TOPOLOGICAL ENTROPY AT AN $\Omega$-EXPLOSION

BY

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Abstract. In this paper an example is given of a $C^2$ map $g$ from the circle onto itself, which permits an $\Omega$-explosion. It is shown that topological entropy (considered as a map from $C^2(S^1, S^1)$ to the nonnegative real numbers) is continuous at $g$.

1. Introduction. Let $g$ denote any $C^2$ mapping of the circle onto itself which satisfies the following properties (see Figure 1):

- $g$ has an expanding fixed point $e$ and a contracting fixed point $c$, and these are the only fixed points of $g$.
- $g$ preserves orientation at $e$ and $c$.
- $g$ has nondegenerate singularities $t$ and $s$, and these are the only singularities of $g$.
- The points $e$, $t$, $s$, $g(s)$, and $c$ are distinct and in order on the circle in the counterclockwise direction.
- $g$ is one-to-one on each of the intervals $(e, t)$, $(t, s)$, and $(s, e)$. Here we
use the notation \((a, b)\) to denote the open arc from a counterclockwise to \(b\).

(6) \(g(t) = e\).

These properties imply that \(\Omega(g) = \{e, c\}\), where \(\Omega(g)\) denotes the nonwandering set (see [2], [5], or [7] for definition).

It is easy to see that \(g\) permits an \(\Omega\)-explosion. By this we mean that for any neighborhood \(N\) of \(g\) in \(C^2(S^1, S^1)\), there is a map \(f \in N\) with \(\Omega(f)\) infinite. See Proposition 9 in §4 for a proof.

Let \(\text{ent}\) denote topological entropy (see §2 for the definition). Our main result is the following:

**Theorem A.** The map \(\text{ent}: C^2(S^1, S^1) \to \mathbb{R}\) is continuous at \(g\).

Theorem A implies that for any bifurcation through \(g\), at the map \(g\) there is no sudden jump in the amount of action.

To prove Theorem A we first, in §3, obtain an upper bound for entropy (Theorem 6). Then, in §4, we construct a sequence \((f_n)\) of maps with \(\text{ent}(f_n) \to 0\) as \(n \to \infty\). Finally, in §5, we prove Theorem A by using Theorem 6 to show that for arbitrarily large \(n\) if \(f\) is close enough to \(g\) in \(C^2(S^1, S^1)\), then \(\text{ent}(f) < \text{ent}(f_n)\).

2. Preliminary definitions and results. We first review the definition of topological entropy given in [1]. Let \(X\) be a compact space and \(f: X \to X\) a continuous map. For any two open covers \(\mathcal{U}\) and \(\mathcal{V}\) of \(X\), let \(\mathcal{U} \lor \mathcal{V}\) denote \(\{ A \cap B: A \in \mathcal{U} \text{ and } B \in \mathcal{V}\}\), and let \(f^{-n}(\mathcal{U})\) denote \(\{ f^{-1}(A): A \in \mathcal{U}\}\). Let \(M_n(f, \mathcal{U})\) denote the minimum cardinality of a subcover of \(X\) of

\[\mathcal{U} \lor f^{-1}(\mathcal{U}) \lor \cdots \lor f^{-n+1}(\mathcal{U}).\]

We set

\[\text{ent}(f, \mathcal{U}) = \lim_{n \to \infty} \frac{1}{n} (\ln(M_n(f, \mathcal{U}))),\]

where \(\ln\) denotes the natural logarithm. It is easy to see that this limit exists and is finite (see [1]). Finally, we define the topological entropy of \(f\) by

\[\text{ent}(f) = \sup(\text{ent}(f, \mathcal{U}))\]

where the supremum is taken over all open covers \(\mathcal{U}\) of \(X\). If \(X\) is a metric space it suffices to consider any sequence of open covers whose diameter approaches zero (see [1]). By the diameter of an open cover \(\mathcal{U}\) we mean the supremum of the diameters of the open sets in \(\mathcal{U}\).

We now state some basic facts about topological entropy which will be used later. In each of the four propositions, \(f\) is a continuous map of a compact space \(X\) into itself. Proposition 2 follows immediately from the definition, and Proposition 4 follows from Proposition 3.

