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KNOTTING A k-CONNECTED CLOSED PL m-MANIFOLD IN E^{2m-k} (1)

BY

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ABSTRACT. Embeddings of a k-connected closed PL m-manifold ($0 < k < m - 3$) in $(2m - k)$-dimensional euclidean space are classified up to isotopy. Thus this paper completes the results stated, and partly proved, in J. F. P. Hudson's *Piecewise linear topology*.

1. **Introduction.** We work entirely within the PL category and so all our spaces and maps are assumed PL. Throughout let M denote a k-connected closed m-manifold, $0 < k \leq m - 3$, and let $q = 2m - k$.

Penrose, Whitehead, and Zeeman [7] proved that M embeds in E^q, the q-dimensional euclidean space, and Zeeman (cf. [12]) proved that any two embeddings of M in E^{q+1} are isotopic (or ambiently isotopic, which is the same by the isotopy extension theorem [5, 6.12, p. 147]). A natural problem is, therefore, to describe the set of isotopy classes of embeddings of M in E^q. This problem has been solved for many cases; we give the references below. In the present paper a complete solution of the problem is obtained using a method which resembles the method used by Haefliger and Hirsch [2] in the differentiable case.

Denote by $\mathcal{I}$ the set of isotopy classes (= the set of ambient isotopy classes) of embeddings of M into E^q (this will be valid until §7; in §8, however, $\mathcal{I}$ will denote the set of isotopy classes of embeddings of M into a given q-manifold Q). Our goal is to characterize the set $\mathcal{I}$ in terms of the $(k + 1)$st, i.e. the first (possibly) nontrivial, homology group of M with appropriate coefficients; which coefficients we take depends on the orientability type of M and on $m - k$. We distinguish four cases as follows:

OO: M orientable, $m - k$ odd,

OE: M orientable, $m - k$ even,

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NO: M nonorientable, m $(= m - k)$ odd, and

NE: M nonorientable, m even.

Z will denote the additive group of integers, and Z_2 will mean $Z/2Z$. When we use homology or cohomology without specifying the coefficient group, we mean integer coefficients. The following theorem describes the set $\mathcal{G}$.

THEOREM 1.1. *There exists a bijective map* $\mathcal{G} \to \mathcal{H}_{k+1}(M)$, *where* $\mathcal{H}_{k+1}(M)$ *is the abelian group equal to* $H_{k+1}(M)$ *in case* OO, *to* $H_{k+1}(M; Z_2)$ *in cases* OE *and* NO, *and to*

$$H_1(M, x_0; Z^t)/2\,\mathrm{im}\big[H_1(M; Z^t) \to H_1(M, x_0; Z^t)\big]$$

in case NE; *here* x_0 *is an arbitrary point of* M, *and* Z^t *are the twisted integer coefficients.*

The reader may not be very familiar with the group $\mathcal{H}_1(M)$ in case NE. Therefore we will say a few words about it here so that we can state Theorem 1.2. It will be convenient for us to replace the coefficient system Z^t with the isomorphic system Γ, the orientation sheaf of M, whose groups are $\Gamma(x) = H_m(M, M - x)$ $(x \in M)$; see e.g. [1, §22] for details. Correspondingly we change the definition of the group $\mathcal{H}_1(M)$ (in case NE), which will sometimes be more correctly denoted by $\mathcal{H}_1(M, x_0)$.

Choose an arbitrary point $x_0 \in M$ and let j: $(M, \varnothing) \subset (M, x_0)$. Since $H_0(x_0; \Gamma(x_0)) \approx \Gamma(x_0) \approx Z$ and $H_0(M; \Gamma) \approx H^m(M) \approx Z_2$ we get from the homology sequence of (M, x_0) the following exact sequence

$$0 \to H_1(M; \Gamma) \xrightarrow{j_*} H_1(M, x_0; \Gamma) \xrightarrow{\partial_*} 2H_0(x_0; \Gamma(x_0)) \to 0.$$

It follows that $\mathrm{im}\, j_*$ is a direct summand of $H_1(M, x_0; \Gamma)$ with the complementary direct summand isomorphic to Z, and hence

$$\begin{aligned}\mathcal{H}(M, x_0) &= H_1(M, x_0; \Gamma)/2\,\mathrm{im}\, j_* \\ &\approx \big((\mathrm{im}\, j_*) \otimes Z_2\big) \oplus Z \approx (H_1(M; \Gamma) \otimes Z_2) \oplus Z\end{aligned}$$

(this splitting is not natural, though). The group $H_1(M; \Gamma) \otimes Z_2$ can be written in several other forms. For instance, it is naturally isomorphic to the subgroup of $H_1(M; Z_2)$ generated by orientation preserving loops.

It is not difficult to prove that for any $x_0, x_1 \in M$ the groups $\mathcal{H}_1(M, x_0)$ and $\mathcal{H}_1(M, x_1)$ are isomorphic. Moreover, with each pair of local orientations u_0, u_1 of M at x_0 and x_1, respectively (i.e. u_i is a generator of $\Gamma(x_i)$), we can associate an isomorphism $\lambda(-; u_0, u_1)$ of $\mathcal{H}_1(M, x_0)$ onto $\mathcal{H}_1(M, x_1)$ which satisfies the following "transitivity" formula

$$\lambda(-; u_1, u_2)\lambda(-; u_0, u_1) = \lambda(-; u_0, u_2)$$

and the condition $\lambda(-; u_0, u_0) = \text{id}$. Indeed, suppose that $x_0 \neq x_1$ and choose an oriented arc A from x_0 to x_1 such that u_1 is u_0 transported along A. The following commutative diagram, in which all homomorphisms are induced by inclusions

$$\begin{tikzcd} H_1(M;\Gamma) \arrow[r, "\text{id}"] \arrow[d] & H_1(M;\Gamma) \arrow[d] & H_1(M;\Gamma) \arrow[l, "\text{id}"'] \arrow[d] \\ H_1(M, x_0;\Gamma) \arrow[r, "\approx"] & H_1(M, A;\Gamma) & H_1(M, x_1;\Gamma) \arrow[l, "\approx"'] \end{tikzcd}$$

induces an isomorphism $\lambda = \lambda(-; A)\colon \mathcal{K}_1(M, x_0) \to \mathcal{K}_1(M, x_1)$. Now choose another oriented arc B from x_0 to x_1 such that u_1 is u_0 transported along B. If we take any $\alpha \in \mathcal{K}_1(M, x_0)$ and any representative $a \in H_1(M, x_0; \Gamma)$ for α, then $\lambda(\alpha; B) - \lambda(\alpha; A)$ is represented by $2n(B - A)$ where n is the integer such that $\partial_*(a) = (2nu_0)x_0 \in H_0(x_0; \Gamma(x_0))$. Now B − A is an orientation preserving 1-cycle (over Z) and hence $B - A$ represents an element of $H_1(M; \Gamma)$. It follows that $\lambda(\alpha; B) = \lambda(\alpha; A)$. This means that λ depends only on u_0 and u_1 and therefore we write $\lambda(-; u_0, u_1)$ instead of $\lambda(-; A)$. In a very similar manner we can prove the transitivity formula stated above where u_i is a generator of $\Gamma(x_i)$ $(i = 0, 1, 2)$ for x_0, x_1, x_2 any three distinct points. If we then define $\lambda(-; u_0, u_0) = \text{id}$ and $\lambda(-; u_0, -u_0) = -\text{id}$, the transitivity formula becomes valid generally.

Now we can state our next theorem, of which 1.1 is an immediate consequence.

THEOREM 1.2. *In cases* OO, OE, *and* NO *there exists a map* $\delta\colon \mathcal{G} \times \mathcal{G} \to \mathcal{K}_{k+1}(M)$ *with the following properties*:

(1) $\delta(\varepsilon, \varepsilon) = 0$ *for each* $\varepsilon \in \mathcal{G}$,

(2) $\delta(\varepsilon_0, \varepsilon_1) + \delta(\varepsilon_1, \varepsilon_2) = \delta(\varepsilon_0, \varepsilon_2)$ *for all* $\varepsilon_0, \varepsilon_1, \varepsilon_2 \in \mathcal{G}$, *and*

(3) $\delta(\varepsilon_0, -)\colon \mathcal{G} \to \mathcal{K}_{k+1}(M)$ *is bijective for each* $\varepsilon_0 \in \mathcal{G}$.

In case NE *there exists a function which associates with each* $x_0 \in M$ *and each local orientation* u_0 *of* M *at* x_0 *a map*

$$\delta = \delta(-, -; u_0)\colon \mathcal{G} \times \mathcal{G} \to \mathcal{K}_1(M, x_0)$$

which satisfies (1), (2), *and* (3) *above and is such that for any two points* x_0, $x_1 \in M$ *and for any local orientations* u_i *of* M *at* x_i $(i = 0, 1)$ *we have*

$$\delta(-, -; u_1) = \lambda(-; u_0, u_1)\delta(-, -; u_0).$$

For many cases Theorems 1.1 and 1.2 have been known. As far as this author knows, Hudson [4] was the first to prove a version of 1.1 in the PL category; he dealt only with the manifolds $M = S^{k+1} \times S^{m-k-1}$ $(m > 2k + 2)$. Haefliger and Hirsch [2] proved 1.1 in the smooth category for the cases

OO and OE under the assumption $0 < k < \frac{1}{2}(m-4)$. In his book [5, Chapter 11], Hudson defined a map $d_2\colon \mathcal{G} \times \mathcal{G} \to H_{k+1}(M; Z_2)$ (he called it d) and proved that d_2 has properties analogous to (1) and (2) of 1.2 and that if $0 < k < m-4$ then $d_2(\varepsilon_0, -)\colon \mathcal{G} \to H_{k+1}(M; Z_2)$ is surjective for any $\varepsilon_0 \in \mathcal{G}$. He also asserted without proof that, again for $0 < k < m-4$, $d_2(\varepsilon_0, -)$ is injective in cases OE and NE; note that this assertion for the case NE disagrees with our Theorem 1.1. Hudson made no further comment about the case NO. He mentioned, however, that in the case OO one can define, imitating the definition of d_2, a map $d\colon \mathcal{G} \times \mathcal{G} \to H_{k+1}(M)$ and that for $0 < k < m-4$ the map $d(\varepsilon_0, -)\colon \mathcal{G} \to H_{k+1}(M)$ is bijective.

Our definition of δ is not similar to Hudson's definition of d_2 or d; nevertheless they give essentially the same result:

THEOREM 1.3. *In all cases, the composition of* $\delta\colon \mathcal{G} \times \mathcal{G} \to \mathcal{H}_{k+1}(M)$ *and the natural projection* $\mathcal{H}_{k+1}(M) \to H_{k+1}(M; Z_2)$ *is equal to Hudson's map* d_2. *In case* OO, δ *agrees with* d *up to sign.*

Observe that we allow $k < m-3$ where Hudson requires $k < m-4$. However, the only two interesting cases with $k = m-3$ are $m = 3$, $k = 0$ and $m = 4$, $k = 1$. For, if $m \geqslant 5$, then every closed $(m-3)$-connected m-manifold is $[\frac{1}{2}m]$-connected and hence it is a homotopy m-sphere, by Poincaré duality and the Hurewicz theorem. But for $m \geqslant 5$ a homotopy m-sphere is a real m-sphere (cf. [8, p. 109]), and therefore the corresponding set $\mathcal{G}$ has only one element, i.e. Theorem 1.1 is automatically true in this case.

It is natural to ask whether there is a generalization of 1.1 in which the ambient space E^q is replaced with a q-manifold. In fact, Theorem 1.1 is a special case of the following theorem.

