LATTICE-VALUED BOREL MEASURES. II

BY

SURJIT SINGH KHURANA

ABSTRACT. Let $T$ be a completely regular Hausdorff space, $C_b(T)$ the set of all bounded real-valued continuous functions on $T$, $E$ a boundedly monotone complete ordered vector space, and $\varphi: C_b(T) \to E$ a positive linear map. It is proved that under certain conditions there exist $\sigma$-additive, $\tau$-smooth or tight $E$-valued measures on $T$ which represent $\varphi$.

Let $T$ be a completely regular Hausdorff space, $C_b(T)$ the vector-lattice of all bounded, real-valued functions on $T$. Let $E$ be a boundedly complete partially ordered vector space and $\varphi: C_b(T) \to E$ a positive linear map, i.e., $f \in C_b(T), f > 0$ implies $\varphi(f) > 0$. In case $T$ is compact, it is known ([2], [5]) that a quasi-regular Borel measure $\mu$ on $T$ which represents $\varphi$, i.e., $\varphi(f) = \int f d\mu$, $\forall f \in C(T)$, all continuous real-valued functions on $T$ (see [4], [5] for details). The more general case of a Hausdorff completely regular space is the aim of study in this paper.

For a topological space $Y$ let $\mathcal{B}(Y)$ be the $\sigma$-algebra of all Borel subsets of $Y$ and $\mathcal{B}_0(Y)$ the $\sigma$-algebra of all Baire subsets of $Y$ (that is, the smallest $\sigma$-algebra which makes each bounded continuous function on $Y$ measurable). Let $B(Y)$ ($B_0(Y)$) be the space of all bounded Borel (Baire) measurable functions on $Y$. For basic facts about vector lattices we refer to [1] (see also [2], [4]-[9]). We shall make use of the result proved in [2], that if a boundedly $\sigma$-complete vector lattice $E$ contains a vector subspace $F$ which is monotone order $\sigma$-closed and if $F$ contains a vector sublattice $G$ of $E$, then $F$ contains the order $\sigma$-closure of $G$. If $S$ is a Stonian ($\sigma$-Stonian) compact Hausdorff space we define a mapping $\psi: B(S) \to C(S)$ ($\psi_1: B_0(S) \to C(S)$), $\psi(f) = f$ ($\psi_1(f) = f$) except on a meagre subset of $S$. It is easy to verify that $\psi$ and $\psi_1$ are positive order $\sigma$-continuous linear maps and for any increasing net $\{f_\alpha\} \subset C(S)$, with $\sup f_\alpha = f \in B(S)$, $\psi(f) = \sup \psi(f_\alpha)$, $\sup$ being taken in the boundedly complete vector lattice $C(S)$ [2] (these maps are called Loomis-Sikorski maps [5]).

Throughout the paper any $E$-valued measure $\mu$ on any $\sigma$-algebra is required

Received by the editors July 7, 1975 and, in revised form, April 28, 1976.


Key words and phrases. $\tau$-smooth measures, tight measures, weakly $\sigma$-distributive lattices, monotone order $\sigma$-continuous, order $\sigma$-continuous, monotone order $\sigma$-closed, order $\sigma$-closed.
to be nonnegative and $\sigma$-additive with respect to the order of $E$, i.e., whenever \( \{F_n\}, 1 \leq n < \infty \), is a monotone sequence in the $\sigma$-algebra then $\mu(\bigcup_{n=1}^{\infty} F_n) = \sqrt{\sum_{n=1}^{\infty} \mu(F_n)}$. Integration with respect to these measures is taken in the sense of [4], [5]. Throughout this paper $\phi: C_b(T) \to E$ is assumed to be a given positive linear map, $T$ being a Hausdorff completely regular space, and $E$ a boundedly monotone complete vector lattice assumed to be over the field of real numbers. Denoting by $X$ the Stone-Čech compactification of $T$, we get a positive linear map $\overline{\phi}: C(X) \to E$, $\overline{\phi}(f) = \phi(f|_T)$, which is represented by a quasi-regular Borel measure $\overline{\mu}$ on $X$, by the

**Proposition 1 (Wright [4], [5]).** Given a positive linear map $\phi: C_b(T) \to E$, there exists a unique quasi-regular $E$-valued Borel measure $\overline{\mu}$ on $X$ such that

$$\phi(f|_X) = \int f d\overline{\mu}, \text{ for all } f \in C(X).$$

**Proof.** This theorem is proved in [5]. By quasi-regularity we mean that for any open subset $V$ of $X$, $\overline{\mu}(V) = \sup\{ \mu(C): C \text{ compact}, C \subseteq V \}$.

