \cup \sigma(R) = \sigma(T) \cup \sigma(\bar{T}) = \sigma(R) \cup \sigma(\bar{T}).$$

If $\lambda \in \sigma(R) \setminus \sigma(T)$ *or* $\lambda \in \sigma(\bar{T}) \setminus \sigma(T)$, *then* $\lambda \in \sigma(R) \cap \sigma(\bar{T})$, $R - \lambda$ *and* $\bar{T} - \lambda$ *are semi-Fredholm operators and* $\operatorname{ind}(\bar{T} - \lambda) = -\operatorname{ind}(R - \lambda) > 0$. *In particular,* $\sigma(T) = \sigma(R) \cup \sigma(\bar{T})$ *if and only if* $\mathfrak{M} \in \operatorname{Lat} \mathcal{Q}_T^a$.

PROOF. Assume that $0 \notin \sigma(R) \cup \sigma(\bar{T})$. Since $\bar{T}$ is invertible, given $x \in \mathfrak{X}$ there exists $y \in \mathfrak{X}$ such that $\pi x = \bar{T}\pi y = \pi T y$; hence, $z = Ty - x \in \mathfrak{M}$. Since R is invertible, $z = Rw$ for some $w \in \mathfrak{M}$. It follows that $x = Ty - Rw = T(y - w)$; therefore, T maps $\mathfrak{X}$ *onto* $\mathfrak{X}$.

On the other hand, if $Tx = 0$, then $\pi T x = \bar{T}\pi x = 0$, and the invertibility of $\bar{T}$ implies that $\pi x = 0$, i.e., $x \in \mathfrak{M}$. Finally, since R is also invertible, $Rx = Tx = 0$ implies that $x = 0$. We conclude that T is invertible. Replacing T by $T - \lambda$ for each $\lambda \notin \sigma(R) \cup \sigma(\bar{T})$, it follows that $\sigma(T) \subset \sigma(R) \cup \sigma(\bar{T})$. A fortiori $\sigma(T) \cup \sigma(R)$ and $\sigma(T) \cup \sigma(\bar{T})$ are also contained in $\sigma(R) \cup \sigma(\bar{T})$.

Assume that $\lambda \in \sigma(R) \cup \sigma(\bar{T}) \setminus \sigma(T)$; then either $\lambda \in \sigma(R) \setminus \sigma(T)$ or $\lambda \in \sigma(\bar{T}) \setminus \sigma(T)$. In both cases the conclusion is the same: $\mathfrak{M} \notin \operatorname{Lat} \mathcal{Q}_T^a$ and $\lambda \in \sigma(R) \cap \sigma(\bar{T})$ (see [5, Lemmas 2.2 and 6.3]). Therefore $\sigma(T) \cup \sigma(R) \supset \sigma(R) \cup \sigma(\bar{T})$ and $\sigma(T) \cup \sigma(\bar{T}) \supset \sigma(R) \cup \sigma(\bar{T})$, whence we obtain the equalities of the first statement; moreover, the same arguments show that $\sigma(T) = \sigma(R) \cup \sigma(\bar{T})$ if and only if $\mathfrak{M} \in \operatorname{Lat} \mathcal{Q}_T^a$.

Since $\lambda \notin \sigma(T)$, $(T - \lambda)\mathfrak{M}$ is closed; in fact, it is a *proper* subspace of $\mathfrak{M}$, and therefore $(R - \lambda)$ is a semi-Fredholm operator of negative index because $\ker(R - \lambda) \subset \ker(T - \lambda) = \{0\}$. On the other hand, $(T - \lambda)$ maps $\mathfrak{X}$ *onto* $\mathfrak{X}$ and, therefore, $(\bar{T} - \lambda)$ maps $\mathfrak{X}/\mathfrak{M}$ onto $\mathfrak{X}/\mathfrak{M}$, i.e., $(\bar{T} - \lambda)$ is a semi-Fredholm operator of positive index. Finally, observe that $\ker(\bar{T} - \lambda) = (T - \lambda)^{-1}\mathfrak{M}/\mathfrak{M}$ is isomorphic to $\mathfrak{M}/(T - \lambda)\mathfrak{M} = \mathfrak{M}/(R - \lambda)\mathfrak{M}$ and, therefore, $\operatorname{ind}(\bar{T} - \lambda) = -\operatorname{ind}(R - \lambda) > 0$. □