THEOREM 1.4. *If Q is an $(m-k)$-connected q-manifold without boundary, then there exists a bijective map of the set of isotopy classes of embeddings $M \to Q$ onto the group $\mathcal{H}_{k+1}(M) \oplus \pi_m(Q)$.*

Theorem 1.2 admits a similar generalization, but we shall not state it. A by-product of the proof of 1.4 is the following

COROLLARY 1.5. *Suppose that M is orientable, $m-k$ is odd, and Q is an $(m-k)$-connected q-manifold without boundary. Let e_1, e_2 be two embeddings of M into Q such that e_1 and e_2 agree everywhere except possibly on an m-ball $D \subset M$. If e_1 and e_2 are isotopic, then for every regular neighborhood W of* $e_1(M - \operatorname{int} D)$ mod $e_1(D) \cup e_2(D)$ *there exists an isotopy of Q which is fixed on W and carries e_1 to e_2.*

We conclude the introduction with a brief indication of how the map δ of Theorem 1.2 is constructed. Let $e_0, e_1\colon M \to E^q$ be two arbitrary embeddings and let $\varepsilon_i \in \mathcal{I}$ be the isotopy class of e_i ($i = 0, 1$). Choose an m-ball $D \subset M$ and an orientation for D and let N be the closure of $M - D$. Then e_1 is isotopic to an embedding $e: M \to E^q$ such that $e|N = e_0|N$. Considering $e_0(D)$ and $e(D)$ as m-chains in $E^q - e_0(\operatorname{int} N)$ form the m-cycle $e(D) - e_0(D)$ and then let $\theta(e_0, e) \in H^p(N)$, where $p = m - k - 1$, be the image under e_0^* of the Alexander dual of $e(D) - e_0(D)$. If we divide the group $H^p(N)$ by a suitable subgroup S, then the image of $\theta(e_0, e)$ in $H^p(N)/S$ depends only on ε_0 and ε_1, and moreover, the function θ^*: $\mathcal{I} \times \mathcal{I} \to H^p(N)/S$ induced by θ is such that $\theta^*(\varepsilon_0, -)$ maps $\mathcal{I}$ bijectively onto $H^p(N)/S$; we have to take S trivial in case OO, equal to twice $H^p(N)$ in cases OE and NO, and equal to twice the image of $H^p(M) \to H^p(N)$ in case NE. The map δ is then defined to be the composition of θ^* and Poincaré duality.

2. **Notation and conventions, general.** Let N be an n-manifold. By int N and ∂N we denote the interior and the boundary of N, respectively. If c, d are integral chains in N such that their dimensions add up to n and such that the support of c misses the support of ∂d and the support of d misses the support of ∂c, then $c \# d$ denotes the intersection number of c with d; here, of course, an orientation for N must be specified.

Let X be a space and Y a subspace of X. By Cl Y we shall denote the closure of Y in X; it will always be clear from the context in which space the closure is taken. By id_X we denote the identity mapping $X \to X$; the subscript X is omitted whenever evident from the context. By a slight abuse of notation, id will often stand for the inclusion of Y into X, e.g. if $f: X \to X$ is a map, then by $f|Y = \mathrm{id}$ we mean that $f(y) = y$ for $y \in Y$.

By I we shall denote the closed interval $[0, 1]$ of the real line. If X and Y are spaces and $F: X \times I \to Y \times I$ is a level preserving map (i.e. such that $F(X \times t) \subset Y \times t$ for each t), then F_t denotes the map $X \to Y$ such that $F(x, t) = (F_t(x), t)$. Similarly, if $F: X \times I \to Y$ is a homotopy, we denote by F_t the map $X \to Y$ defined by $F_t(x) = F(x, t)$. An *isotopy of X in Y* is a level preserving embedding $X \times I \to Y \times I$. An *isotopy of Y* is a level preserving homeomorphism $F: Y \times I \to Y \times I$ such that $F_0 = \mathrm{id}$; if $U \subset Y$ and $F|U \times I = \mathrm{id}$, we say that F is *fixed on U* (or that *F keeps U fixed*); if U and V are subsets of Y and if $F_1(U) = V$, we say that *F carries U onto V*; finally, if $f, g: X \to Y$ are maps and $F_1 f = g$, then F is said to *carry f to g*.

For each positive integer n let E^n be the n-dimensional euclidean space, i.e. the product of n copies of the real line, and let $I^n \subset E^n$ be the product of n copies of the closed interval $[-1, 1]$ (thus $I^1 \neq I$). By 0^n we denote the origin of E^n. S^{n-1} is an alternative symbol for ∂I^n. For each n we identify E^n with the subspace of E^{n+1} consisting of points with zero last coordinate. This induces canonical inclusions $E^n \subset E^{n+k}$, $I^n \subset I^{n+k}$, $S^n \subset S^{n+k}$ for each $k > 0$.

Whenever we deal with oriented manifolds we assume the manifolds E^n, I^n, S^n oriented "in the standard way" as follows. The standard orientation for E^n is determined with the natural order of the coordinate axes. In I^n, the standard orientation is the one induced by the standard orientation of E^n. More generally, if P is a coordinate plane in E^n, i.e. the linear subspace spanned by some subcollection of the coordinate axes of E^n, then the standard orientation of P is given by the natural order of the coordinate axes of E^n which are contained in P, and $I^n \cap P$ is oriented accordingly. The standard orientation of S^n is the one coherent with the standard orientation of I^{n+1}. If Q is an oriented manifold and N is a codimension 0 submanifold of Q, then N is always assumed to have the orientation induced by Q.

3. **The first step in the construction of δ.** The next paragraph introduces some specific conventions and notation which will be valid throughout this section and will be frequently referred to in subsequent sections.

3.1. Choose and fix an embedding e_0: $M \to E^q$. Take an m-ball $D \subset M$ and an orientation for D; then let $N = M - \operatorname{int} D$. Choose a regular neighborhood W of $e_0(N)$ mod $e_0(D)$ in E^q and let $V = E^q - \operatorname{int} W$. Denote by $\mathfrak{E}$ the set of all embeddings of M into E^q and let $\mathfrak{E}_0$ be the subset of $\mathfrak{E}$ consisting of those embeddings e: $M \to E^q$ for which $e|N = e_0|N$ and $e(\operatorname{int} D) \subset \operatorname{int} V$; call two such embeddings equivalent if one can be moved to the other by an isotopy of E^q which keeps W fixed; let $\mathfrak{I}_0$ be the set of the corresponding equivalence classes in $\mathfrak{E}_0$. Let $p = m - k - 1$.

Obviously the pair $(N, \partial N)$ is k-connected and hence N unknots in E^q [5, 10.3, p. 201]. This and uniqueness of relative regular neighborhoods easily imply

LEMMA 3.2. *The natural map $\mathfrak{I}_0 \to \mathfrak{I}$ induced by the inclusion $\mathfrak{E}_0 \subset \mathfrak{E}$ is onto.*

We state here another lemma that will be used in later sections; it follows from uniqueness of regular neighborhoods and from Hudson's "concordance implies isotopy" theorem [6, 1.1].

LEMMA 3.3. *Let N' be a regular neighborhood of N in M and let W' be a regular neighborhood of $W \cup e_0(N')$ mod $e_0(M - \operatorname{int} N')$ in E^q. Then each class in $\mathfrak{I}_0$ contains an embedding e: $M \to E^q$ such that $e|N' = e_0|N'$ and $e(M - N') \subset E^q - W'$.*

Take any $e \in \mathfrak{E}_0$. Thinking of $e(D)$ and $e_0(D)$ as (integral) m-chains we can form the difference $e(D) - e_0(D)$, which is then an m-cycle and thus represents an element of $H_m(E^q - \operatorname{int} e_0(N)) \approx H_m(E^q - e_0(N))$. By

Alexander duality, $H_m(E^q - e_0(N)) \approx H^p(N)$. Let $\theta(e) \in H^p(N)$ be the element corresponding to $e(D) - e_0(D)$. Clearly, $\theta(e)$ depends only on the class of e in $\mathfrak{I}_0$.

PROPOSITION 3.4. *The map* θ: $\mathfrak{S}_0 \to H^p(N)$ *induces a bijection* θ_0: $\mathfrak{I}_0 \to H^p(N)$.

PROOF. To simplify the notation we will assume that M is a submanifold of E^q and that e_0 is the inclusion. Clearly it suffices to show that the map $\mathfrak{I}_0 \to H_m(E^q - \operatorname{int} N)$ which assigns to each $\varepsilon \in \mathfrak{I}_0$ the homology class of $e(D) - e_0(D)$, where $e \in \mathfrak{S}_0$ represents ε, has a bijective right inverse γ: $H_m(E^q - \operatorname{int} N) \to \mathfrak{I}_0$.

By a general position argument, $E^q - N$ is 1-connected, and by means of Alexander and Poincaré duality we can show that $H_i(E^q - N) = 0$ for $0 < i < m$. Therefore $E^q - N$, and hence $E^q - \operatorname{int} N$ and V, are $(m - 1)$-connected, and by the Hurewicz theorem we have a natural isomorphism $H_m(E^q - \operatorname{int} N) \approx \pi_m(V)$. This isomorphism is the first step in the construction of our bijective map γ.

Consider the exact homotopy sequence of the pair $(V, \partial D)$:

$$\pi_m(\partial D) \to \pi_m(V) \to \pi_m(V, \partial D) \to \pi_{m-1}(\partial D).$$

The homomorphism $\pi_m(\partial D) \to \pi_m(V)$ is trivial since it factors through $\pi_m(D)$. Therefore $\pi_m(V)$ is mapped isomorphically onto the kernel of ∂: $\pi_m(V, \partial D) \to \pi_{m-1}(\partial D)$. Alternatively, with each $\alpha_0 \in \pi_m(V, \partial D)$ we can associate a natural bijective map of $\pi_m(V)$ onto $\partial^{-1}(\partial\alpha_0)$.

Let F be the set of maps $D \to V$ which are identity on ∂D. Fix a point $b \in \partial D$. The chosen orientation of D determines uniquely a bijective correspondence between the elements of $\pi_m(V, \partial D)$ and homotopy classes of maps $(D, \partial D, b) \to (V, \partial D, b)$. Let $\alpha_0 \in \pi_m(V, \partial D)$ be the element corresponding to the inclusion $D \subset V$. Then $\partial^{-1}(\partial\alpha_0)$ can be identified with the quotient set $F/\sim$ where $\sim$ is the equivalence relation in F given by: $f \sim g$ if $f \simeq g$ as maps of $(D, \partial D, b)$ into $(V, \partial D, b)$. The bijective map $\pi_m(V) \to F/\sim$ obtained in this way is the second step in the construction of γ.

Define another equivalence relation, $\approx$, in F by: $f \approx g$ if $f \simeq g$ (rel ∂D). One can easily show that $F/\sim = F/\approx$. On the other hand, Irwin's embedding theorem [5, 8.1, p. 174] and Zeeman's unknotting theorem [5, 10.1, p. 198] imply that the map $\mathfrak{S}_0 \to F$ which assigns to each $e \in \mathfrak{S}_0$ its restriction $e|D \in F$ induces a bijection $\mathfrak{I}_0 \to F/\approx$. The inverse map $F/\approx \to \mathfrak{I}_0$ is the last step in the construction of γ.

Now it easily follows from the construction of γ that if u is any element of $H_m(E^q - \operatorname{int} N)$ and if $e \in \mathfrak{S}_0$ represents $\gamma(u) \in \mathfrak{I}_0$, then u is the homology class of $e(D) - e_0(D)$. Proposition 3.4 is proved.

4. Proof of Theorem 1.1 in case OO and of Theorem 1.3. In §7 the function δ of Theorem 1.2 will be defined and Theorem 1.2 will be proved. However, owing to a gap in our knowledge of handle decompositions of $(m-3)$-connected closed m-manifolds, that proof will be valid only if $k=0$ or $m-k>4$. This means that we have to give a different proof for the case $m=4$, $k=1$. The essential part of this proof will be given here; it will cover all cases with M orientable and $m-k$ odd. At the same time we shall do the essential part of the proof of Theorem 1.3.

Adopt the notation and conventions of 3.1. For every M we have a natural isomorphism π_2: $H^p(N; Z_2) \to H_{k+1}(M; Z_2)$ which is essentially Poincaré duality; more precisely, π_2 is the composition of isomorphisms

$$H^p(N; Z_2) \approx H_{k+1}(N, \partial N; Z_2) \approx H_{k+1}(M, D; Z_2) \approx H_{k+1}(M; Z_2).$$

Similarly, if M is orientable, the orientation of M induced by the chosen orientation of D determines an isomorphism π: $H^p(N) \to H_{k+1}(M)$. As in §1 let d_2: $\mathcal{I} \times \mathcal{I} \to H_{k+1}(M; Z_2)$ and d: $\mathcal{I} \times \mathcal{I} - H_{k+1}(M)$ (in case OO) be the functions defined by Hudson. Let $\varepsilon_0 \in \mathcal{I}$ be the isotopy class of e_0.