To get a measure on $T$ we first note the following result.

**Lemma 2.** (i) $\mathfrak{B}_0(T) = \{ E \cap T: E \in \mathfrak{B}_0(X) \}$.

(ii) $\mathfrak{B}(T) = \{ E \cap T: E \in \mathfrak{B}(X) \}$.

**Proof** is similar to Lemma C [6].

**Corollary 3.** There is a unique $E$-valued Baire measure $\mu$ on $X$ which represents $\phi$, i.e.,

$$\phi(f) = \int f d\mu, \text{ for all } f \in C_b(T),$$

if and only if $\overline{\mu}(A) = 0$ for any $A = \mathfrak{B}_0(X)$ with $A \cap T = \emptyset$.

**Proof.** If $\overline{\mu}(A) = 0, \forall A \in \mathfrak{B}_0(X)$, with $A \cap T = \emptyset$, then defining $\mu(P) = \overline{\mu}(P_0)$, where $P_0 \cap T = P$ for a $P_0 \in \mathfrak{B}_0(X)$, for any $P \in \mathfrak{B}_0(T)$, it is easy to see that $\mu$ is well defined, is countably additive, and $\int f d\mu = \int f|_X d\mu, \forall f \in C(X)$. Conversely, if there is such a $\mu$, then $\int f d\mu = \int f|_X d\mu, \forall f \in C(X)$. This means $\{ f \in B_0(X): \int f d\mu = \int f|_X d\mu \}$ contains $C(X)$ and is monotone order $\sigma$-closed. Thus $\int f d\mu = \int f|_X d\mu, \forall f \in \mathfrak{B}_0(X)$, and so the result follows (cf. [6, Theorem E]).

For each positive $e$ in $E$, let

$$E[e] = \{ a \in E: \exists \lambda > 0 \text{ such that } -\lambda e < a < \lambda e \}.$$

Thus $E[e]$ is an order-unit space and can be equipped with the order-unit norm. Since $\phi$ and $\overline{\mu}$ take their values in $E (\phi(1))$ there is no loss of generality in supposing $E$ is an order-unit space with order unit $e = \phi(1)$. Let $E$ be equipped with order-unit norm.
Proposition 4. A sufficient condition for the existence of a unique $E$-valued Baire measure $\mu$ on $T$ which represents $\varphi$ is that there exists a weakly $\sigma$-distributive [6], boundedly $\sigma$-complete vector lattice $W$ such that $E$ can be embedded (without alternation of suprema) in $W$ and, whenever $\{f_n\}$ is a monotonic decreasing sequence in $C_b(T)$ with pointwise infimum 0, then $\bigwedge_{n=1}^{\infty} \varphi(f_n) = 0$.

Proof. In this case, $\tilde{\mu}: B_0(X) \to E$ is regular [6]. The given condition gives $\tilde{\mu}(Z) = 0$ for any zero set $Z$ of $X$, $Z \cap T = \emptyset$ (by zero-set we mean $f^{-1}\{0\}$, for some $f \in C(X)$). By regularity $\mu(P) = 0$ for any $P \in B_0(X)$, $P \cap T = \emptyset$. Corollary 3 now gives the result.

Proposition 5. A sufficient condition for the existence of a unique $E$-valued Baire measure $\mu$ on $T$ which represents $\varphi$ is that whenever $\{f_n\}$ is a monotone decreasing sequence in $C_b(T)$ which pointwise converges to 0 then $\|\varphi(f_n)\| \to 0$.

Proof. Proceeding as in [2] we see that $\tilde{\varphi}(C(X))$ is embedded, as an ordered vector space, in $C(S)$ for some Stonean compact Hausdorff space $S$, preserving arbitrary suprema and infima and $\tilde{\varphi}(1)$ being the constant function 1 in $C(S)$ (this can also be done by taking MacNeille-Dedekind completion of $E$ [4], [9]). This gives us a positive linear map $\varphi: C_b(T) \to C(S) \subset B_0(S)$, with pointwise order in $B_0(S)$. Since $B_0(S)$ is trivially weakly $\sigma$-distributive, it follows from [8, Theorem 3.4] that $\varphi$ extends to a linear, positive, monotone order $\sigma$-continuous map $\varphi: B_0(T) \to B_0(S)$. Since $(\psi_1 \circ \varphi)^{-1}(E)$ is monotone order $\sigma$-closed and contains the lattice $C_b(T)$, we get $(\psi_1 \circ \varphi)^{-1}(E) = B_0(T)$. Defining $\mu = \psi_1 \circ \varphi|_{B_0(T)}$ we get the desired Baire measure. The uniqueness is easy to verify.