THEOREM 4. *Let* $T \in \mathcal{L}(\mathfrak{X})$ *and let* $\mathfrak{M}_1, \mathfrak{M}_2 \in \operatorname{Lat} T$. *Assume that* $\mathfrak{M}_3 = \mathfrak{M}_1 + \mathfrak{M}_2$ *is closed in* $\mathfrak{X}$ *and let* $\mathfrak{M}_0 = \mathfrak{M}_1 \cap \mathfrak{M}_2$. *Then*

$$\begin{aligned}\sigma(T_0) \cup \sigma(T_3) &= \sigma(T_0) \cup \sigma(T_1) \cup \sigma(T_2)\\ &= \sigma(T_1) \cup \sigma(T_2) \cup \sigma(T_3),\end{aligned}$$

where $T_j = T|\mathfrak{M}_j$, $j = 0, 1, 2, 3$. *Moreover, if* $\lambda \in [\sigma(T_0) \cup \sigma(T_3)] \setminus [\sigma(T_1) \cup \sigma(T_2)]$, *then* $\lambda \in \sigma(T_0) \cap \sigma(T_3)$, $T_0 - \lambda$ *and* $T_3 - \lambda$ *are semi-Fredholm operators and* $\operatorname{ind}(T_3 - \lambda) = -\operatorname{ind}(T_0 - \lambda) > 0$.

PROOF. Let $R = T|\mathfrak{M}_3$, let $\bar{T}$ be the operator induced by T on $\mathfrak{X}/\mathfrak{M}_0$ (as in Theorem 3 above) and let $\bar{R} = \bar{T}|\mathfrak{M}_3/\mathfrak{M}_0$. Then the fact that $\mathfrak{M}_3$ is closed implies that $\mathfrak{M}_3/\mathfrak{M}_0 = \mathfrak{M}_1/\mathfrak{M}_0 \oplus \mathfrak{M}_2/\mathfrak{M}_0$ (*algebraic direct sum*).

According to [5, Theorems 6.1 and 6.2], $\mathfrak{M}_1, \mathfrak{M}_2 \in \operatorname{Lat} R$ and $\mathfrak{M}_1/\mathfrak{M}_0, \mathfrak{M}_2/\mathfrak{M}_0 \in \operatorname{Lat} \bar{R}$; moreover, it is clear that $\bar{R}$ commutes with the projection of $\mathfrak{M}_3/\mathfrak{M}_0$ onto $\mathfrak{M}_1/\mathfrak{M}_0$ along $\mathfrak{M}_2/\mathfrak{M}_0$ and, therefore, $\mathfrak{M}_1/\mathfrak{M}_0, \mathfrak{M}_2/\mathfrak{M}_0 \in \operatorname{Lat} \mathcal{Q}_T'' \subset \operatorname{Lat} \mathcal{Q}_T^a$ (where $\mathcal{Q}_L'' = \{S \in \mathcal{L}(\mathcal{X}): SV = VS$ for all $V \in \mathcal{L}(\mathcal{X})$ commuting with $L\}$ denotes the double commutant of $L \in \mathcal{L}(\mathcal{X})$; see [5, Lemma 2.3]). Hence

$$\sigma(\bar{R}) = \sigma\left(\bar{R}|\mathfrak{M}_1/\mathfrak{M}_0\right) \cup \sigma\left(\bar{R}|\mathfrak{M}_2/\mathfrak{M}_0\right).$$

By applying Theorem 3 to T_1, T_2 and T_3, we obtain

$$\begin{aligned}\sigma(T_0) \cup \sigma(T_3) &= \sigma(T_0) \cup \sigma(\bar{R})\\ &= \sigma(T_0) \cup \sigma\left(\bar{R}|\mathfrak{M}_1/\mathfrak{M}_0\right) \cup \sigma\left(\bar{R}|\mathfrak{M}_2/\mathfrak{M}_0\right)\\ &= \bigcup_{j=0}^{2} \sigma(T_j) = \bigcup_{j=0}^{3} \sigma(T_j).\end{aligned}$$