PROPOSITION 4.1. *In case OO, the following diagram is commutative up to sign*

$$\begin{tikzcd} \mathcal{I}_0 \arrow[r, "\theta_0", "\approx"'] \arrow[d] & H^p(N) \arrow[d, "\pi", "\approx"'] \\ \mathcal{I} \arrow[r, "d(\epsilon_0, -)"] & H_{k+1}(M) \end{tikzcd}$$

PROPOSITION 4.2. *In all cases the following diagram is commutative*

$$\begin{tikzcd} \mathcal{I}_0 \arrow[r, "\theta_0", "\approx"'] \arrow[d] & H^p(N) \arrow[r] & H^p(N; Z_2) \arrow[d, "\pi_2", "\approx"'] \\ \mathcal{I} \arrow[rr, "d_2(\epsilon_0, -)"] & & H_{k+1}(M; Z_2) \end{tikzcd}$$

3.2 and 4.1 clearly imply that $\mathcal{I}_0 \to \mathcal{I}$ and $d(\varepsilon_0, -)$ are bijective in case OO. Therefore Theorem 1.1 is true in case OO. At the same time we get 1.5 for $Q = E^q$.

When we define our function δ it will be obvious that 4.1 and 4.2 imply 1.3. Therefore we shall not return to 1.3 any more.

We will give only the proof of 4.1. The proof of 4.2 is very similar, only easier. First we will recall the definition of d.

Suppose that M is orientable and $m - k$ is odd. Let e: $M \to E^q$ be an arbitrary embedding and let $\varepsilon \in \mathcal{I}$ be its isotopy class. We define, following Hudson [5, Chapter 11], the difference class $d(\varepsilon_0, \varepsilon) \in H_{k+1}(M)$ as follows. Take a general position map G: $M \times I \to E^q \times I$ such that

$$G(x, 0) = (e_0(x), 0) \text{ and } G(x, 1) = (e(x), 1) \quad \text{for each } x \in M$$

(such a map can be constructed from a homotopy of M in E^q from e_0 to e). The singular set $S(G)$ of G has dimension $\leqslant k + 1$. Choose triangulations of $M \times I$ and $E^q \times I$ with respect to which G is simplicial. Then $S(G)$ is a subcomplex of $M \times I$. Choose an oriented $(k + 1)$-simplex σ in $S(G)$. Since the set of triple points of G has dim $< k + 1$ there exists a unique oriented $(k + 1)$-simplex $\sigma_1 \neq \sigma$ in $S(G)$, and there exists an oriented $(k + 1)$-simplex τ in $E^q \times I$ such that $G(\sigma) = G(\sigma_1) = \tau$, with matching orientations. Choose an orientation for $M \times I$. Let σ^*, σ_1^*, τ^* be the dual cells of σ, σ_1, τ, respectively, oriented so that the intersection numbers $\sigma \# \sigma^*$, $\sigma_1 \# \sigma_1^*$, and $\tau \# \tau^*$ are all equal to 1 (here $E^q \times I$ is oriented as a subspace of E^{q+1}). It can easily be seen that G embeds σ^* and σ_1^* properly into τ^* and that $G(\partial\sigma^*) \cap G(\partial\sigma_1^*) = \emptyset$. Let $n(\sigma)$ denote the intersection number $G(\sigma^*) \# G(\sigma_1^*)$ in τ^* (note that $\dim \sigma^* = \dim \sigma_1^* = m - k$ and $\dim \tau^* = 2(m - k)$). Obviously $n(\sigma)$ is independent of the orientation of $M \times I$. The sum $w = \sum n(\sigma)\sigma$ as σ runs over all $(k + 1)$-simplices of $S(G)$ and in each some orientation is chosen turns out to be a $(k + 1)$-cycle in $M \times I$, and the image of the class of this cycle under the projection $H_{k+1}(M \times I) \to H_{k+1}(M)$ is then by definition the difference class $d(\varepsilon_0, \varepsilon)$. It can be proved that $d(\varepsilon_0, \varepsilon)$ really depends only on ε_0 and ε.

Proof of 4.1. As in the proof of 3.4 we shall think of M as a submanifold of E^q and of e_0 as the inclusion. Take a regular neighborhood N' of N in M and let $D' = M - \operatorname{int} N'$. Choose an arbitrary $\varepsilon \in \mathcal{I}_0$. By 3.3 we can represent ε by an embedding $e \in \mathcal{E}_0$ such that $e|N' = \mathrm{id}$.

Choose a triangulation K for N. Then let $K \times I$ be the cell complex on $N \times I \subset E^q \times I$ consisting of convex linear cells $\alpha \times I$, for $\alpha \in K$, and their faces.

Choose a homotopy H: $D' \times I \to E^q$ (rel $\partial D'$) from the inclusion to $e|D'$ such that H is a general position map and in general position with respect to K in the following sense: $H(D' \times I)$ misses the $(p - 1)$-skeleton of K and intersects the p-skeleton of K in a finite set, and each intersection point of $H(D' \times I)$ with a p-simplex α of K has a neighborhood U in E^q such that $H|H^{-1}(U)$ is an embedding and there exists a homeomorphism $U \to I^q$ which maps $\alpha \cap U$, $N' \cap U$, and $H(H^{-1}(U))$ onto I^p, I^m, and $0^p \times I^{m+1}$, respectively. Now define G: $M \times I \to E^q \times I$ by: $G|N' \times I = \mathrm{id}$ and

$$G(x,t) = (H(x,t),t) \quad \text{for } x \in D', t \in I.$$

Use this map G to evaluate Hudson's difference class $d(\varepsilon_0, \varepsilon)$ (by a slight abuse of notation we denote the image of ε under $\mathcal{G}_0 \to \mathcal{G}$ again by ε). First construct the $(k+1)$-cycle $w = \sum n(\sigma)\sigma$ in $M \times I$ as described above and then let z be the $(k+1)$-cycle in M which is the image of w under the chain map induced by the projection $M \times I \to M$. Thus $d(\varepsilon_0, \varepsilon)$ is the class of z.

We assert that $\pi^{-1}d(\varepsilon_0, \varepsilon) = (-1)^{m+1}\theta_0(\varepsilon)$. To prove this we shall represent $\pi^{-1}d(\varepsilon_0, \varepsilon)$ and $\theta_0(\varepsilon)$ by explicit p-cocycles: $\pi^{-1}d(\varepsilon_0, \varepsilon)$ is the class of the cocycle which assigns to each oriented p-simplex α of K the intersection number $\alpha \# z$ in M (where the orientation of M is induced by the chosen orientation of D), and $\theta_0(\varepsilon)$ is the class of the cocycle which assigns to α the intersection number $\alpha \# H(D' \times I)$ in E^q (here $D' \times I$ is oriented coherently with $D' \times 1$). Therefore it suffices to prove that if α is an arbitrary oriented p-simplex of K, then $\alpha \# z = (-1)^{m+1}\alpha \# H(D' \times I)$.

The support $|w|$ of w is contained in the singular set $S(G)$. Therefore $(N \times I) \cap |w| \subset (N \times I) \cap G(D' \times I)$ and in particular $(\alpha \times I) \cap |w| \subset (\alpha \times I) \cap G(D' \times I)$; the latter inclusion is actually an equality, which follows from the manner in which $H(D' \times I)$ intersects α. Consequently $\alpha \cap |z| = \alpha \cap H(D' \times I)$. Therefore it suffices to show that if $x \in \alpha \cap H(D' \times I)$, then the local intersection number at x of α with z in M is equal to $(-1)^{m+1}$ times the local intersection number at x of α with $H(D' \times I)$ in E^q.

By hypothesis there exist a neighborhood U of x in E^q and a homeomorphism $h\colon U \to I^q$ mapping $\alpha \cap U$, $N' \cap U$, and $H(H^{-1}(U))$ onto I^p, I^m, and $0^p \times I^{m+1}$, respectively. Choose h so that h, $h|\alpha \cap U$, and $h|N' \cap U$ have degree 1 (as homeomorphisms onto their respective images). Suppose that $h|H(H^{-1}(U))$ has degree $(-1)^s$. Then the local intersection number at x of α with $H(D' \times I)$ in E^q is $(-1)^s$. Since

$$h(|z| \cap U) = h(N' \cap H(D' \times I) \cap U) = 0^p \times I^{k+1},$$

the local intersection number at x of α with z in M is equal to the coefficient with which the oriented $(k+1)$-ball $h^{-1}(0^p \times I^{k+1})$ appears in z. We are now going to determine this coefficient.

Obviously there exists a unique $t \in \operatorname{int} I$ such that $(x, t) \in (\alpha \times I) \cap G(D' \times I)$. Choose a homeomorphism $f\colon I \to I^1$ which sends $0, t, 1$ to $-1, 0, 1$, respectively. Then $h \times f$ is a homeomorphism of $U \times I$ onto I^{q+1} mapping $(\alpha \cap U) \times I$ onto $I^p \times 0^{m+1} \times I^1$ and $(N' \cap U) \times I$ onto $I^m \times 0^{p+1} \times I^1$. By a suitable isotopy of I^{q+1} which moves points only in the "vertical" direction we can move $h \times f$ to a homeomorphism $g\colon U \times I \to I^{q+1}$ which maps $(N' \cap U) \times I$ onto $A = I^m \times 0^{p+1} \times I^1$ and $G(D' \times I) \cap (U \times I)$ onto $B = 0^p \times I^{m+1} \times 0$ and makes the following diagram commutative

$$\begin{array}{ccc} U \times I & \xrightarrow{g} & I^q \times I^1 \\ \downarrow & & \downarrow \\ U & \xrightarrow{h} & I^q \end{array}$$

(the vertical arrows mean the natural projections). Clearly g has degree 1, and $g^{-1}|B$ has degree $(-1)^s$ (again, as a homeomorphism onto its image). Since we agreed to orient $D' \times I$ coherently with $D' \times 1$ we must accordingly orient $N' \times I$ coherently with $N' \times 1$. Hence $g^{-1}|A$ has degree $(-1)^m$.

Let $R = A \cap B = 0^p \times I^{k+1} \times 0^{p+2}$ and let A', B', C' be the dual cubes (i.e. orthogonal complements) of R in A, B, and I^{q+1}, respectively. Note that if all cubes have standard orientations, then the intersection numbers $R \# A'$ in A, $R \# B'$ in B, and $R \# C'$ in I^{q+1} are all equal to 1, and the intersection number $A' \# B'$ in C' is -1.

The coefficient with which $h^{-1}(0^p \times I^{k+1})$ appears in z is clearly equal to the coefficient with which $g^{-1}(R)$ appears in w; the latter coefficient is, as follows from the definition of d, $(-1)^{m+s}$ times the intersection number $A' \# B'$ in C', which is further equal to $(-1)^{m+s+1}$. This proves the relation $\alpha \# z = (-1)^{m+1}\alpha \# H(D' \times I)$ and concludes the proof of 4.1.

5. **Handle decompositions of M and W.** Suppose that $k = 0$ or $k \leqslant m - 4$. Adopt the notation and conventions of 3.1. In the proof of Theorem 1.2 we shall need some special handle decompositions of M and W. The objective of §5 is to construct these handle decompositions and to introduce further notation.

LEMMA 5.1. *If $k = 0$ or $k \leqslant m - 4$, then M has a handle decomposition with the following properties*:

(1) *handles of the same index are disjoint*;

(2) *D is the only m-handle*;

(3) *all handles except D have index $\leqslant p$*;

(4) *in case* NO *all $(m-1)$-handles are nonorientable* (*where an $(m-1)$-handle A_i is called* (*non*)*orientable if $A_i \cup D$ is a* (*non*)*orientable manifold*); *in case* NE *precisely one $(m-1)$-handle is nonorientable.*

PROOF. We will show how to construct a handle decomposition of M such that the dual decomposition has the properties required in 5.1. It is well known that M possesses handle decompositions in which D is the only 0-handle (see e.g. [5, 4.3, p. 234]). It is also easy to arrange that handles of the same index are disjoint. This takes care of (1) and (2) of 5.1. If $k = 0$, then (3) is automatically satisfied. If $1 \leqslant k \leqslant m - 4$, then (since the pair $(N, \partial N)$ is

k-connected) we can apply [11, Theorem 3] to construct a handle decomposition of M in which D is a 0-handle and all other handles have index $> k$. So we get (3) in all cases.

Now suppose that M is nonorientable. Take an arbitrary handle decomposition of M in which D is the only 0-handle. Clearly, at least one of the 1-handles is nonorientable, i.e. such that its union with D is a nonorientable manifold. Denote by M_1 the union of D and all 1-handles. Let A_0 be a nonorientable 1-handle and let A be any other 1-handle. It is easy to see that sliding one component of the attaching tube of A isotopically across A_0 changes the orientability type of A, while M_1 remains the same up to homeomorphism [5, 2.1, p. 227]. It follows that we can always find decompositions in which there is only one nonorientable 1-handle and also decompositions in which all 1-handles are nonorientable. This concludes the proof of 5.1.