**Proposition 1** (see [5]). If \(X\) is a metric space, \(\text{ent}(f) = \text{ent}(f|\Omega(f))\).
Proposition 2. If $X$ is finite, $\text{ent}(f) = 0$.

Proposition 3 (see [1]). If $X_1$ and $X_2$ are closed subsets of $X$, with $X_1 \cup X_2 = X$ and $f(X_1) \subset X_1$ and $f(X_2) \subset X_2$, then
\[
\text{ent}(f) = \max\{\text{ent}(f|X_1), \text{ent}(f|X_2)\}.
\]

Proposition 4. If $K$ is a closed subset of $X$, with $f(K) \subset K$, then $\text{ent}(f|K) \leq \text{ent}(f)$.

We will assume the reader is familiar with the following terminology (see [2] or [7]); nonwandering set, expanding fixed point, contracting fixed point, and stable manifold of a contracting fixed point (denoted $W^s(c)$). For any point $x \in W^s(c)$, we will use the notation $s\text{slsm}(x)$ to denote the component of $W^s(c)$ which contains $x$.

3. An upper bound for entropy. The proof of Theorem 6 (which modifies a theorem of [2]) uses the following lemma from [2] (see [2, §3, Lemma 5]).

Lemma 5. Let $f \in C^0(S^1, S^1)$. Let $K_1, \ldots, K_n$ be proper closed intervals of $S^1$, such that for each $i = 1, \ldots, n - 1$, $f|K_i$ is a homeomorphism and $f(K_i) \subset K_{i+1}$. Let $\mathcal{E}$ be a covering of $K_1 \cup \cdots \cup K_n$ by finitely many open intervals, such that $\forall A \in \mathcal{E}$ and $i = 1, \ldots, n$, $A \cap K_i$ is an interval (or empty). Let $\text{card}(\mathcal{E}) = k$ (card denotes cardinality). Then there is a subset $\mathcal{E}'$ of $(\ast)$ which covers $K_1$, with $\text{card}(\mathcal{E}') < n \cdot k$.

In Theorems 6 and A we will use the following definition. Let $\mathcal{C}$ be a finite collection of closed intervals on $S^1$, $\mathcal{C} = \{I(1), \ldots, I(p)\}$, and let $f \in C^0(S^1, S^1)$. We denote by $K_n(f, \mathcal{C})$ the number of distinct nonempty sets of the form
\[
I(j_1) \cap f^{-1}(I(j_2)) \cap \cdots \cap f^{(-n+1)}(I(j_n)),
\]
where $j_i \in \{1, \ldots, p\}$ for $i = 1, \ldots, n$.

Theorem 6. Let $f \in C^1(S^1, S^1)$, and let $\mathcal{C} = \{I(1), \ldots, I(p)\}$ be a finite collection of proper closed intervals on $S^1$. Let $W = S^1$ or $W = S^1 - (O_1 \cup O_2 \cup \cdots \cup O_m)$ where $O_i$ is a component of the stable manifold of a contracting periodic point $c_i$ for $i = 1, \ldots, m$. Suppose the following conditions hold.

1. $I(1) \cup \cdots \cup I(p) = W$.
2. For $j = 1, \ldots, p$, $f$ maps $I(j)$ homeomorphically onto its image.
3. For any $i = 1, \ldots, p$ and $j = 1, \ldots, p$, $f(I(i)) \cap I(j)$ is an interval.

Then
\[
\text{ent}(f) \leq \lim_{n \to \infty} \frac{1}{n} \left( \ln(K_n(f, \mathcal{C})) \right).
\]

Proof. Let $\delta$ denote the minimum length of the intervals $S^1 - I(j)$ where $j = 1, \ldots, p$. Let $\mathcal{E}$ be any finite cover of $I(1) \cup \cdots \cup I(p)$ by open
intervals with the diameter of $\mathcal{O}$ less than $\delta$. Let $k = \text{card}(\mathcal{O})$.