Remark. It is enough to assume in Proposition 4 and 5 that $E$ is boundedly monotone order $\sigma$-complete.

Definition. (a) An $E$-valued measure $\mu: B(T) \to E$ is said to be $\tau$-smooth if, whenever $\{U_n\}$ is an increasing net of open sets, $\mu(\bigcup U_n) = \bigvee \mu(U_n)$.

(b) An $E$-valued measure $\mu: B(T) \to E$ is said to be tight if for any open set $U$, $\mu(U) = \bigvee \{ \mu(C): C \text{ compact, } C \subseteq U \}$.

We list some properties of these measures.

Proposition 6. (i) If $\mu$ is $\tau$-smooth then for any decreasing net $\{g_a\}$ of bounded, upper semicontinuous, real-valued functions with pointwise inf $g_a = g \in B(X_0)$, $\int g d\mu = \bigwedge a \int g_a d\mu$; also for an open set $V$, in $T$, $\mu(V) = \bigvee \{ \mu(P): P \subseteq V, P \text{ closed in } T \}$.

(ii) If $\mu$ is tight, then the following statements hold:

(a) $\mu$ is $\tau$-smooth.

(b) If a net $\{f_n\}$, in $B(T)$, converges to $f \in B(T)$, uniformly on compact
subsets of $T$, $\|f_n\| < 1$, $\forall \alpha$, and $E$ is a boundedly complete vector lattice, then $\int f_n d\mu \to 0$ (order convergence).

(c) If $V_1$ and $V_2$ are open subsets of $T$, then $\mu(V_1 \setminus V_2) = \sqrt{\mu(C)}: C$ compact, $C \subset V_1 \setminus V_2$.

(d) For a bounded, nonnegative, lower semicontinuous function $f$ on $T$, $\int f d\mu = \sqrt{\{ g \mu: 0 \leq g \leq f, g$ simple and a combination of characteristic functions of disjoint compact subsets of $T\}.$

**Proof.** (i) This follows by using the inequality

$$\frac{1}{n} \sum_{i=1}^{n} \mu \left( \left\{ x \in T: f(x) > \frac{i}{n} \right\} \right) \leq \int f d\mu \leq \frac{1}{n} \mu(T) + \frac{1}{n} \sum_{i=1}^{n} \mu \left( \left\{ x \in T: f(x) > \frac{i}{n} \right\} \right),$$

valid for any $n > 1$ and for any measurable $f$, $0 < f < 1$. The regularity property follows from the fact that every point of $T$ has a nbd. base consisting of closed sets.

(ii) (a) is trivially obvious. To prove (b), we have, for any compact subset $C$ of $T$,

$$\int f d\mu = \int_C f d\mu + \int_{T \setminus C} f d\mu \leq \lim \int_C f_n d\mu + \mu(T \setminus C)$$

$$= \lim \left( \int_C f_n d\mu - \int_{T \setminus C} f_n d\mu \right) + \mu(T \setminus C)$$

$$\leq \lim \int f_n d\mu + 2\mu(T \setminus C).$$

In a similar way we get $\int f d\mu > \liminf \int f_n d\mu - 2\mu(T \setminus C)$, from which the result follows. Proof of (c) is straightforward. To prove (d), assume $0 < f < 1$, fix a positive integer $n$ and let $V_i = \{ x \in T: f(x) > i/n \}, 0 < i < n$. $V_i$'s are open and

$$\sum_{i=1}^{n} \frac{i-1}{n} \chi_{V_{i-1}\setminus V_i} < f \leq \sum_{i=1}^{n} \frac{i}{n} \chi_{V_{i-1}\setminus V_i}.$$

Using (c) and the fact

$$\int \left( \sum_{i=1}^{n} \frac{i}{n} \chi_{V_{i-1}\setminus V_i} - \sum_{i=1}^{n} \frac{i-1}{n} \chi_{V_{i-1}\setminus V_i} \right) d\mu = \frac{1}{n} \mu(V_0) = \frac{1}{n} \mu(T)$$

we get the result.