Choose, and fix, a handle decomposition of M satisfying (1)–(4) of 5.1. For each i denote by M_i the union of handles of index $\leqslant i$. As is well known we can represent M as a CW complex X such that, for each i, M_i is a regular neighborhood in M of the i-skeleton X^i of X and such that $\mathrm{Cl}(X^i - M_{i-1})$ is the union of core disks of i-handles of M. Observe that $M_p = N$ and let $N_0 = M_{p-1}$, $D_0 = \mathrm{Cl}(M - N_0)$, $P = X^p$, $P_0 = X^{p-1}$. Let $A = N \cap D_0$ and let $A_1, \ldots, A_n$ be the components of A, i.e. the p-handles of M. Further let $C = A \cap P$ and $C_j = A_j \cap P$; thus C_j is the core disk of A_j.

Take the handle decomposition of $e_0(M)$ corresponding to the chosen decomposition of M. Then construct a handle decomposition of W such that each handle of W intersects $e_0(N)$ in a handle of $e_0(N)$ having the same core disk. Let W_0 be the union of handles of W of index $\leqslant p - 1$ and let $V_0 = E^q - \mathrm{int}\, W_0$. Then $e_0(N_0)$ is properly embedded in W_0, and $e_0(D_0)$ is properly embedded in V_0. Finally let $B = W \cap V_0$ and let B_j be the p-handle of W with core disk $e_0(C_j)$ ($1 \leqslant j \leqslant n$).

6. **Improving isotopies of** M **in** E^q. The main result of this section is 6.3, which is an extension of Theorem 1.1 in [10]. In the following lemma we consider all our I^n and S^n oriented in the standard way, and each product $I^p \times S^n$ is given the orientation induced from S^{p+n}.

LEMMA 6.1. *Consider the standard inclusions* $I^p \times S^r \subset I^p \times S^{p+r}$ *and* $I^p \times a \subset I^p \times S^{p+r}$, *where* $p \geqslant 3$, $r \geqslant 0$, *and* $a \in S^{p+r} - S^r$. *For any embedding* $e\colon I^p \times S^r \to I^p \times S^{p+r}$ *such that* $e|S^{p-1} \times S^r = \mathrm{id}$ *let* $s(e)$ *denote the intersection number* $(I^p \times a) \# e(I^p \times S^r)$ *in* $I^p \times S^{p+r}$. *Then we have*:

(1) *if* p *is odd, then* $s(e) = 0$;

(2) *if* p *is even, then* $s(e)$ *is even and, moreover, for any integer* n *there exists*

an embedding e_n: $I^p \times S^r \to I^p \times S^{p+r}$ (*with* $e_n|S^{p-1} \times S^r = \mathrm{id}$) *such that* $s(e_n) = 2n$.

PROOF. First observe that $s(e)$ is independent of a: if $a, b \in S^{p+r} - S^r$, then there exists an isotopy H of proper embeddings of I^p in $I^p \times S^{p+r}$ such that $H_0 = \mathrm{id} \times a$, $H_1 = \mathrm{id} \times b$, and $H_t(S^{p-1})$ misses $e(I^p \times S^r)$ for every t; thus $I^p \times a$ and $I^p \times b$ have the same intersection number with $e(I^p \times S^r)$.

Lemma 6.1 will be proved by induction on r. Take first $r = 0$. Let $u = (1, 0^p) \in S^p$. For each integer n choose an embedding e_n: $I^p \times S^0 \to I^p \times S^p$ as follows: take any map g_n: $(I^p, S^{p-1}) \to (S^p, u)$ representing $n \in \pi_p(S^p, u)$ (i.e. g_n has degree n with respect to the standard orientations of I^p and S^p) and set $e_n(x, i) = (x, ig_n(x))$ for $x \in I^p$, $i = \pm 1$. Denote by $s_+(e_n)$ and $s_-(e_n)$ the intersection numbers of $I^p \times a$ with $e_n(I^p \times u)$ and $e_n(I^p \times (-u))$, respectively. Then $s_+(e_n) = n$ and $s(e_n) = s_+(e_n) - s_-(e_n)$.

The number $s_-(e_n)$ is, like $s(e_n)$, independent of a. Thus $s_-(e_n) = (I^p \times (-a)) \# e_n(I^p \times (-u))$. The autohomeomorphism h of $I^p \times S^p$ defined by $h(x, y) = (x, -y)$ has degree $(-1)^{p+1}$ and maps $I^p \times a$, $e_n(I^p \times u)$ onto $I^p \times (-a)$, $e_n(I^p \times (-u))$, respectively, in an orientation preserving way. Hence the intersection number of $h(I^p \times a)$ with $he_n(I^p \times u)$ in $I^p \times S^p$ is $(-1)^{p+1}$ times the same intersection number evaluated in $h(I^p \times S^p)$; in symbols, $s_-(e_n) = (-1)^{p+1}s_+(e_n)$. It follows that $s(e_n) = (1 + (-1)^p)s_+(e_n)$, i.e. $s(e_n) = 0$ if p is odd and $s(e_n) = 2n$ if p is even.

Take now an arbitrary embedding e: $I^p \times S^0 \to I^p \times S^p$ such that $e|S^{p-1} \times S^0 = \mathrm{id}$. Then $e|I^p \times u \simeq e_n|I^p \times u$ (rel $S^{p-1} \times u$) for some n. By Zeeman's unknotting theorem [5, 10.1, p. 198] we can get an isotopy F of $I^p \times S^p$ which is fixed on the boundary and carries $e|I^p \times u$ to $e_n|I^p \times u$. Let U be a regular neighborhood of $e_n(I^p \times u)$ in $I^p \times S^p$ such that U misses $e_n(I^p \times (-u))$ and $F_1 e(I^p \times (-u))$. Since the pair $(I^p \times S^p, e_n(I^p \times u))$ is homeomorphic to $(I^p \times S^p, I^p \times u)$ (e.g. by the n-isotopy extension theorem [5, 6.13, p. 154]), V is a $2p$-ball. Therefore the restrictions of $F_1 e$ and e_n to $I^p \times (-u)$ are homotopic (rel $S^{p-1} \times (-u)$) in V. We thus see that $e \simeq e_n$ (rel $S^{p-1} \times S^0$), and hence $s(e) = s(e_n)$. This finishes the case $r = 0$.

Now suppose that, for some $r > 0$, Lemma 6.1 is true for embeddings of $I^p \times S^{r-1}$ in $I^p \times S^{p+r-1}$. Take an embedding e: $I^p \times S^r \to I^p \times S^{p+r}$ such that $e|S^{p-1} \times S^r = \mathrm{id}$. Let D_1 be the hemisphere of S^{p+r} consisting of points in S^{p+r} with nonnegative first coordinate; let $D_2 = S^{p+r} - \mathrm{int}\, D_1$ and $B_i = S^r \cap D_i$ $(i = 1, 2)$. Using regular neighborhood theory (cf. the proof of [10, 2.1]) we can show that e is isotopic, via an isotopy of $I^p \times S^{p+r}$ which keeps the boundary fixed, to an embedding which maps $I^p \times \mathrm{int}\, B_i$ into $I^p \times \mathrm{int}\, D_i$ $(i = 1, 2)$; therefore we will assume that $e(I^p \times \mathrm{int}\, B_i) \subset I^p \times \mathrm{int}\, D_i$. The restriction $e' = e|I^p \times \partial B_1$ is then a proper embedding of $I^p \times \partial B_1$ into $I^p \times \partial D_1$ and $e'|S^{p-1} \times \partial B_1 = \mathrm{id}$. Conversely, any proper embedding e': $I^p \times \partial B_1 \to I^p \times \partial D_1$ with $e'|S^{p-1} \times \partial B_1 = \mathrm{id}$ can be extended to an embedding e of $I^p \times S^r$ into $I^p \times S^{p+r}$ such that $e|S^{p-1} \times S^r = \mathrm{id}$. If we choose our point a in $\partial D_1 - \partial B_1$, if we orient ∂B_1 coherently with B_1 and ∂D_1 coherently with D_1, and if we denote by $s'(e')$ the intersection number of $I^p \times a$ with $e'(I^p \times \partial B_1)$ in $I^p \times \partial D_1$, then $s(e) = s'(e')$. This, together with our hypothesis that 6.1 is true for $r - 1$, proves 6.1 for r.

The first part of the proof of 6.1 implies

COROLLARY 6.2. *Take the situation of* 6.1 *with* $r = 0$. *If we reverse the orientation in one component of* $I^p \times S^0$, *then we have*:

(1) *if* p *is even, then* $s(e) = 0$;

(2) *if* p *is odd, then* $s(e)$ *is even and, moreover, every even integer is obtained as* $s(e)$ *for some* e.

PROPOSITION 6.3. *Suppose that* $k = 0$ *or* $k \leqslant m - 4$. *Adopt the notation and conventions of* 3.1 *and of* §5. *Then, given any* $e \in \mathcal{E}_0$ *representing the same class of* $\mathcal{G}$ *as* e_0, *there exists an isotopy* F *of* E^q *such that*

$$F_1 e_0 = e, \quad F|(W_0 \cup C) \times I = \mathrm{id}, \quad \textit{and} \quad F(B \times I) = B \times I.$$

By 1.5 a stronger version of 6.3 holds in case OO (and hence also for $m = 4$, $k = 1$, which is essentially the only case not satisfying the hypotheses of 6.3). Nevertheless we will include the case OO in our proof since it will be discussed together with the case OE. We also remark that Proposition 6.3 and its proof remain valid if E^q is replaced with any $(m - k)$-connected q-manifold Q.

PROOF OF 6.3. We will again assume that M is a submanifold of E^q and that e_0 is the inclusion. If $k > 0$, 6.3 follows directly from [10, 1.1 and 2.6] (we let P_0, W_0, B have the roles of R, W, and W', respectively). Suppose, therefore, that $k = 0$ (and hence $q = 2m$ and $p = m - 1$). For this case [10, 1.1] implies that there exists an isotopy H of E^q, fixed on W_0, such that $H_1|M = e$. Thus our task is only to correct H, keeping it unchanged on $W_0 \times I$, so that the corrected isotopy will be fixed on C and will map $B \times I$ onto itself.

To get the desired "correction" of H it suffices to find a proper embedding g: $(D \cap A) \times I \to (V \cap B) \times I$ such that g is identity on $\partial((D \cap A) \times I)$ and the m-sphere

$$g((D \cap A) \times I) \cup ((D \cap N_0) \times I) \cup H(D \times \partial I)$$

is homologous to 0 in V. Indeed, suppose that we have found such an embedding g. By [6, 1.5 and 4.1] we can assume that g is the restriction to $(D \cap A) \times I$ of an isotopy G' of $V \cap B$ which is fixed on $\partial(V \cap B)$. Extend G' to an isotopy G of W, fixed on $W_0 \cup C$ and satisfying $G_1|A = \mathrm{id}$; the extension $G|B_i \times I$ of $G'|(B_i \cap V) \times I$ can be constructed as follows:

using a level preserving homeomorphism

$$(B_i, A_i, C_i, B_i \cap V) \times I \to (I^q, I^m, I^p, I^p \times S^m) \times I$$

we can transfer our extension problem to $I^q \times I$, where it can be solved exploiting the cone structure of $I^q \times I$ with base $(I^q \times 0) \cup (S^{q-1} \times I)$ and vertex $(0^q, 1)$. Let $g'\colon \partial(D \times I) \to \partial(V \times I)$ be the embedding which agrees with g on $(D \cap A) \times I$ and with H on the rest of $\partial(D \times I)$. By hypothesis, $g'(\partial(D \times I)) \sim 0$ in $V \times I$ (the symbol $\sim$ means "is homologous to"). It follows that $g' \simeq 0$ in $V \times I$, for $V \times I$ is $(m-1)$-connected, by a general position argument. Now Irwin's embedding theorem [5, 8.1, p. 174] implies that g' extends to a proper embedding $f'\colon D \times I \to V \times I$. By [6, 1.5 and 4.1] we can assume that there exists an isotopy F' of V which agrees with f' on $D \times I$ and with G on $\partial V \times I$. Then the isotopy F of E^q which is equal to F' on $V \times I$ and to G on $W \times I$ satisfies the assertions of 6.3.