Let

$$I(j_1, j_2, \ldots, j_n) = I(j_1) \cap f^{-1}(I(j_2)) \cap \cdots \cap f^{-n+1}(I(j_n))$$

where $j_i \in \{1, 2, \ldots, p\} \ \forall i = 1, \ldots, n$. Then each nonempty $I(j_1, j_2, \ldots, j_n)$ is a closed interval and $f$ maps $I(j_1, j_2, \ldots, j_n)$ homeomorphically into $I(j_2, \ldots, j_n)$. For any fixed $I(j_1, j_2, \ldots, j_n)$, by Lemma 5 (with $K_i = I(j_1, \ldots, j_n)$), there is a subset of $(\ast)$ which covers $I(j_1, j_2, \ldots, j_n)$ of cardinality at most $n \cdot k$.

Let $X = S^1 - (\bigcup_{i=1}^{n} W^s(c_i))$. Then $X$ is a compact set with $f(X) \subset X$ and $f^{-1}(X) \subset X$.

Let $\mathcal{U}(X)$ be the open cover of $X$ defined by $\mathcal{U}(X) = \{A \cap X: A \in \mathcal{U}\}$. Then for any $I(j_1, j_2, \ldots, j_n)$, the minimal number of open sets of

$$\mathcal{U}(X) \cup f^{-1}(\mathcal{U}(X)) \cup \cdots \cup f^{-n+1}(\mathcal{U}(X))$$

needed to cover $X \cap I(j_1, j_2, \ldots, j_n)$ is equal to the minimal number of open sets of $(\ast)$ needed to cover $X \cap I(j_1, j_2, \ldots, j_n)$. Also $X$ is contained in the union of all the $I(j_1, j_2, \ldots, j_n)$. Hence

$$M_n(f|X, \mathcal{U}(X)) < (K_n(f, \mathcal{O})) \cdot n \cdot k.$$  

This implies that

$$\text{ent}(f|X, \mathcal{U}(X)) < \lim_{n \to \infty} \frac{1}{n} \left( \ln(K_n(f, \mathcal{O})) \right).$$

Since the diameter of $\mathcal{U}(X)$ may be taken to be arbitrarily small, we have

$$\text{ent}(f|X) < \lim_{n \to \infty} \frac{1}{n} \left( \ln(K_n(f, \mathcal{O})) \right).$$

But $X$ contains all nonwandering points of $f$ except for the finite set \{c_1, c_2, \ldots, c_m\}. Thus, using Propositions 1–4,

$$\text{ent}(f) = \text{ent}(f|\mathcal{U}(f)) = \text{ent}(f|X) < \lim_{n \to \infty} \frac{1}{n} \left( \ln(K_n(f, \mathcal{O})) \right).$$  

Q.E.D.

4. Construction of the sequence $f_n$. Let $f_n$ be any map in $C^2(S^1, S^1)$ which satisfies properties (1)–(4) of $g$ in §1 and the following:

$(5')$ There are points $l \in (e, t)$ and $k \in (t, s)$ with $f(l) = f(k) = e$.

$(6')$ $g$ is one-to-one on each of the intervals $(e, l)$, $(l, t)$, $(t, s)$, and $(s, e)$.

$(7')$ $f_n(t) \in \text{slsm}(c^{-n})$ where $c^{-n}$ is defined as follows. Let $c^0 = c$. Then for $i = 1, \ldots, n$ let $c^{-i}$ denote the unique inverse image (under $f_n$) of $c^{-i+1}$ in $(e, l)$. Recall that $\text{slsm}(c^{-n})$ denotes the component of $W^s(c)$ which contains $c^{-n}$.

The map $f_2$ is pictured in Figure 2. Let $H_1, H_2, H_3, H_4, H_5$ be the disjoint
closed intervals which form the complement of
\[ \text{slsm}(c) \cup \text{slsm}(c^{-1}) \cup \text{slsm}(c^{-2}) \cup \text{slsm}(c^{-3}) \cup \text{slsm}(t). \]
Note that \( \text{slsm}(c) = (k, e) \). We can define a 5 \( \times \) 5 matrix \( A_3 \) by \( A_3(i, j) = 1 \) if \( f_3(H_i) \cap H_j \neq \emptyset \) and \( A_3(i, j) = 0 \) otherwise. Note that \( f_3(H_i) \cap H_j \neq \emptyset \) implies \( f_3(H_i) \supset H_j \). It is easy to see from Figure 2 that

\[
A_3 = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

We can define a matrix \( A_n \) analogously, and \( A_n \) is the \((n + 2) \times (n + 2)\) matrix

\[
A_n = \begin{bmatrix}
1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0
\end{bmatrix}
\]
where the missing rows have ones on the super diagonal and zeros elsewhere.