Let $\lambda \in [\sigma(T_0) \cup \sigma(T_3)] \setminus [\sigma(T_1) \cup \sigma(T_2)]$. Since $\bigcup_{j=0}^{2}\sigma(T_j) = \bigcup_{j=0}^{3}\sigma(T_j)$, it follows that $\lambda \in \sigma(T_0)$. On the other hand, $(T - \lambda)\mathfrak{M}_0$ is closed and $\ker(T_0 - \lambda) = \{0\}$, because $T_1 - \lambda$ is invertible and $\mathfrak{M}_0$ is a subspace of $\mathfrak{M}_1$; therefore $T_0 - \lambda$ is a semi-Fredholm operator of negative index.

Consider the map W: $\mathfrak{M}_1 \oplus \mathfrak{M}_2 \to \mathfrak{M}_3$ defined by $W(x_1, x_2) = x_1 - x_2$. Clearly, $\lambda \in \sigma(T_1 \oplus T_2) = \sigma(T_1) \cup \sigma(T_2)$, $\ker W = \{(x_0, x_0)\colon x_0 \in \mathfrak{M}_0\}$ is an "isometrically isomorphic copy" of $\mathfrak{M}_0$ and $\operatorname{ind}(T_1 \oplus T_2|_{\ker W} - \lambda) = \operatorname{ind}(T_0 - \lambda)$. By using the canonical isomorphism between $\mathfrak{M}_3 = \operatorname{ran} W$ and $\mathfrak{M}_1 \oplus \mathfrak{M}_2/\ker W$ and applying Theorem 3, we conclude that $\lambda \in \sigma(T_3)$ and, moreover, that $T_3 - \lambda$ is a semi-Fredholm operator with index $\operatorname{ind}(T_3 - \lambda) = -\operatorname{ind}(T_0 - \lambda) > 0$. This proves, in particular, that $\bigcup_{j=0}^{2}\sigma(T_j) = \bigcup_{j=1}^{3}\sigma(T_j)$. $\square$

COROLLARY 5. *Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be two invariant subspaces of T satisfying the hypotheses of* Theorem 4 *and assume, moreover, that $\mathfrak{M}_0, \mathfrak{M}_3 \in \operatorname{Lat} \mathcal{Q}_T^a$. Then $\mathfrak{M}_1, \mathfrak{M}_2 \in \operatorname{Lat} \mathcal{Q}_T^a$.*

PROOF. Our hypotheses on $\mathfrak{M}_0, \mathfrak{M}_3$, Theorem 4 and Lemma 2.2 of [5] imply that $\sigma(T_1) \cup \sigma(T_2) \subset \sigma(T_0) \cup \sigma(T_3) \subset \sigma(T)$. Hence $\sigma(T_j) \subset \sigma(T)$, $j = 1, 2$, and (according to [5, Lemma 2.2]) this is equivalent to $\mathfrak{M}_1, \mathfrak{M}_2 \in \operatorname{Lat} \mathcal{Q}_T^a$. $\square$

The arguments of the proof of Theorem 4 can be applied to other situations; namely, we have

THEOREM 6. *Let $\{\mathfrak{M}_\nu\colon \nu \in \Phi\}$ be an arbitrary family of invariant subspaces of $T \in \mathcal{L}(\mathcal{X})$ and assume that $\mathcal{X} = \sum_{\nu\in\Phi}\mathfrak{M}_\nu$, the algebraic sum of the $\mathfrak{M}_\nu$'s. Then every $\lambda \in \sigma(T) \setminus \bigcup_\nu \sigma(T|\mathfrak{M}_\nu)$ is an interior point of $\sigma(T)$ such that*

$T - \lambda$ is a semi-Fredholm operator of positive index.