To construct an embedding g as indicated above we shall have to divide our discussion into three cases. But first we construct a couple of new objects. Let $c\colon \partial W_0 \times I \to W_0$ be a collar on ∂W_0 such that $c(\partial N_0 \times I) \subset N_0$ and $c((P \cap \partial W_0) \times I) \subset P$. Let $B' = B \cup c((B \cap W_0) \times I)$ and let B_i' be the component of B' containing B_i $(1 \leqslant i \leqslant n)$. For each i choose a homeomorphism $h_i\colon (I^q, I^m, I^p) \to (B_i', B_i' \cap M, B_i' \cap P)$. Consider the parallelepiped $L = I^m \times I \times 0^p \subset I^q$. The cube I^m is a face of L and thus $\partial L - \operatorname{int} I^m$ is an m-ball. Let

$$M^* = (M - B') \cup \bigcup h_i(\partial L - \operatorname{int} I^m).$$

Then $M^* \cap C = \varnothing$. Choose a homeomorphism $M \to M^*$ which is identity on $M - B'$ and maps A onto $M^* \cap B$; for an arbitrary subset $U \subset M$ let $U^* \subset M^*$ be the image of U. Now we can start with

Cases OO *and* OE. Choose an i $(1 \leqslant i \leqslant n)$. By a general position argument V_0 is m-connected, and therefore Zeeman's unknotting theorem [5, 10.1, p. 198] implies that there exists an isotopy J of $V_0 \times I$, fixed on $\partial(V_0 \times I)$, which carries $H|C_i \times I$ to inclusion. Let

$$Y = (V_0 \times 0) \cup (\partial V_0 \times I) \cup H(M \times 1) \subset \partial(V_0 \times I).$$

Then $(B_i \times I, (B_i \times I) \cap Y)$ and $(J_1H(B_i \times I), (B_i \times I) \cap Y)$ are two regular neighborhoods of $C_i \times I$ in $(V_0 \times I, Y)$. Hence (cf. [10, 1.5]) there exists an isotopy K of $V_0 \times I$, fixed on $(C_i \times I) \cup Y$, which carries $J_1H(B_i \times I)$ onto $B_i \times I$. Let $G_i = K_1J_1H|B_i \times I\colon B_i \times I \to B_i \times I$. Now suppose that we have constructed such a G_i for each i. Define $G\colon W \times I \to W \times I$ by $G|W_0 \times I = \mathrm{id}$ and $G|B_i \times I = G_i$ $(1 \leqslant i \leqslant n)$, and then let $g\colon (D \cap A) \times I \to (B \cap V) \times I$ be the restriction of G.

Consider the embedding $g'\colon \partial(D \times I) \to \partial(V \times I)$ defined by $g'|\partial D \times I = G|\partial D \times I$ and $g'|D \times \partial I = H|D \times \partial I$. We have to show that

$$g'(\partial(D \times I)) \sim 0$$

in $V \times I$. Choose an orientation for $D_0^* \times I$. Then, as $(m+1)$-chains, $D_0^* \times I = (D \times I) + (A^* \times I)$, and consequently

$$\begin{aligned}\partial(D_0^* \times I) &= \partial(D \times I) + \partial(A^* \times I)\\ &= (D \times \partial I) + ((D \cap N_0) \times I) + (A^* \times \partial I)\end{aligned}$$

and

$$\begin{aligned}H(\partial(D_0^* \times I)) &= g'(D \times \partial I) + g'((D \cap N_0) \times I) + G(A^* \times \partial I)\\ &= g'(\partial(D \times I)) + G(\partial(A^* \times I)).\end{aligned}$$

We have $G(\partial(A^* \times I)) \sim 0$ in $(V_0 - C) \times I$, and hence in $V \times I$, since $A^* \times I \subset (V_0 - C) \times I$. Therefore it suffices to show that $H(\partial(D_0^* \times I)) \sim 0$ in $(V_0 - C) \times I$.

For each i, $H|C_i \times I \simeq \mathrm{id}$ (rel $\partial(C_i \times I)$) in $V_0 \times I$. Therefore

$$H(D_0^* \times I) \# (C_i \times I) = H(D_0^* \times I) \# H(C_i \times I) = 0.$$

Now we can apply Whitney's lemma [8, 5.12] to push $H(D_0^* \times I)$ isotopically off $C \times I$, keeping $H(\partial(D_0^* \times I))$ fixed. This proves that $H(\partial(D_0^* \times I)) \sim 0$ in $(V_0 - C) \times I$ and finishes the proof of 6.3 for the cases OO and OE.

Case NO. Divide $D \times I$ into four parts–R, S, T, and E–as follows (see Figure 1). Let R be a regular neighborhood of

$$(A \cap D) \times I \bmod \partial((A \cap D) \times I) \text{ in } D \times I$$

and let $E_1 = \mathrm{Cl}((D \times I) - R)$; let S be a regular neighborhood of $R \cap E_1$ in E_1 and let $E_2 = \mathrm{Cl}(E_1 - S)$; let T be a regular neighborhood of $S \cap E_2$ mod $\partial(S \cap E_2)$ in E_2 and let $E = \mathrm{Cl}(E_2 - T)$. For each i let R_i', R_i'' be the two components of R intersecting A_i and let $R_i = R_i' \cup R_i''$, $R' = \bigcup R_i'$, $R'' = \bigcup R_i''$. Similarly, and consistently, define S_i', S_i'', S_i, S', S'' and T_i', T_i'', T_i, T', T''. Now we are going to define a map $f\colon D \times I \to V_0 \times I$ by defining separately $f|E$, $f|T$, $f|S$, and $f|R$.

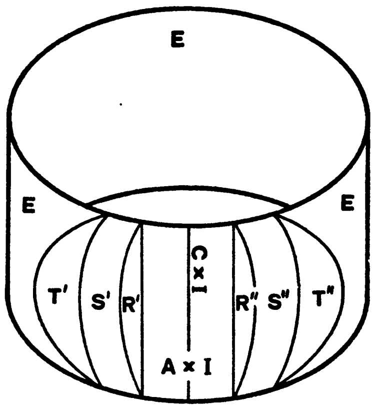

FIGURE 1

$D_0 \times I$, and hence $H(D_0^* \times I)$, can be thought of as the quotient space of E obtained by gluing $E \cap T_i'$ to $E \cap T_i''$ for each i. Let

$$f|E: E \to H(D_0^* \times I)$$

be the corresponding quotient map. More precisely, let f map E onto $H(D_0^* \times I)$ in such a way that $f(E \cap (D \times j)) = H(D_0^* \times j)$ ($j = 0, 1$), for each point $x \in H(C_i^* \times I)$ the inverse image $f^{-1}(x)$ contains exactly two points, one from each component of $E \cap T_i$, and $f^{-1}(x)$ is a single point for any other $x \in H(D_0^* \times I)$.

For each i let $f|S_i' \cap T_i'$ be a homeomorphism onto $C_i^* \times I$ extending $f|\partial(S_i' \cap T_i')$ (the latter map has already been defined since $\partial(S \cap T) = \partial(T \cap E)$). Then let $f|T'$: $T' \to V_0 \times I$ be an arbitrary extension of $f|\partial T'$; at least one such extension exists because V_0 is m-connected. Take a homeomorphism h: $T'' \to T'$ such that $f|E \cap T'' = (f|E \cap T')h$ and then let $f|T'' = (f|T')h$.

Now extend the map $f|S \cap T$: $S \cap T \to C^* \times I$ to a map $f|S$ of S onto $A^* \times I$ such that $f|S - T$ is a homeomorphism onto $(A^* - C^*) \times I$ and $f|\partial(R \cap S) = \mathrm{id}$.

The map $f|R$ will be defined a little later. It will depend on the intersection numbers of the balls $C_i \times I$ with $f(E_1) = f(E \cup S \cup T)$ (here $C \times I$ is oriented arbitrarily and E, S, T are given orientations induced by some orientation of $D \times I$). So let us see what we can say about these intersection numbers.

We can assume without loss of generality that $(C \times I) \cap H(C^* \times I) = \varnothing$. Then for each i the intersection numbers of $C_i \times I$ with $f(E)$ and $f(T)$ are defined and their sum is $(C_i \times I) \# f(E_1)$. In cases OO and OE we proved that $(C_i \times I) \# H(D_0^* \times I)$ was 0 for each i; now we can prove, using the same argument, only that the intersection number modulo 2 of $C_i \times I$ with $H(D_0^* \times I)$ is 0. This implies that $(C_i \times I) \# H(D_0^* \times I)$ is even. Observe that it follows from the manner in which $f|T''$ was defined that

$$(C_i \times I) \# f(T_j'') = (C_i \times I) \# f(T_j')$$

for each i and j; hence $(C_i \times I) \# f(T)$ is even. We conclude that $(C_i \times I) \# f(E_1) = 2n_i$ for some integer n_i.

Orient $(A \cap D) \times I = (A \times I) \cap R$ coherently with R and let $C^* \times I$ have the orientation corresponding to the chosen orientation of $C \times I$. Since

$$(B_i \cap V, A_i \cap D, C_i^*) \times I \approx \left(I^m \times S^m, I^m \times S^0, I^m \times (0, 1, 0^{m-1})\right)$$

and since each A_i is a nonorientable $(m - 1)$-handle of M it follows from 6.2 that there exists a proper embedding g: $(A \cap D) \times I \to (B \cap V) \times I$ such that $g|\partial((A \cap D) \times I) = \mathrm{id}$ and $(C_i^* \times I) \# g((A \cap D) \times I) = -2n_i$ for each i.

Now let $f|(A \cap D) \times I = g$ and then let $f|R\colon R \to B \times I$ be an arbitrary extension of the already defined map $f|\partial R\colon \partial R \to (B \cap V) \times I$. We have $(C_i \times I) \# f(R) = (C_i^* \times I) \# f(R \cap (A \times I)) = -2n_i$ for each i, and consequently $(C_i \times I) \# f(D \times I) = 0$. It easily follows that $f(\partial(D \times I)) \sim 0$ in $(V_0 - C) \times I$ and hence in $V \times I$. But $f|\partial(D \times I)$ is obviously homotopic in $\partial(V \times I)$ to the map g' which agrees with g on $(A \cap D) \times I$ and with H on the rest of $\partial(D \times I)$. Therefore $g'(\partial(D \times I)) \sim 0$ in $V \times I$, what we had to prove.

Case NE. By construction all $(m-1)$-handles A_i of M are orientable except one, say A_1. We can assume without loss of generality that H is fixed on C_1 and maps $B_1 \times I$ onto itself; for if H fails to have these two properties, we can find an isotopy J of $E^q \times I$, fixed on $E^q \times 0$, $W_0 \times I$, $H(M \times 1)$, such that $J_1 H|C_1 \times I = \mathrm{id}$ and $J_1 H(B_1 \times I) = B_1 \times I$ (cf. the beginning of the discussion of cases OO and OE above), and then we can replace H with $J_1 H$.

Define subsets R, S, T, E of $D \times I$ and maps $f|E$, $f|T - T_1$, $f|S - S_1$ as in case NO; let $f|T_1\colon T_1 \to H(C_1^* \times I)$ be the composition of a retraction $T_1 \to T_1 \cap E$ and of $f|T_1 \cap E$, and let $f|R_1\colon R_1 \to H((A_1 \cap D) \times I)$ be the composition of a retraction $R_1 \to R_1 \cap (A \times I)$ and of H; then let $f|S_1\colon S_1 \to (B_1 \cap V) \times I$ be an extension of $f|S_1 \cap (R_1 \cup T_1)$ such that $f|S_1 - T_1$ is a homeomorphism onto $H((A^* - C^*) \times I)$. Now we can show as in case NO that for each i there exists an integer n_i such that $(C_i \times I) \# f(E_1) = (C_i \times I) \# f(E) = 2n_i$; clearly $n_1 = 0$. By 6.1 we can now define $f|R - R_1$ in such a way that $(C_i \times I) \# f(D \times I) = 0$ for each i; then we can take $g = f|(A \cap D) \times I$. Proposition 6.3 is proved.

7. The definition of δ and the proof of 1.2. We will now prove that for some subgroup S of $H^p(N)$ there exists a bijective map $\theta_1\colon \mathcal{G} \to H^p(N)/S$ such that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{G}_0 & \xrightarrow{\theta_0} & H^p(N) \\ \downarrow & & \downarrow \\ \mathcal{G} & \xrightarrow{\theta_1} & H^p(N)/S \end{array}$$

This is an immediate consequence of the following

PROPOSITION 7.1. *Adopt the notation and conventions of* 3.1 *and* 3.4. *There exists a subgroup* $S \subset H^p(N)$ *such that for each* $\varepsilon \in \mathcal{G}$

$$\{\theta(e_2) - \theta(e_1) | e_1, e_2 \in \varepsilon \cap \mathcal{E}_0\} = S.$$

Specifically, $S = 0$ *in case* OO, $S = 2H^p(N)$ *in cases* OE *and* NO, *and* $S = 2\,\mathrm{im}[H^p(M) \to H^p(N)]$ *in case* NE.