The following proposition follows from Theorem D of [2].

**Proposition 7.** \(\text{ent}(f_n) = \ln(\lambda_n)\) where \(\lambda_n\) denotes the largest eigenvalue of \(A_n\).

**Theorem 8.** \(\text{ent}(f_n) \to 0\) as \(n \to \infty\).

**Proof.** A straightforward calculation shows that for \(n \geq 3\) the characteristic polynomial of \(A_n\) is \(p_n(x) = (-1)^n(x)(x^{n+1} - x^n - 2)\). Now \(\lambda_n\) is the largest root of \(p_n\), and it is easy to see that \(\lambda_n \to 1\) as \(n \to \infty\). The theorem now follows from Proposition 7. Q.E.D.

**Proposition 9.** For any neighborhood \(N\) of \(g\) in \(C^2(S^1, S^1)\), there is a map in \(N\) with positive entropy (and hence by Propositions 1 and 2 infinite nonwandering set).

**Proof.** Let \(N\) be any neighborhood of \(g\) in \(C^2(S^1, S^1)\). There is (for large enough \(n\)) a map \(h \in N\) satisfying properties (1)-(4) and (5')-(7') of the map \(f_n\) in the sequence defined above. Hence \(\text{ent}(h) = \ln(\lambda_n)\) where \(\lambda_n\) is the largest root of \(p_n(x) = (-1)^n(x)(x^{n+1} - x^n - 2)\). Clearly \(\lambda_n > 1\), so \(\ln(\lambda_n) > 0\). Q.E.D.

5. Proof of Theorem A.

**Theorem A.** The map \(\text{ent}: C^2(S^1, S^1) \to R\) is continuous at \(g\).

**Proof.** Let \(\varepsilon > 0\). Choose \(N\) large enough that \(\text{ent}(f_N) < \varepsilon\) where \(f_N\) is the \(N\)th term of the sequence defined in §4. Choose \(\delta > 0\) such that if \(d(g, f) < \delta\), where \(d\) denotes a metric on \(C^2(S^1, S^1)\), then the following hold.

1. \(f\) has an expanding fixed point \(e(f)\) and a contracting fixed point \(c(f)\) and these are the only fixed points of \(f\).
2. \(f\) preserves orientation at \(e(f)\) and \(c(f)\).
3. \(f\) has nondegenerate singularities \(t(f)\) and \(s(f)\) and these are the only singularities of \(f\).
4. The points \(e(f), t(f), s(f), f(s(f)), \) and \(c(f)\) are distinct and in order on the circle in the counterclockwise direction.
5. Either (5A) holds or (5B) and (5C) hold.

5A. \(f(t(f)) \in [c(f), e(f)]\) and \(f\) is one-to-one on each of the intervals \((e(f), t(f)), (t(f), s(f))\) and \((s(f), e(f))\).
5B. There are points \(l(f) \in (e(f), t(f))\) and \(k(f) \in (t(f), s(f))\) with \(f(l(f)) = f(k(f)) = e(f)\).
5C. \(f(t(f)) \in [e(f), c^{-N}(f)]\) where \(c^{-N}(f)\) is defined as follows. Let
$c^0(f) = c(f)$. Then for $i = 1, \ldots, N$ let $c^{-i}(f)$ denote the unique inverse image (under $f$) of $c^{-i+1}(f)$ in $(e(f), l(f))$.

Let $f \in C^2(S^1, S^1)$ with $d(g, f) < \delta$. We will show that $\text{ent}(f) < \epsilon$. If property (5A) above holds, it follows that $\Omega(f) = \{e(f), c(f)\}$, and $\text{ent}(f) = 0$. Hence we may assume that (5B) and (5C) hold.