PROOF. Let $\lambda \in \sigma(T) \setminus \bigcup_\nu \sigma(T|\mathfrak{M}_\nu)$. Then $(T-\lambda)\mathfrak{M}_\nu = \mathfrak{M}_\nu$ for all $\nu \in \Phi$ and, therefore,

$$(T-\lambda)\mathfrak{X} = (T-\lambda)\sum_\nu \mathfrak{M}_\nu = \sum_\nu (T-\lambda)\mathfrak{M}_\nu = \sum_\nu \mathfrak{M}_\nu = \mathfrak{X},$$

i.e. $T-\lambda$ maps $\mathfrak{X}$ *onto* $\mathfrak{X}$. Hence, $T-\lambda$ is a semi-Fredholm operator of positive index and, therefore (see [6, Chapter IV]), λ is an interior point of $\sigma(T)$. □

EXAMPLE B. Let T be as in Example A, let $S = T|H^2$ (the unilateral shift) and set $L = T^* \oplus S^*$ (where L^* denotes the adjoint of the operator L) acting in the usual fashion on the orthogonal direct sum $\mathfrak{X} = L^2 \oplus H^2$. Then we can decompose

$$\begin{aligned}\mathfrak{X} &= \left[(H^2)^\perp \oplus H^2\right] \oplus H^2\\ &= (H^2)^\perp \oplus \{(f,f): f \in H^2\} \oplus \{(f,-f): f \in H^2\}\\ &= \mathfrak{M}_1 + \mathfrak{M}_2,\end{aligned}$$

where $\mathfrak{M}_1 = (H^2)^\perp \oplus \{(f,f): f \in H^2\}$ and $\mathfrak{M}_2 = (H^2)^\perp \oplus \{(f,-f): f \in H^2\}$. Straightforward computations show that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are invariant under L; $L|\mathfrak{M}_1$ and $L|\mathfrak{M}_2$ are similar to T and, therefore, $\sigma(T|\mathfrak{M}_1) = \sigma(T|\mathfrak{M}_2) = \partial D$. However, $\mathfrak{M}_0 = \mathfrak{M}_1 \cap \mathfrak{M}_2 = (H^2)^\perp \oplus \{0\}$ and $L|\mathfrak{M}_0$ is unitarily equivalent to S; hence $\sigma(L|\mathfrak{M}_0) = \sigma(L|\mathfrak{M}_3) = D^-$ ($\mathfrak{M}_3 = \mathfrak{X}$).

EXAMPLE C. Now set $B = T \oplus T$ acting on $L^2 \oplus L^2$. Then $\sigma(B) = \partial D$; $\mathfrak{M}_1 = L^2 \oplus H^2$ and $\mathfrak{M}_2 = H^2 \oplus L^2$ belong to $\operatorname{Lat} B \setminus \operatorname{Lat} \mathfrak{A}_B^a$, because $\sigma(B|\mathfrak{M}_1) = \sigma(B|\mathfrak{M}_2) = D^-$ is not included in ∂D. This example shows that, in general, from $\mathfrak{X} = \mathfrak{M}_1 + \mathfrak{M}_2$, $\mathfrak{M}_1 \neq \mathfrak{X} \neq \mathfrak{M}_2$, we cannot conclude that $\sigma(B) \supset \sigma(B|\mathfrak{M}_1)$.

3. Algebras between $\mathfrak{A}_T$ and $\mathfrak{A}_T^a$.

Let $T \in \mathfrak{L}(\mathfrak{X})$, let Λ be a subset of $\mathbf{C}$ containing at most one point of each bounded component of $\rho(T)$, and let $\mathfrak{A}_T(\Lambda)$ denote the weakly closed algebra generated by T and $\{(T-\lambda)^{-1}: \lambda \in \Lambda\}$ (for instance, $\mathfrak{A}_T = \mathfrak{A}_T(\varnothing)$). Then part of the results of [5, §6] and the above theorems can be extended to the algebras $\mathfrak{A}_T(\Lambda)$ by using the same kind of arguments. Thus, we shall establish without proof the following:

THEOREM 7. (i) *If $\mathfrak{M} \in \operatorname{Lat} T$, $R = T|\mathfrak{M}$ and $\bar{T}$ is the operator induced by T on $\mathfrak{X}/\mathfrak{M}$, then the following are equivalent*: (a) $\mathfrak{M} \in \operatorname{Lat} \mathfrak{A}_T(\Lambda)$; (b) $\Lambda \subset \rho(R)$; (c) $\Lambda \subset \rho(\bar{T})$; (d) $\Lambda \subset \rho(R) \cup \rho(\bar{T})$; (e) $\sigma(T; \mathfrak{A}_T(\Lambda)) = \sigma(R) \cup \sigma(\bar{T})$.