PROOF. We shall assume that $k = 0$ or $k < m - 4$. As we know, this assumption excludes only the case $m = 4$, $k = 1$, for which 7.1 has already been proved (see 4.1). Of course, for the same reason we could now omit the whole case OO. But we will include this case in our proof since it requires no extra work.

Again we will think of M as a submanifold of E^q and of e_0 as the inclusion. Adopt the notation and conventions of §5. Then P is a deformation retract of N, and therefore θ induces a map θ': $\mathcal{E}_0 \to H^p(P)$; for an arbitrary $e \in \mathcal{E}_0$, $\theta'(e) \in H^p(P)$ is the Alexander dual of the class of $e(D) - e_0(D)$ in $H_m(E^q - P)$. Clearly we may prove the assertion of 7.1 for $H^p(P)$ and θ' instead of $H^p(N)$ and θ.

We know that $\theta'(e)$ (for $e \in \mathcal{E}_0$) depends only on the class of e in $\mathcal{I}_0$. Our plan for proving 7.1 is to replace every given pair e_1, $e_2 \in \mathcal{E}_0 \cap \varepsilon$ with another pair e_1', e_2' such that e_i and e_i' are in the same class of $\mathcal{I}_0$ ($i = 1, 2$) and such that the pair e_1', e_2' is sufficiently "nice" that an explicit evaluation of the difference $\theta'(e_2') - \theta'(e_1')$ is possible. For this purpose we must first propose a way for explicit evaluation of θ'. This is done in the next two paragraphs.

Let cP denote the cone on P; for every subset $T \subset P$ let cT be the subcone of cP with base T. Extend the inclusion $P \subset E^q$ to a map f: $cP \to E^q$; for every $T \subset P$ denote $f(cT)$ by γT. We can assume that f satisfies the following conditions:

(a) the singular set of f is empty or finite and misses $P \cup cP_0$;

(b) $f^{-1}(M) - P$ is empty or finite and misses cP_0;

(c) $(W, W \cap \gamma P)$ is a regular neighborhood of P in $(E^q, \gamma P)$;

(d) $B \cap \gamma P_0 = \varnothing$;

(e) $(B_i, A_i, C_i, B_i \cap \gamma P) \approx (I^q, I^m, I^p, I^p \times 0^{q-p-1} \times I)$ for each i.

In our proof of 7.1 we shall use the following explicit definition of θ'. Let $e \in \mathcal{E}_0$. Referring to the cellular cohomology of P (with respect to the CW structure of P as the p-skeleton of the CW complex X introduced in §5) consider the p-cochain which assigns to each oriented p-cell E of P the intersection number $(e(D) - e_0(D)) \# \gamma E$ where γE is oriented coherently with E. This cochain is automatically a cocycle, and its class in $H^p(P)$ is $\theta'(e)$.

Choose a regular neighborhood $(W', N', W' \cap \gamma P)$ of W in $(E^q, M, \gamma P)$ and let $V' = E^q - \operatorname{int} W'$, $D' = M - \operatorname{int} N'$. Choose a homeomorphism of $\partial W \times I$ onto $W' \cap V$ which maps $\partial N \times I$ onto $N' \cap D$ and $(\partial W \cap \gamma P) \times I$ onto $W' \cap V \cap \gamma P$. Let B' be the union of B and the image of $(B \cap V) \times I$; analogously define A', A_i', B_i' ($1 \leqslant i \leqslant n$). Denote by $\mathcal{E}_0'$ the set of all embeddings e: $M \to E^q$ such that $e|N' = \mathrm{id}$ and $e(\operatorname{int} D') \subset \operatorname{int} V'$. By 3.3 each class of $\mathcal{I}_0$ has a representative in $\mathcal{E}_0'$.

For each $\varepsilon \in \mathcal{I}$ denote by $T(\varepsilon)$ the set of all pairs $(e_1, e_2) \in \mathcal{E}_0 \times \mathcal{E}_0$ such that $e_1 \in \mathcal{E}_0' \cap \varepsilon$, e_2 agrees with e_1 on $D' \cup N \cup N_0'$, and $e_2(A' \cap D)$ is properly embedded in $B' \cap V$. We assert that for each $\varepsilon \in \mathcal{I}$ the set

$$S(\varepsilon) = \{\theta'(e_2) - \theta'(e_1) | e_1, e_2 \in \mathcal{E}_0 \cap \varepsilon\},$$

in which we are interested, coincides with the set

$$S'(\varepsilon) = \{\theta'(e_2) - \theta'(e_1) | (e_1, e_2) \in T(\varepsilon)\}.$$

First we will show that $S'(\varepsilon) \subset S(\varepsilon)$. Take $(e_1, e_2) \in T(\varepsilon)$. The only thing to prove is that e_2 is isotopic to e_1. We construct an isotopy G of E^q carrying e_2 to e_1 as follows. Let G be fixed on $V' \cup W_0'$. Let $G|(B' \cap V) \times I$ be an isotopy of $B' \cap V$ which carries $e_2|A' \cap D$ to inclusion ($= e_1|A' \cap D$), keeping $B' \cap V'$ and $B' \cap V \cap W_0'$ fixed (existence of such an isotopy is guaranteed by [6, 1.1]). To extend G over $B \times I$ choose for each i a level preserving homeomorphism $h_i\colon (B_i, A_i, C_i) \times I \to (I^q, I^m, I^p) \times I$. Thinking of $I^q \times I$ as the rectilinear cone with base $U = (I^q \times 0) \cup (S^{q-1} \times I)$ and vertex $(0^q, 1)$ extend the level preserving autohomeomorphism $h_i G h_i^{-1}|U$ of U conewise to a (level preserving) autohomeomorphism of $I^q \times I$. Then transfer this extension back to $B_i \times I$ and let the result, by definition, by $G|B_i \times I$. Note that $G_1|A = \mathrm{id}$ since $G_1|\partial A = \mathrm{id}$ and since $G_1|A_i$ is the conewise extension of $G_1|\partial A_i$. This means that $G_1 e_2|A = e_1|A$; clearly $G_1 e_2 = e_1$ elsewhere. The inclusion $S'(\varepsilon) \subset S(\varepsilon)$ is proved.

Now we will prove the other inclusion, i.e. $S(\varepsilon) \subset S'(\varepsilon)$. Take $e_1, e_2 \in \mathcal{E}_0 \cap \varepsilon$ and consider $\theta'(e_2) - \theta'(e_1) \in S(\varepsilon)$. Since $\mathcal{E}_0'$ intersects each class of $\mathcal{I}_0$ and since $\theta'(e_i)$ depends only on the class of e_i in $\mathcal{I}_0$ ($i = 1, 2$) we can assume without loss of generality that $e_1, e_2 \in \mathcal{E}_0' \cap \varepsilon$. Now, since e_1 and e_2 are isotopic it follows from 6.3 that there exists an isotopy F of E^q such that $F_1 e_2 = e_1$, $F|(W_0' \cup C) \times I = \mathrm{id}$, and $F(B' \times I) = B' \times I$. Define isotopies K and L of E^q as follows. Identify $W' \cap V$ with $\partial W' \times I$ in such a way that $\partial W'$ is identified with $\partial W' \times 0$, ∂W with $\partial W' \times 1$, and $B' \cap V$ with $(B' \cap V') \times I$. Let $K|W \times I = \mathrm{id}$ and $K|V' \times I = F|V' \times I$; on $W' \cap V = \partial W' \times I$ define K by

$$K_t(x, s) = \begin{cases} (x, s) & \text{if } t \leqslant s \\ (F_{t-s}(x), s) & \text{if } t \geqslant s \end{cases} \qquad (x \in \partial W'; s, t \in I).$$

Let L be fixed on $V' \cup W_0'$; on $W' \cap V = \partial W' \times I$ define L by

$$L_t(x, s) = \begin{cases} (x, s) & \text{if } t \leqslant 1 - s \\ (F_t F_{1-s}^{-1}(x), s) & \text{if } t \geqslant 1 - s \end{cases} \qquad (x \in \partial W'; s, t \in I);$$

then extend L conewise over each $B_i \times I$ (cf. the definition of $G|B \times I$ above).

Let $e = K_1 e_2$. Then $L_1 e = e_1$, and hence $(e_1, e) \in T(\varepsilon)$. Since K is fixed on W, e and e_2 are in the same class of $\mathcal{G}_0$. Therefore $\theta'(e_2) - \theta'(e_1) = \theta'(e) - \theta'(e_1) \in S'(\varepsilon)$. This completes the proof that $S(\varepsilon) = S'(\varepsilon)$.

The set $S'(\varepsilon)$ will be relatively easy to describe explicitly. For each i denote by E_i the p-cell of the CW complex $X^p = P$ that contains C_i; choose an orientation for E_i. Let $(e_1, e_2) \in T(\varepsilon)$. According to what was said above, $\theta'(e_2) - \theta'(e_1) \in H^p(P)$ is the class of the p-cocycle u which assigns to each E_i the intersection number $(e_2(D) - e_1(D)) \# \gamma E_i$. Note that $e_2(D) - e_1(D) = e_2(A' \cap D) - (A' \cap D)$. Since $A' \cap D$ misses γP and since $e_2(A_j' \cap D)$ misses γE_i for $i \neq j$, $u(E_i) = e_2(A_i' \cap D) \# \gamma E_i$. It will be easier for us to see the result if we bring everything to a canonical situation. For each i we have

$$\begin{aligned}(B_i' \cap V, A_i' \cap D, B_i' \cap V \cap \gamma E_i) &\\ \approx (I^p \times S^{q-p-1}, I^p \times S^{m-p-1}, I^p \times a) \times I &\\ \approx (I^{m-k} \times S^m, I^{m-k} \times S^k, I^{m-k} \times a)&\end{aligned}$$

where $a = (0^m, 1) \in S^m$. Choose, and fix, homeomorphisms

$$\begin{aligned}g_i\colon (B_i' \cap V, A_i' \cap D, B_i' \cap V \cap \gamma E_i)&\\ \to (I^{m-k} \times S^m, I^{m-k} \times S^k, I^{m-k} \times a).&\end{aligned}$$

Then each $f_i = g_i e_2 g_i^{-1} | I^{m-k} \times S^k$ is a proper embedding of $I^{m-k} \times S^k$ into $I^{m-k} \times S^m$ satisfying $f_i | S^{m-k-1} \times S^k = \mathrm{id}$. Clearly,

$$u(E_i) = f_i(I^{m-k} \times S^k) \# (I^{m-k} \times a).$$

Now, as (e_1, e_2) ranges over all $T(\varepsilon)$ we get for each f_i all possible embeddings of $I^{m-k} \times S^k$ into $I^{m-k} \times S^m$ such that $f_i | S^p \times S^k = \mathrm{id}$. Therefore 6.1 and 6.2 will give us the complete information about the set $S'(\varepsilon)$.

Case OO. It follows from 6.1 that all $f_i(I^{m-k} \times S^k) \# (I^{m-k} \times a)$ are 0 for any choice of the f_i. Thus $S'(\varepsilon) = 0$ for any ε.

Case OE. Again by 6.1, $f_i(I^{m-k} \times S^k) \# (I^{m-k} \times a)$ is necessarily even and, moreover, it can be any even integer. Hence $S'(\varepsilon) = 2H^p(P)$ for any ε.

Case NO. Now each A_i is a nonorientable handle of M and therefore 6.2 implies that the set of all possible values for each $f_i(I^m \times S^0) \# (I^m \times a)$ is $2Z$. Thus we have again $S'(\varepsilon) = 2H^p(P)$ for each ε.

Case NE. Now one $(m-1)$-handle of M, say A_1, is nonorientable and all other $(m-1)$-handles are orientable. Therefore 6.2 implies that

$$f_1(I^m \times S^0) \# (I^m \times a)$$

is necessarily 0 while the possible values for any other $f_i(I^m \times S^0) \# (I^m \times$

a) are precisely all even integers. The description of $S'(\varepsilon)$ is now a little more involved.