We define a collection of proper closed intervals $\mathcal{C}(f) = \{I_1, \ldots, I_{N+2}\}$ as follows. Let $I_1, \ldots, I_N$ be the components of the complement in $[e(f), c^{-1}(f)]$ of

$$\text{sln}(c^{-1}(f)) \cup \cdots \cup \text{sln}(c^{-N}(f)).$$

Let $I_{N+1} = [I(f), t(f)]$ and $I_{N+2} = [t(f), k(f)]$. Then if $W$ is the complement in $S^1$ of

$$\text{sln}(c(f)) \cup \text{sln}(c^{-1}(f)) \cup \cdots \cup \text{sln}(c^{-N}(f))$$

we have $I_1 \cup \cdots \cup I_{N+2} = W$. Hence by Theorem 6,

$$\text{ent}(f) < \lim_{n \to \infty} \frac{1}{n} \left(\ln(K_n(f, c(f)))\right).$$

Let $H_1, \ldots, H_{N+2}$ be the components of the complement in $S^1$ of the following set (defined with respect to $f_N$):

$$\text{sln}(c) \cup \text{sln}(c^{-1}) \cup \cdots \cup \text{sln}(c^{-N}) \cup \text{sln}(t).$$

Let $h = f_N$ and $\Omega(h) = \{H_1, \ldots, H_{N+2}\}$.

It will be helpful for the reader to see Figure 2, in which $N = 3$ and $H_1, H_2, H_3, H_4,$ and $H_5$ are as indicated. In the case $N = 3$ one may also use Figure 2 for a picture of the intervals $I_1, I_2, I_3, I_4,$ and $I_5$. To do this, of course, we must replace $e, t, c,$ etc., by $e(f), t(f), c(f),$ etc. Then in the modified figure, $I_1, I_2,$ and $I_3$ are intervals corresponding to $H_1, H_2,$ and $H_3,$ while $I_4 = [t(f), k(f)]$ and $I_5 = [t(f), k(f)]$.

We may assume that the $H_i$ are numbered as in Figure 2, and the $I_i$ are numbered analogously. We claim that for each positive integer $n$, $K_n(f, c(f)) \leq K_n(h, \Omega(h))$. To prove this claim, suppose that

$$I(j_i) \cap f^{-1}(I(j_2)) \cap \cdots \cup f^{(-1)}(I(j_n)) \neq \emptyset.$$

Then $f(I(j_i)) \cap I(j_{i+1}) \neq \emptyset$ for $i = 1, \ldots, n - 1$. By construction, whenever $f(I(i)) \cap I(k) \neq \emptyset$, $A_N(i, k) = 1$ where $A_N$ is the matrix defined in §4.

Hence $h(H(j_i)) \supset H(j_{i+1})$ for $i = 1, \ldots, n - 1$. This implies that

$$H(j_i) \cap h^{-1}(H(j_2)) \cap \cdots \cap h^{(-1)}(H(j_n)) \neq \emptyset.$$

This proves our claim that for each positive integer $n$, $K_n(f, c(f)) < K_n(h, \Omega(h))$.

Let $X$ be the complement in $S^1$ of the stable manifold of $c$ (with respect to $h = f_N$). Let $\Omega(X) = \{H_1 \cap X, \ldots, H_{N+2} \cap X\}$. Then $\Omega(X)$ is an open
cover of $X$, and for each positive integer $n$ (since the $H_i$ are pairwise disjoint),

$$K_n(h, \mathcal{D}(h)) = M_n(h^{|X}, \mathcal{D}(X)).$$

We have for each positive integer $n$,

$$K_n(f, \mathcal{C}(f)) < K_n(h, \mathcal{D}(h)) = M_n(h^{|X}, \mathcal{D}(X)).$$

Also,

$$\text{ent}(f) < \lim_{n \to \infty} \frac{1}{n} \left( \ln(K_n(f, \mathcal{C}(f))) \right).$$

Hence

$$\text{ent}(f) < \text{ent}(h^{|X}, \mathcal{D}(X)) < \text{ent}(h) < \epsilon. \quad \text{Q.E.D.}$$

**References**


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