(ii) *If $\mathfrak{M} \in \operatorname{Lat} \mathfrak{A}_T(\Lambda)$ and $\mathfrak{N}$ is a subspace of $\mathfrak{M}$, then $\mathfrak{N} \in \operatorname{Lat} \mathfrak{A}_R(\Lambda)$ implies that $\mathfrak{N} \in \operatorname{Lat} \mathfrak{A}_T(\Lambda)$.*

(iii) *If* $\mathfrak{M} \in \operatorname{Lat} \mathcal{Q}_T(\Lambda)$ *and* $\overline{\mathfrak{N}}$ *is a subspace of* $\mathfrak{X}/\mathfrak{M}$, *then* $\overline{\mathfrak{N}} \in \operatorname{Lat} \mathcal{Q}_{\overline{T}}(\Lambda)$ *implies that* $\mathfrak{N} = \pi^{-1}(\overline{\mathfrak{N}}) \in \operatorname{Lat} \mathcal{Q}_T(\Lambda)$.

(iv) *If* Λ *is finite, there exists a constant* $s(T, \Lambda) > 0$, *such that* $\hat{d}(\mathfrak{M}, \mathfrak{N}) > s(T, \Lambda)$ *for all* $\mathfrak{M} \in \operatorname{Lat} \mathcal{Q}_T(\Lambda)$ *and all* $\mathfrak{N} \in \operatorname{Lat} T \setminus \operatorname{Lat} \mathcal{Q}_T(\Lambda)$. *In particular*, $\operatorname{Lat} \mathcal{Q}_T(\Lambda)$ *is a clopen sublattice of* $\operatorname{Lat} T$. *Moreover, both results remain true under the weaker assumption*: Λ *only intersects finitely many components of* $\mathbf{C} \setminus \sigma(T; \mathcal{Q}_T(\Lambda))$.

(v) *If* $\mathfrak{M}_1, \mathfrak{M}_2 \in \operatorname{Lat} T$, $\mathfrak{M}_0 = \mathfrak{M}_1 \cap \mathfrak{M}_2 \in \operatorname{Lat} \mathcal{Q}_T(\Lambda)$ *and* $\mathfrak{M}_3 = \mathfrak{M}_1 + \mathfrak{M}_2$ *is closed and belongs to* $\operatorname{Lat} \mathcal{Q}_T(\Lambda)$, *then* $\mathfrak{M}_1, \mathfrak{M}_2 \in \operatorname{Lat} \mathcal{Q}_T(\Lambda)$.

Our last result says that each of the lattices $\operatorname{Lat} \mathcal{Q}_T(\Lambda)$ is invariant under "small perturbations of the dimension". In fact, we have

THEOREM 8. *If* $\mathfrak{M}, \mathfrak{N} \in \operatorname{Lat} T$, $\mathfrak{M} \subset \mathfrak{N}$ *and* $\dim \mathfrak{N}/\mathfrak{M} < \infty$, *then* $\mathfrak{M} \in \operatorname{Lat} \mathcal{Q}_T(\Lambda)$ *if and only if* $\mathfrak{N} \in \operatorname{Lat} \mathcal{Q}_T(\Lambda)$. *Moreover*, $\sigma(T|\mathfrak{N}) = \sigma(T|\mathfrak{M}) \cup \{\lambda_1, \ldots, \lambda_n\}$, *where* $\lambda_1, \ldots, \lambda_n$ *are the eigenvalues of the finite dimensional operator induced by* $T|\mathfrak{N}$ *on* $\mathfrak{N}/\mathfrak{M}$ *and* $\sigma(\overline{T}_{\mathfrak{M}}) = \sigma(\overline{T}_{\mathfrak{N}}) \cup \{\lambda_1, \ldots, \lambda_n\}$, *where* $\overline{T}_{\mathfrak{M}}$ *and* $\overline{T}_{\mathfrak{N}}$ *denote the operators induced by* T *on* $\mathfrak{X}/\mathfrak{M}$ *and* $\mathfrak{X}/\mathfrak{N}$, *respectively. In particular, if* $\mathfrak{M} \in \operatorname{Lat} T$ *and* $\dim \mathfrak{M} < \infty$ *or* $\operatorname{codim} \mathfrak{M} < \infty$, *then* $\mathfrak{M} \in \operatorname{Lat} \mathcal{Q}_T^a$.