Represent M as the CW complex X as described in §5. The cellular chain complex $C_*(X)$, regarded as a graded group, is the direct sum of graded subgroups $C_*^{(1)}$ and $C_*^{(2)}$ which are defined as follows: $C_j^{(1)} = 0$ if $j > m$ or $j < m-1$, $C_m^{(1)} = C_m(X)$ = the infinite cyclic group generated by the unique m-cell of X, and $C_{m-1}^{(1)}$ = the cyclic subgroup of $C_{m-1}(X)$ generated by E_1; $C_j^{(2)} = 0$ if $j \geqslant m$, $C_j^{(2)} = C_j(X)$ if $j < m-1$, and $C_{m-1}^{(2)}$ = the subgroup of $C_{m-1}(X)$ generated by $E_2, \ldots, E_n$. Clearly $C_*^{(2)}$ is a subcomplex of $C_*(X)$. Now consider $C_*^{(1)}$. The boundary of the generator of $C_m^{(1)}$ is $2E_1$; hence E_1 is a cycle and hence the boundary homomorphism maps $C_{m-1}^{(1)}$ to 0. It follows that $C_*^{(1)}$, too, is a subcomplex of $C_*(X)$. For the subcomplex $X^{m-1} = P$ of X we have $C_j(X^{m-1}) = 0$ for $j \geqslant m$ and $C_j(X^{m-1}) = C_j^{(1)} \oplus C_j^{(2)}$ for $j < m$.

The cochain complex $C^*(X) = \mathrm{Hom}(C_*(X), Z)$ is the direct sum of cochain complexes $C_{(1)}^* = \mathrm{Hom}(C_*^{(1)}, Z)$ and $C_{(2)}^* = \mathrm{Hom}(C_*^{(2)}, Z)$. Observe that the coboundary homomorphism $\delta\colon C_{(1)}^{m-1} \to C_{(1)}^m$ is injective; in particular, it maps the generator of $C_{(1)}^{m-1}$ to twice the generator of $C_{(1)}^m$. It follows that $H^{m-1}(X) = C_{(2)}^{m-1}/\delta C_{(2)}^{m-2}$. Similarly we find out that $H^{m-1}(X^{m-1}) = C_{(1)}^{m-1} \oplus (C_{(2)}^{m-1}/\delta C_{(2)}^{m-2})$. We also see that the restriction homomorphism $H^{m-1}(X) \to H^{m-1}(X^{m-1})$ is precisely the inclusion $H^{m-1}(X) \subset C_{(1)}^{m-1} \oplus H^{m-1}(X) = H^{m-1}(X^{m-1})$.

Now it easily follows from what we said at the beginning of the discussion of the case NE that $S'(\varepsilon) = 2H^{m-1}(X) \subset H^{m-1}(X^{m-1})$, or, if we do not wish to refer to a particular CW structure of M and P, $S'(\varepsilon)$ is twice the image of $H^{m-1}(M) \to H^{m-1}(P)$. This finishes the case NE and concludes the proof of 7.1.

As we observed at the beginning of this section, 7.1 implies that θ induces a bijective map $\theta_1\colon \mathcal{G} \to H^p(N)/S$. Let $\varepsilon_0 \in \mathcal{G}$ be the class of e_0. We will now show that the map θ_1, which apparently depends on e_0, depends actually only on ε_0. Take an arbitrary $e_0' \in \varepsilon_0$. Then e_0' determines a bijective map θ_1': $\mathcal{G} \to H^p(N)/S$ analogous to θ_1. Choose an isotopy F of E^q such that $e_0' = F_1 e_0$. Then the following diagram is commutative

$$\begin{tikzcd}
H_m(E^q - e_0(N)) \arrow[r, "\approx"] \arrow[d, "F_{1*}"] & H^p(e_0(N)) \arrow[r, "e_0^*"] & H^p(N) \arrow[r] & H^p(N)/S \\
H_m(E^q - e_0'(N)) \arrow[r, "\approx"] & H^p(e_0'(N)) \arrow[r, "e_0'^*"] \arrow[u, "F_1^*"] & H^p(N) \arrow[r] \arrow[u, "\mathrm{id}"] & H^p(N)/S \arrow[u, "\mathrm{id}"]
\end{tikzcd}$$

Now it will be easy to show that $\theta_1' = \theta_1$. Take any $\varepsilon \in \mathcal{G}$. In order to evaluate $\theta_1(\varepsilon)$ represent ε by an embedding $e\colon M \to E^q$ which agrees with e_0

on a neighborhood of N; for evaluating $\theta_1'(\varepsilon)$ we will use the embedding $e' = F_1 e \in \varepsilon$, which agrees with e_0' on a neighborhood of N. Now, $\theta_1(\varepsilon)$ is the image of $[e(D) - e_0(D)] \in H_m(E^q - e_0(N))$ under the composition of the homomorphisms in the first row of the above diagram, and $\theta_1'(\varepsilon)$ is the image of $[e'(D) - e_0'(D)] \in H_m(E^q - e_0'(N))$ under the composition of the homomorphisms in the second row of the diagram. Thus the commutativity of the diagram implies that, indeed, $\theta_1'(\varepsilon) = \theta_1(\varepsilon)$.

We shall now write $\theta^*(\varepsilon_0, \varepsilon)$ for $\theta_1(\varepsilon)$. Since ε_0 is arbitrary we get in this way a map $\theta^*\colon \mathcal{G} \times \mathcal{G} \to H^p(N)/S$. Now we have arrived at the point where we can define the map $\delta\colon \mathcal{G} \times \mathcal{G} \to \mathcal{H}_{k+1}(M)$ and prove that it has the desired properties.

Case OO. Let $\delta\colon \mathcal{G} \times \mathcal{G} \to H_{k+1}(M)$ be the composition of $\theta^*\colon \mathcal{G} \times \mathcal{G} \to H^p(N)$ and the isomorphism $\pi\colon H^p(N) \to H_{k+1}(M)$ (as defined at the beginning of §4). In principle δ could depend on D and its orientation, but in fact it depends on neither. It is very easy to see that δ is independent of the orientation of D: if we reverse the orientation of D, then both θ^* and π are multiplied by -1 and hence δ remains unchanged.

Now let D, D' be two arbitrary m-balls in M and let $\delta, \delta'\colon \mathcal{G} \times \mathcal{G} \to H_{k+1}(M)$ correspond to D and D', respectively. We will show that $\delta' = \delta$. Let $N = M - \operatorname{int} D$, $N' = M - \operatorname{int} D'$. Pick any two elements $\varepsilon_0, \varepsilon \in \mathcal{G}$. Represent ε_0 by an arbitrary $e_0 \in \mathcal{E}$ and then choose $e \in \varepsilon$ so that e and e_0 agree on a neighborhood of N in M. There exists an isotopy G of M such that $G_1(D) = D'$. Then $(e_0 \times \mathrm{id})G\colon M \times I \to E^q \times I$ is an isotopy of M in E^q and hence, by the isotopy extension theorem, there exists an isotopy F of E^q such that $F_t e_0 = e_0 G_t$ for each t. Note that the following diagram is commutative.

$$\begin{tikzcd}
H_m(E^q - e_0(N)) \arrow[r, "\approx"] \arrow[d, "F_{1*}"] & H^p(e_0(N)) \arrow[r, "\approx"] & H^p(N) \arrow[r, "\approx"] & H_{k+1}(M) \arrow[d, "G_{1*} = \mathrm{id}"] \\
H_m(E^q - e_0(N')) \arrow[r, "\approx"] & H^p(e_0(N')) \arrow[r, "\approx"] \arrow[u, "F_1^*"] & H^p(N') \arrow[r, "\approx"] \arrow[u, "G_1^*"] & H_{k+1}(M)
\end{tikzcd}$$

By definition, $\delta(\varepsilon_0, \varepsilon)$ is the image of $[e(D) - e_0(D)] \in H_m(E^q - e_0(N))$ under the composition of the isomorphisms in the first row of the diagram. Since $e' = F_1 e G_1^{-1}$ is isotopic to e and since e_0 and e' agree on a neighborhood of N' in M, $\delta'(\varepsilon_0, \varepsilon)$ is the image of

$$[e'(D') - e_0(D')] \in H_m(E^q - e_0(N'))$$

under the composition of the isomorphisms in the second row of the diagram. Since

$$\begin{aligned}[e'(D') - e_0(D')] &= [F_1 e G_1^{-1}(D') - F_1 e_0 G_1^{-1}(D')] \\ &= F_{1*}([e(D) - e_0(D)]),\end{aligned}$$

the commutativity of the diagram implies that $\delta'(\varepsilon_0, \varepsilon) = \delta(\varepsilon_0, \varepsilon)$.

Now that we know that δ is independent of all the arbitrary choices involved in its definition it is trivial to check that δ has properties (1) and (2) of 1.2; we proved before that δ satisfies (3). Thus Theorem 1.2 is proved for the case OO.

Cases OE *and* NO. We showed that $S = 2H^p(N)$. Hence $H^p(N)/S \approx H^p(N) \otimes Z_2 \approx H^p(N;\ Z_2)$. The map $\delta\colon\ \mathcal{G} \times \mathcal{G} \to H_{k+1}(M;\ Z_2)$ is now defined to be the composition of θ^* and the isomorphism $\pi_2\colon H^p(N;\ Z_2) \to H_{k+1}(M;\ Z_2)$ which was defined at the beginning of §4. The proof that δ is independent of D and that it has the right properties is practically the same as in case OO.

Case NE. Now S is twice the image of $H^{m-1}(M) \to H^{m-1}(N)$. To simplify the notation we shall identify (in this proof) $H^{m-1}(M)$ with its image in $H^{m-1}(N)$, and analogously we shall do so for every natural monomorphism. From the following diagram, in which all homomorphisms are induced by inclusions,

$$\begin{tikzcd} H^{m-1}(M, D) \arrow[r, "\approx"] \arrow[d, "\approx"] & H^{m-1}(M) \arrow[d] \\ H^{m-1}(N, \partial N) \arrow[r] & H^{m-1}(N) \end{tikzcd}$$

we see that the images of $H^{m-1}(M)$ and $H^{m-1}(N, \partial N)$ in $H^{m-1}(N)$ are equal. We have, therefore, a map $\theta^*\colon\ \mathcal{G} \times \mathcal{G} \to H^{m-1}(N)/2H^{m-1}(N, \partial N)$.

The manifold M has a preferred orientation over the orientation sheaf Γ of M: Take a triangulation for M and orient all m-simplices of this triangulation. If σ is an arbitrary (oriented) m-simplex of M, let b_σ be its barycenter and let u_σ be the generator of $\Gamma(b_\sigma)$ determined by the orientation of σ. The sum of all $u_\sigma \sigma$ is an m-cycle over Γ, and the corresponding element of $H_m(M;\ \Gamma)$, which is a generator of $H_m(M;\ \Gamma)$, is, by definition, the "positive" orientation of M over Γ. Analogously we define the positive orientation of N over $\Gamma|N$.

Choose a point $x_0 \in \operatorname{int} D$ and consider the following commutative diagram, in which the two leftmost horizontal maps are Poincaré duality isomorphisms determined by the positive orientation of N over $\Gamma|N$ and all other homomorphisms are induced by inclusions.

$$\begin{tikzcd}
H^{m-1}(N, \partial N) \arrow[r, "\approx"] \arrow[d] & H_1(N; \Gamma|N) \arrow[r, "\approx"] \arrow[d] & H_1(M; \Gamma) \arrow[d] & H_1(M; \Gamma) \arrow[l, "\mathrm{id}"'] \arrow[d] \\
H^{m-1}(N) \arrow[r, "\approx"] & H_1(N, \partial N; \Gamma|N) \arrow[r, "\approx"] & H_1(M, D; \Gamma) & H_1(M, x_0; \Gamma) \arrow[l, "\approx"']
\end{tikzcd}$$

Since all horizontal maps in the diagram are isomorphisms we get a natural isomorphism

(7.2) $\qquad H^{m-1}(N)/2H^{m-1}(N, \partial N) \to H_1(M, x_0; \Gamma)/2H_1(M; \Gamma).$

Now define δ: $\mathcal{G} \times \mathcal{G} \to H_1(M, x_0; \Gamma)/2H_1(M; \Gamma) = \mathcal{H}_1(M, x_0)$ to be the composition of θ^* and (7.2).

Clearly δ depends on x_0; it also depends on the orientation of D (or, better to say, on the local orientation of M at x_0 which is determined by the chosen orientation of D), but not on D itself. The proof that δ is independent of D is similar to the corresponding proof in case OO and we shall omit it.