**Corollary 7.** There exists an $E$-valued $\tau$-smooth measure $\mu$ on $T$ which represents $\varphi$, if and only if $\mu(P) = 0$, for each Borel set $P \in \mathcal{B}(X)$, $P \cap T = \emptyset$.
Proof. If \( \tilde{\mu}(P) = 0 \) for any \( P \in \mathfrak{B}(X) \), \( P \cap T = \emptyset \), we define \( \mu(Q) = \tilde{\mu}(Q) \) for any \( Q \in \mathfrak{B}(T) \), \( \tilde{\mu} \) being in \( \mathfrak{B}(X) \) such that \( Q \cap T = Q \). It is easily verified that \( \mu \) is well defined and is countably additive. Also, since \( \tilde{\mu} \) is \( \tau \)-smooth and \( \tilde{\mu}(P) = 0 \), \( \forall P \in \mathfrak{B}(X) \), \( P \cap T = \emptyset \), it easily follows that \( \mu \) is \( \tau \)-smooth. Further, \( \int f d\tilde{\mu} = \int f|_X d\mu \), \( \forall f \in C(X) \). Conversely, if there is such a \( \mu \), then

\[
\int f d\tilde{\mu} = \int f|_X d\mu, \forall f \in C(X).
\]

This means \( H = \{ f \in B(X) : \int f d\tilde{\mu} = \int f|_X d\mu \} \) is a monotone order \( \sigma \)-closed subspace of \( B(X) \) and contains all bounded upper semicontinuous on \( X \) (using Proposition 6). Since the subspace, of \( B(X) \), generated by upper semicontinuous functions on \( X \) is a vector sublattice of \( B(X) \) (simple verification) we have \( H = B(X) \) and so the result follows (cf. Corollary 3).

Proposition 8. A sufficient condition for the existence of a unique \( \tau \)-smooth \( E \)-valued Borel measure \( \mu \) on \( T \) which represents \( \varphi \) is that

(i) whenever \( \{ f_\alpha \} \) is a decreasing net in \( C_b(T) \) with \( \inf f_\alpha = 0 \) (pointwise order) than \( \wedge \varphi(f_\alpha) = 0 \), and

(ii) \( E \) is embedded, as an ordered vector space, in a weakly \( (\sigma, \infty) \)-distributive vector lattice \([7]\), preserving arbitrary suprema and infima.

Proof. Idea of proof is same as Proposition 4. The measure \( \tilde{\mu} : \mathfrak{B}(X) \rightarrow E \) is regular in this case. Proceeding as in Proposition 4 and using Corollary 7, we get the result.

Proposition 9. A sufficient condition for the existence of a unique \( \tau \)-smooth \( E \)-valued Borel measure \( \mu \) on \( T \) which represents \( \varphi \) is that whenever \( \{ f_\alpha \} \) is a decreasing net in \( C_b(T) \) with \( \inf f_\alpha = 0 \) (pointwise order), then \( \| \varphi(f_\alpha) \| \rightarrow 0 \).

Proof. As in Proposition 5, \( \tilde{\varphi}(C(X)) \) can be considered embedded, as an ordered vector space, in \( C(S) \) for a Stonian compact Hausdorff space \( S \), preserving arbitrary suprema and infima. This gives us a positive linear map \( \varphi : C_b(T) \rightarrow C(S) \subset B_1(S) \), \( B_1(S) \) being all bounded real-valued functions on \( S \) with pointwise order. Since \( B_1(S) \) is boundedly complete and weakly \( (\sigma, \infty) \)-distributive, using Proposition 8, we get a \( \tau \)-smooth \( B_1(S) \)-valued measure \( \mu_0 \) on \( T \) representing \( \varphi \). Now \( H = \{ f \in B(T) : \int f d\mu_0 \in B(S) \} \) is a monotone order \( \sigma \)-closed subspace on \( T \) and so \( H = B(T) \) (same argument as in Corollary 7). The required measure is \( \mu = \psi \circ \mu_0 \mid \mathfrak{B}(T) \). To prove it is \( E \)-valued let \( H_1 = \{ f \in B(T) : \int f d\mu \in E \} \). Then \( H_1 \) is a monotonic order \( \sigma \)-closed subspace of \( B(T) \) and contains upper semicontinuous bounded functions on \( T \). Arguing as in Corollary 7, we prove \( H_1 = B(T) \). This proves \( \mu \) is \( E \)-valued. Uniqueness is easily verified.
Proposition 10. A necessary and sufficient condition for \( \varphi \) being representable by a unique tight Borel measure \( \mu \) on \( T \) is that \( \varphi(1) = \mu(X) = \sqrt{\{ \mu(C) : C \text{ compact, } C \subset T \}} \). If \( E \) is a boundedly complete vector lattice then this will happen if and only if for any uniformly bounded net \( \{ f_\alpha \} \subset C_b(T) \) such that \( f_\alpha \to 0 \) uniformly on compact subsets of \( X_0 \), \( \varphi(f_\alpha) \to 0 \) in \( E \) (order convergence).