PROOF. By applying Theorem 3 to $T|\mathfrak{N}$, we obtain

$$\sigma(T|\mathfrak{N}) \cup \sigma(T|\mathfrak{M}) = \sigma(T|\mathfrak{M}) \cup \{\lambda_1, \ldots, \lambda_n\} = \sigma(T|\mathfrak{N}) \cup \{\lambda_1, \ldots, \lambda_n\}$$

Since $\{\lambda_1, \ldots, \lambda_n\}$, the spectrum of the operator $(T|\mathfrak{N})^-$ induced by $T|\mathfrak{N}$ on $\mathfrak{N}/\mathfrak{M}$, is a finite set, we have that $\sigma((T|\mathfrak{N})^-) = \partial\sigma((T|\mathfrak{N})^-) \subset \sigma(T|\mathfrak{N})$ (see [5, §2]), we conclude that $\sigma(T|\mathfrak{N}) = \sigma(T|\mathfrak{M}) \cup \{\lambda_1, \ldots, \lambda_n\}$.

Similarly, by applying Theorem 3 to $\overline{T}_{\mathfrak{M}}$, we obtain the equality $\sigma(\overline{T}_{\mathfrak{M}}) = \sigma(\overline{T}_{\mathfrak{N}}) \cup \{\lambda_1, \ldots, \lambda_n\}$ (to see this, observe that $(T|\mathfrak{N})^-$ coincides with $\overline{T}_{\mathfrak{M}}|(\mathfrak{N}/\mathfrak{M})$). We have proved that the symmetric difference $\sigma(T|\mathfrak{N}) \triangle \sigma(T|\mathfrak{M})$ is contained in the finite set $\{\lambda_1, \ldots, \lambda_n\}$; thus, since $\sigma(T|\mathfrak{N}) \triangle \sigma(T|\mathfrak{M})$ cannot contain a component of $\rho(T)$, $\mathfrak{M}$ and $\mathfrak{N}$ belong to exactly the same lattices $\operatorname{Lat} \mathcal{Q}_T(\Lambda)$ (where Λ runs over all possible sets of the above described type).

If $\dim \mathfrak{M} < \infty$ ($\operatorname{codim} \mathfrak{M} < \infty$, resp.), then $\mathfrak{M} \in \operatorname{Lat} \mathcal{Q}_T^a$ because $\dim \mathfrak{M}/\{0\} < \infty$ and $\{0\} \in \operatorname{Lat} \mathcal{Q}_T^a$ ($\dim \mathfrak{X}/\mathfrak{M} < \infty$ and $\mathfrak{X} \in \operatorname{Lat} \mathcal{Q}_T^a$, resp.) □

It is interesting to observe that not every finite dimensional (or finite codimensional) invariant subspace of T is *bi-invariant*, i.e., invariant under the double commutant $\mathcal{Q}_T''$ of T. Indeed, we have the following counterexample, inspired in a paper of A. L. Lambert and T. R. Turner:

EXAMPLE D. Let S_α and S_β be injective unilateral weighted shifts in l^2 such that the operator $L = S_\alpha^* \oplus S_\beta^* \in \mathcal{L}(l^2 \oplus l^2)$ satisfies the relations $\mathcal{Q}_L = \mathcal{Q}_L^a \neq \mathcal{Q}_L'' = \mathcal{Q}_L'$ (for a concrete numerical example, see [7]). Assume that S_α (S_β, resp.) is defined in the usual way with respect to the orthonormal basis $\{e_n\}_{n=0}^\infty$ of $l^2 \oplus \{0\}$ ($\{f_n\}_{n=0}^\infty$ of $\{0\} \oplus l^2$, resp.). Then $\ker L = \{\lambda e_0 + \mu f_0 \colon \lambda, \mu \in \mathbf{C}\} \in \operatorname{Lat} \mathcal{Q}_L' = \operatorname{Lat} \mathcal{Q}_L''$ (i.e. $\ker L$ is actually a *hyperinvariant* subspace), and the orthogonal projection P of $l^2 \oplus l^2$ onto $l^2 \oplus \{0\}$ belongs to $\mathcal{Q}_L''$. Clearly, every one-dimensional subspace of $\ker L$ belongs to $\operatorname{Lat} \mathcal{Q}_L^a$. On the other hand, since $P \in \mathcal{Q}_L''$, $\operatorname{Lat} \mathcal{Q}_L''$ splits with respect to the above direct sum decomposition, i.e., if $\mathfrak{M} \in \operatorname{Lat} \mathcal{Q}_L''$, then $\mathfrak{M} = P\mathfrak{M} \oplus (I - P)\mathfrak{M}$ (see [5, §5]). It is not hard to conclude that L has exactly two one-dimensional bi-invariant subspaces: the ones generated by e_0 and by f_0.