We thus have a function which assigns to each point x_0 of M and to each generator u_0 of $\Gamma(x_0)$ a map

$$\delta = \delta(-, -; u_0): \mathcal{G} \times \mathcal{G} \to \mathcal{H}_1(M, x_0).$$

Let us now examine the dependence of δ on x_0 and u_0. Choose two distinct points $x_0, x_1 \in M$ and generators $u_i \in \Gamma(x_i)$ $(i = 0, 1)$. Take an arc A from x_0 to x_1 such that u_1 is u_0 transported along A. Let D be a regular neighborhood of A in M and let D have the orientation induced by u_0 (or u_1). Then we can use this D for evaluating both $\delta(-, -; u_0)$ and $\delta(-, -; u_1)$, and thus the definition of δ implies that $\delta(-, -; u_0)$ and $\delta(-, -; u_1)$ differ by the isomorphism $\mathcal{H}_1(M, x_0) \to \mathcal{H}_1(M, x_1)$ which is induced by the following diagram

$$\begin{tikzcd}
H_1(M; \Gamma) \arrow[r, "\mathrm{id}"] \arrow[d] & H_1(M; \Gamma) \arrow[d] & H_1(M; \Gamma) \arrow[l, "\mathrm{id}"'] \arrow[d] \\
H_1(M, x_0; \Gamma) \arrow[r, "\approx"] & H_1(M, D; \Gamma) & H_1(M, x_1; \Gamma) \arrow[l, "\approx"']
\end{tikzcd}$$

Obviously this isomorphism is the same as $\lambda(-; u_0, u_1)$ which was defined in §1. Therefore $\delta(-, -; u_1) = \lambda(-; u_0, u_1)\delta(-, -; u_0)$. The latter relation remains valid, as can easily be checked, if we take $x_1 = x_0$. Verification of properties (1) and (2) of 1.2 will again be omitted.

8. **Proof of Theorem 1.4.** We shall assume that $m \geqslant 2k + 2$. For if $m < 2k + 2$, then M is $[\frac{1}{2}m]$-connected and hence $M \approx S^m$; in this case Theorem 1.4 is true by Zeeman's unknotting theorem [5, 10.1, p. 198].

The symbol $\mathcal{G}$ will now denote the set of isotopy classes of embeddings

$M \to Q$. Choose, and fix, an embedding e_0: $M \to Q$ such that $e_0(M)$ lies in the interior of a q-ball $R \subset Q$. We will again use the notation and refer to the conventions introduced in 3.1, except that now E^q has to be replaced with Q in all definitions. Also, we will identify M with $e_0(M)$, i.e. we will assume that M is a submanifold of Q and that e_0 is the inclusion.

PROPOSITION 8.1. *Each orientation of Q determines a natural bijective map θ_0: $\mathcal{I}_0 \to H^p(N) \oplus \pi_m(Q)$.*

PROOF. The same method as used in the proof of 3.4 gives a natural bijective map $\mathcal{I}_0 \to \pi_m(Q - N)$. We assert that $\pi_m(Q - N) \approx H^p(N) \oplus \pi_m(Q)$.

We can show as in the proof of 3.4 that $R - N$ is $(m - 1)$-connected. Consider the exact homotopy sequence of the pair $(Q - N, R - N)$:

$$\pi_m(R - N) \to \pi_m(Q - N) \to \pi_m(Q - N, R - N) \to 0. \tag{8.2}$$

Choose a triangulation T for $Q - \operatorname{int} R$ and let, for each n, T^n be the n-skeleton of T. It is well known, and is easy to see, that if $y \in Q - R$, then $Q - \operatorname{int} R - y$ retracts onto ∂R. It follows that there is a retraction of $(R - N) \cup T^{m+1}$ onto $R - N$ which maps T^{m+1} into ∂R; moreover, since $m + 1 < q - 1$, this retraction is determined uniquely up to homotopy. Therefore the inclusion induced homomorphism

$$\pi_m(R - N) \to \pi_m\big((R - N) \cup T^{m+1}\big)$$

has a canonical left inverse. Since $\pi_m((R - N) \cup T^{m+1}) \to \pi_m(Q - N)$ is an isomorphism we get at the same time a left inverse of $\pi_m(R - N) \to \pi_m(Q - N)$, and it is easy to prove that the latter left inverse is independent of T. Therefore it follows from (8.2) that there exists a natural isomorphism

$$\pi_m(Q - N) \approx \pi_m(R - N) \oplus \pi_m(Q - N, R - N).$$

By the Hurewicz theorem and Alexander duality, $\pi_m(R - N) \approx H^p(N)$, and by the Blakers-Massey theorem [9, Theorem 5, p. 484], $\pi_m(Q - N, R - N) \approx \pi_m(Q, R) \approx \pi_m(Q)$. This proves 8.1.

From the proof of 8.1 we also get the following

ADDENDUM 8.3. For an arbitrary $\varepsilon \in \mathcal{I}_0$, $\theta_0(\varepsilon) \in H^p(N) \oplus \pi_m(Q)$ is obtained as follows. We choose a representative $e \in \mathcal{E}_0$ of ε. Then the projection of $\theta_0(\varepsilon)$ into $\pi_m(Q)$ is the homotopy class of the (oriented) singular m-sphere $e(D) \cup (-D)$, and the projection of $\theta_0(\varepsilon)$ into $H^p(N)$ is the Alexander dual, taken in a regular neighborhood U of R, of $[e'(D) - D] \in H_m(U - \operatorname{int} N)$ where e': $M \to R$ is any map such that $e'|e^{-1}(R) = e|e^{-1}(R)$ and $e'(e^{-1}(Q - R)) \subset \partial R$.

Let θ: $\mathcal{E}_0 \to H^p(N) \oplus \pi_m(Q)$ be the composition of $\mathcal{E}_0 \to \mathcal{I}_0$ and θ_0. Theorem 1.4 and Corollary 1.5 follow from the next

PROPOSITION 8.4. *Take an arbitrary* $\varepsilon \in \mathcal{G}$. *Then*

$$\{\theta(e_2) - \theta(e_1) | e_1, e_2 \in \mathcal{E}_0 \cap \varepsilon\} = S$$

where S is, as in 7.1, *the subgroup of* $H^p(N) \subset H^p(N) \oplus \pi_m(Q)$ *equal to* 0 *in case* OO, *to* $2H^p(N)$ *in cases* OE *and* NO, *and to* 2 im$[H^p(M) \to H^p(N)]$ *in case* NE.

PROOF. If $k = 0$ or $k < m - 4$, the proof of 8.4 is an almost exact repetition of the proof of 7.1. Adopt the notation and conventions of §5 and of the proof of 7.1, with obvious modifications. Construct the singular cone γP in int R. As observed before, 6.3 remains valid if E^q is replaced by Q. Therefore we can prove as in 7.1 that $S(\varepsilon) = S'(\varepsilon)$. Now, if $(e_1, e_2) \in T(\varepsilon)$, then by 8.3 the $\pi_m(Q)$-component of $\theta(e_2) - \theta(e_1)$ is 0, and the $H^p(N)$-component of $\theta(e_2) - \theta(e_1)$ is the Alexander dual of $[e_2'(D) - e_1'(D)]$ where e_i': $M \to R$ agrees with e_i on $e_i^{-1}(R)$ and maps $e_i^{-1}(Q - R)$ into ∂R ($i = 1, 2$). Clearly we can choose e_1' and e_2' so that they agree on

$$e_1^{-1}(Q - R) = e_2^{-1}(Q - R).$$

Then $e_2'(D) - e_1'(D) = e_2(A' \cap D) - (A' \cap D)$. From here on, the proof of 8.4 is the same as that of 7.1.

Now consider the remaining case $m = 4$, $k = 1$. As in the proof of 1.2 for this case we shall exploit here the relation between our θ and Hudson's d (and again our proof will work for the whole case OO). Let e_1, e_2: $M \to Q$ be two embeddings, and suppose that there exists a (general position) map G: $M \times I \to Q \times I$ such that $G(x, i - 1) = (e_i(x), i - 1)$ ($i = 1, 2$; $x \in M$). In §4, where we had $Q = E^q$, a certain construction based on G produced an element of $H_{k+1}(M)$, which depended only on the isotopy classes of e_1 and e_2 and was then defined to be the difference class d between those two isotopy classes. We can perform the very same construction in the present case. We cannot be sure, however, that the resulting element of $H_{k+1}(M)$ is independent of G; therefore we now denote this element by $d'(G)$. As in [5, 11.6, p. 208], $F \simeq G$: $M \times I \to Q \times I$ (rel $M \times \partial I$) implies $d'(F) = d'(G)$.

Take any two isotopic embeddings e_1, e_2: $M \to Q$ (in case OO). Let N' be a regular neighborhood of N in M and let $D' = M - \operatorname{int} N'$. By an analogue of 3.3 we can assume without loss of generality that $e_1|N' = e_2|N' = \mathrm{id}$. Since e_1 and e_2 are isotopic there exists a level preserving embedding F: $M \times I \to Q \times I$ such that $F_0 = e_1$ and $F_1 = e_2$. We wish to replace F with a map G: $M \times I \to Q \times I$ such that $G|N' \times I = \mathrm{id}$ and $G \simeq F$ (rel $M \times \partial I$).

In the first step we shall construct a homotopy K: $(N' \times I) \times I \to Q \times I$ such that $K_0 = F|N' \times I$, $K_1 = \mathrm{id}$, and $K_t|N' \times \partial I = \mathrm{id}$ for each t. The obstructions to constructing such a homotopy K lie in the groups

$$H^n(N' \times I, N' \times \partial I; \pi_n(Q \times I))$$

[3, 7.3.6]. Using the exact cohomology sequence of $(N' \times I, N' \times \partial I)$ and Poincaré duality it is not hard to prove that

$$H^n(N' \times I, N' \times \partial I; A) \approx H^{n-1}(N'; A) \approx \tilde{H}_{m-n+1}(M; A)$$

for any coefficient group A. Thus the obstructions to constructing K lie in the groups $\tilde{H}_{m-n+1}(M; \pi_n(Q))$. But all these groups are 0 since M is k-connected and Q is $(m-k)$-connected. Hence K exists.

Let $Y = (M \times I \times 0) \cup (N' \times I \times I) \cup (M \times \partial I \times I) \subset (M \times I) \times I$. We can extend K to a map K': $Y \to Q \times I$ by sending each point $(x, s; 0) \in (M \times I) \times 0$ to $F(x, s)$ and each point $(x, s; t) \in (M \times \partial I) \times I$ to (x, s). Then, since $(M \times I) \times I$ retracts onto Y, we can extend K' to a map L: $(M \times I) \times I \to Q \times I$. Let $G = L_1$: $M \times I \to Q \times I$. Then G has the desired properties.

From G we can easily construct a homotopy (rel ∂D) between $e_1|D$ and $e_2|D$. Hence, by 8.3, the $\pi_m(Q)$-component of $\theta(e_2) - \theta(e_1)$ is 0. To evaluate the $H^p(N)$-component pick a point $y \in Q - R$ such that

$$(y \times I) \cap G(M \times I) = \varnothing$$

and then choose a retraction r: $Q - y \to R$ such that $r(Q - \text{int } R - y) \subset \partial R$. Consider the maps re_i: $M \to R$ $(i = 1, 2)$ and $(r \times \text{id})G$: $M \times I \to R \times I$. Let W' be a regular neighborhood of $W \cup N'$ mod D' in Q. By Irwin's embedding theorem [5, 8.1, p. 174] there exist embeddings e_i': $M \to \text{int } R$ $(i = 1, 2)$ such that $e_i' \simeq re_i$ (rel N'), as maps of $(M, \text{int } D')$ into $(R, R - W')$. It follows from the homotopy extension theorem that we can deform $(r \times \text{id})G$ to a (general position) map G': $M \times I \to (\text{int } R) \times I$ such that $G'(x, i-1) = (e_i'(x), i-1)$ $(i = 1, 2;\ x \in M)$ and $G'|G^{-1}(W' \times I) = G|G^{-1}(W' \times I)$.

By 8.3 the $H^p(N)$-component of $\theta(e_2) - \theta(e_1)$ is the Alexander dual of

$$[re_2(D) - re_1(D)] = [e_2'(D') - e_1'(D')].$$

By 4.1 this is further equal to $\pm \pi^{-1}d'(G')$. Now, $\pi^{-1}d'(G')$ depends only on the manner in which $G'(D' \times I)$ intersects $N \times I$ (cf. the proof of 4.1). Therefore, since G' and G agree on $G^{-1}(W' \times I)$, we have $\pi^{-1}d'(G') = \pi^{-1}d'(G)$. But $d'(G) = d'(F) = 0$ because $G \simeq F$ (rel $M \times \partial I$) and because F is an embedding. It follows that $\theta(e_2) - \theta(e_1) = 0$, and hence $S = 0$. This concludes the proof of 8.4 and of 1.4.

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