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COMPACT MANIFOLDS AND HYPERBOLICITY

BY

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Abstract. In this paper we establish the strongest possible criterion for the hyperbolicity of a compact complex manifold: such a manifold is hyperbolic if and only if it contains no (nontrivial) complex lines. In addition, we study the behavior of such manifolds under deformation and, in particular, answer the two most natural questions about such deformations: Is the space of hyperbolic complex structures on a given C^∞ manifold open in the space of all its complex structures? (Yes.) Is it closed? (Not in general.) These results answer questions first posed by Kobayashi in [4] and [5].

1. **Preliminaries.** We recall the definition of the Kobayashi pseudodistance on a connected complex manifold M. Denote by $\Delta_r = \{z \in \mathbf{C}: |z| < r\}$ the disc of radius r, and by ω_r the invariant metric $r^4\, dz\, d\bar{z}/(r^2 - |z|^2)^2$, normalized so that $\omega_r(0) = dz\, d\bar{z}$ is the Euclidean metric. We set $\Delta = \Delta_1$, $\omega = \omega_1$, and agree to use $\omega_r(p, q)$ to denote the distance between points p, $q \in \Delta_r$ with respect to the metric ω_r.

A chain $\{f_i\}$ connecting points p, $q \in M$ is a finite collection of holomorphic mappings $f_i\colon \Delta \to M$ and points $p_i \in \Delta$ satisfying $f_0(0) = p, f_0(p_0) = f_1(0), \ldots, f_{n-1}(p_{n-1}) = f_n(0), f_n(p_n) = q$. We define

$$d_M(p, q) = \inf\left\{ \sum_{i=0}^{n} \omega(0, p_i) \right\},$$

where the infimum is taken over all chains connecting p and q. This is the Kobayashi pseudodistance; M is said to be hyperbolic in case d_M is an actual metric.

Given connected complex manifolds M, N and a holomorphic mapping $F\colon M \to N$, a chain $\{f_i\}$ on M induces a chain $\{F \circ f_i\}$ on N. From this it follows that the Kobayashi pseudodistance decreases under F; that is,

$$d_N(F(p), F(q)) \leqslant d_M(p, q), \qquad p, q \in M.$$

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In particular, if f: $M \to M$ is a biholomorphic mapping, then F is an isometry with respect to d_M.

For the disc Δ, the Kobayashi metric coincides with the invariant metric ω. This follows from the Schwarz-Pick lemma, which states that, if f: $\Delta \to \Delta$ is a holomorphic mapping, $\omega(f(p), f(q)) \leqslant \omega(p,q)$, for any p, $q \in \Delta$. This means that $\omega \leqslant d_\Delta$, and the reverse inequality is obvious.

On the other hand, for the complex line $\mathbf{C}$, $d_{\mathbf{C}} \equiv 0$. More generally, two points $p, q \in M$ are said to lie on a complex line if there is an entire holomorphic mapping f: $\mathbf{C} \to M$ with $f(z_1) = p, f(z_2) = q$. In this case as well, $d_M(p,q) = 0$ and M is not hyperbolic. We will later establish the converse result in case M is compact.

We want to give an infinitesimal formulation of hyperbolicity. For this purpose we let M be a connected, complex manifold having a Hermitian metric $|v|$ in the complexified tangent bundle of M. (We will also write $|p,q|$ for the distance, with respect to this metric, between points p, $q \in M$.) For a C^∞ mapping f: $\Delta \to M$, we define

$$|f'(z_0)| = |f_*(\partial/\partial z)_{z=z_0}|.$$

LEMMA 1.1. *M is hyperbolic if, and only if,* $\sup|f'(0)| < \infty, f \in \mathrm{Hol}(\Delta, M)$. ($\mathrm{Hol}(\Delta, M)$ *is the set of all holomorphic mappings of* Δ *into* M.)

PROOF. Because Δ is homogeneous, $\sup|f'(0)| = \sup|f_*(v)|$, $v \in T(\Delta)$, $\omega(v) = 1$. Now suppose that this supremum is equal to $c < \infty$; then $d_M(p,q) \geqslant c^{-1}|p,q| > 0, p \neq q \in M$; therefore M is hyperbolic.