Furthermore, this example answers strongly in the negative a question raised in [5, §6]:

(i) $\ker L \in \operatorname{Lat} \mathcal{Q}_L'$ and $\{\lambda(e_0 + f_0) \colon \lambda \in \mathbf{C}\} \in \mathcal{Q}_{(L|\ker L)}''$. However, $\{\lambda(e_0 + f_0)\} \notin \operatorname{Lat} \mathcal{Q}_L''$.

(ii) Let S_α and S_β be as above and set $Q = S_\alpha \oplus S_\beta \in \mathcal{L}(l^2 \oplus l^2)$. Then the subspace spanned by $\{e_n, f_n\}_{n=1}^\infty$ is equal to $\mathfrak{M} = \text{closure}(\operatorname{ran} Q) \in \operatorname{Lat} \mathcal{Q}_Q'$ and $\overline{\mathfrak{N}} = \mathfrak{M} \oplus \{\lambda(e_0 + f_0) \colon \lambda \in \mathbf{C}\}/\mathfrak{M} \in \operatorname{Lat} \mathcal{Q}_{\overline{Q}}''$, where $\overline{Q} = 0$ is the operator induced by Q on $l^2 \oplus l^2/\mathfrak{M}$. However, $\mathfrak{N} = \pi^{-1}(\overline{\mathfrak{N}}) = \mathfrak{M} \oplus \{\lambda(e_0 + f_0) \colon \lambda \in \mathbf{C}\} \notin \operatorname{Lat} \mathcal{Q}_Q''$.

Let $T \in \mathcal{L}(\mathfrak{X})$ and let $\mathfrak{M}, \mathfrak{N} \in \operatorname{Lat} \mathcal{Q}_T''$, $\mathfrak{M} \subset \mathfrak{N}$. Whether or not $\mathfrak{M} \in \operatorname{Lat} \mathcal{Q}_{T|\mathfrak{N}}''$ (and, similarly, $\mathfrak{N}/\mathfrak{M} \in \operatorname{Lat} \mathcal{Q}_{\overline{T}}''$, where $\overline{T}$ is the operator induced by T on $\mathfrak{X}/\mathfrak{M}$) is an open problem, even under the stronger assumption $\mathfrak{M} \in \operatorname{Lat} \mathcal{Q}_T'$.

REFERENCES

1. R. G. Douglas and C. Pearcy, *On a topology for invariant subspaces*, J. Functional Analysis **2** (1968), 323–341.

2. P. R. Halmos, *A Hilbert space problem book*, Van Nostrand, Princeton, N. J., 1967.

3. H. Helson, *Lectures on invariant subspaces*, Academic Press, New York, 1964.

4. D. A. Herrero and A. L. Lambert, *On strictly cyclic algebras, $\mathcal{P}$-algebras and reflexive operators*, Trans. Amer. Math. Soc. **185** (1975), 229–235.

5. D. A. Herrero and N. Salinas, *Analytically invariant and bi-invariant subspaces*, Trans. Amer. Math. Soc. **173** (1972), 117–136.

6. T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, New York, 1966.

7. A. L. Lambert and T. R. Turner, *The double commutants of invertible shifts*, Duke Math. J. **39** (1972), 385–389.

8. H. Radjavi and P. Rosenthal, *On invariant subspaces and reflexive algebras*, Amer. J. Math. **91** (1969), 683–692.

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