Proof. First suppose that the condition is satisfied. Let \( B \) be any Borel subset of \( X \) disjoint from \( T \) and let \( C \) be any compact subset of \( T \). Then \( \mu(X) \geq \mu(B \cup C) = \mu(B) + \mu(C) \). Thus \( \mu(B) = 0 \) and so by Corollary 7 there exists a well-defined \( \tau \)-smooth \( E \)-valued Borel measure \( \mu \) on \( T \) with \( \mu(B) = \mu(B \cap T) \), \( \forall B \in \mathcal{B}(X) \). In particular, \( \mu(C) = \mu(C) \) for any compact \( C \subset T \). If \( P \) is a closed subset of \( X \), then for any compact \( C \subset T \), we have

\[
\mu(P \setminus P \cap C) = \mu(P \setminus C) < \mu(T \setminus C) < \mu(X \setminus C) = \mu(X) - \mu(C),
\]

and so \( \mu(P) > \sqrt{\{ \mu(P \cap C) : C \text{ compact in } T \}} \). Now for any open set \( U \) in \( T \), \( \mu(U) = \sqrt{\{ \mu(P) : P \subset U, P \text{ closed in } X \}} \) (Proposition 6). Hence \( \mu \) is tight. Converse and uniqueness are easy to verify. Let \( E \) be a boundedly complete vector lattice and suppose that \( \varphi \) satisfies the hypothesis. We define a partial order on \( I = \{(C, \alpha) : C \text{ a compact subset of } T \text{ and } \alpha \text{ a finite subset of } T \setminus C \}, (C_2, \alpha_2) \geq (C_1, \alpha_1) \text{ if } C_2 \supset C_1 \text{ and } \alpha_2 \supset \alpha_1 \setminus C_2 \). \( I \) becomes a directed set. Define \( \forall (C, \alpha) \in I, f_{C, \alpha} \in C(X), 0 < f_{C, \alpha} < 1, 
\]

\[
f_{C, \alpha} = \begin{cases} 0, & \text{on } \alpha, \\ 1, & \text{on } C. \\ \end{cases}
\]

Evidently \( f_{C, \alpha} \rightharpoonup 1 \) uniformly on compact subsets of \( T \) and so \( \mu(f_{C, \alpha}) \to \mu(1) \) (order convergence) in \( E \). For a \( (C_0, \alpha_0) \in I \), \( \inf \{ \mu(f_{C, \alpha}) : (C, \alpha) \geq (C_0, \alpha_0) \} \) is the infimum of \( \{ \mu(C) : C \text{ compact, } C \subset T \} \). The converse is straightforward.

Remark. The second characterization of Proposition 10 is the definition of the tight functional given in [3].

Using similar methods we have the following sufficient condition for the measure extension to hold in any boundedly \( \sigma \)-complete vector lattice.

Proposition 11. Let \( \mathcal{S} \) be an algebra of subsets of a set \( Y \), \( \mathcal{S}^\sigma \) the \( \sigma \)-algebra generated by \( \mathcal{S} \), and \( E \) a boundedly monotone \( \sigma \)-complete partially ordered vector space. Let \( q: \mathcal{S} \to E \) be a positive, finitely additive set function such that whenever \( \{ A_n \} \) is a monotone decreasing sequence in \( \mathcal{S} \) with \( \cap_{n=1}^\infty A_n = \emptyset \), then \( \| q(A_n) \| \to 0 \). Then there exists a countably additive \( E \)-valued measure \( q^\sigma \) on \( \mathcal{S}^\sigma \) which extends \( q \).

Proof. As in Proposition 5, we can consider \( q(\mathcal{S}) \subset C(S) \subset B_0(S) \), for some Stonian compact Hausdorff space \( S \). With pointwise order on \( B_0(S) \), it is weakly \( \sigma \)-distributive and so we have a countably additive measure \( \mu: \mathcal{S}^\sigma \to \).
The desired measure is \( q^\sigma = \psi_1 \circ \mu \). It is easy to verify that \( q^\sigma \) is \( E \)-valued.

I am very grateful to the referee for making many useful suggestions which simplified some proofs.

References


Department of Mathematics, The University of Iowa, Iowa City, Iowa 52242