Conversely, if this supremum is infinite, there is a sequence of mappings $f_n \in \mathrm{Hol}(\Delta, M)$ with $|f_n'(0)| \nearrow \infty$; by compactness we may assume $f_n(0) \to p$. Let U and $V \supset \overline{U}$ be two coordinate neighborhoods of p. The Cauchy estimates tell us that for any positive integer m and sufficiently large n, $f_n(\Delta_{1/m}) \cap \partial U \neq \varnothing$. In particular, we may choose a sequence of points $\{x_m\}$, $x_m \in \partial U$, such that $x_m \in f_n(\Delta_{1/m})$ for some large n and therefore

$$d_M(f_n(0), x_m) \leqslant \omega(0, 1/m) \to 0 \quad \text{as } m \to 0.$$

By the obvious continuity of d_M and the compactness of ∂U, we obtain a point $x \in \partial U$ with $d_M(p,x) = 0$.

In general, Royden has defined the infinitesimal form of the Kobayashi metric to be the function $K(v)$, $v \in T_x(M)$, given by $1/K(v) = \sup r$, where the supremum is taken over those r for which there exists f: $\Delta_r \to M$ with $f(0) = x$ and $f_*(\partial/\partial z)_0 = v$. Royden showed that K is lower semicontinuous on $T(M)$ and so may be used to define a notion of length on piecewise smooth curves, which in turn yields a pseudometric $\tilde{d}_M$ on M. Royden has also shown that $\tilde{d}_M = d_M$ [6].

2. The basic lemma. We now introduce the key lemma, which enables us to extract normal families of mappings from the (a priori) less manageable family $\mathrm{Hol}(\Delta, M)$. The idea is to obtain a bound on $|f'(z)|$ over compact subsets of Δ in terms of $|f'(0)|$. Of course, no such bound can hold for arbitrary functions in $\mathrm{Hol}(\Delta, M)$. However, some preliminary manipulation allows us to obtain functions which satisfy this powerful restriction. We will see later that, for the purpose of handling many questions relating to the Kobayashi metric, we may actually restrict ourselves to consideration of these special functions.

LEMMA 2.1. *Let M be a complex manifold with Hermitian metric $|\ |$. Given $f \in \mathrm{Hol}(\Delta_r, M)$ with $|f'(0)| \geqslant c \geqslant 0$; then there exists $\tilde{f} \in \mathrm{Hol}(\Delta_r, M)$ with*

$$\sup_{z\in\Delta_r} |\tilde{f}'(z)| \left(\frac{r^2 - |z|^2}{r^2}\right) = |\tilde{f}'(0)| = c.$$

PROOF. First we will arrange to have $\sup|\tilde{f}'(z)|((r^2 - |z|^2)/r^2) = c$, and then we will force this supremum to occur at the origin. For $t \in [0, 1]$, let $f_t \colon \Delta_r \to M$ by $f_t(z) = f(tz)$. Let

$$s(t) = \sup_{z\in\Delta_r} |f_t'(z)| \left(\frac{r^2 - |z|^2}{r^2}\right).$$

Since $\sup|f_t'(z)| < \infty$ for $t < 1$, and the term in parentheses goes to zero as z approaches the boundary, $s(t)$ is finite and continuous for $t < 1$. (In fact, $s(t)$ is monotonic increasing.) It is not hard to see that $s(t)$ is continuous even at $t = 1$, although $s(1)$ may be infinite. Furthermore, $s(0) = 0$ and $s(1) \geqslant c$. Thus there exists at least one point $t_0 \in [0, 1]$ with $s(t_0) = c$.

If $t_0 = 1$, we may take $\tilde{f} = f$. If $t_0 < 1$, then by the argument given before, the supremum $s(t_0)$ is actually attained at some interior point z_0. Let L be an automorphism of Δ_r with $L(0) = z_0$ and set $\tilde{f} = f_{t_0} \circ L$: since the quantity $|f'_{t_0}(z)|((r^2 - |z|^2)/r^2)$ measures the derivative with respect to the invariant metric ω_r, it is invariant under L.

Notice that if M is compact, and $0 \leqslant c < \infty$, the family of all C^∞ mappings of Δ into M satisfying the condition of the lemma is relatively compact in the compact-open topology. Since, for any compact $K \subset \Delta_r$, there is a uniform bound on $|f'(z)|$ for all f in this family and all $z \in K$, the family is equicontinuous. Choose a dense sequence $\{z_n\}$ of points of Δ and a sequence of mappings $\{f_m\}$ from this family; by compactness of M and a diagonal process, we may assume that $f_m(z_n)$ converges as $m \to \infty$, for each n. Equicontinuity then tells us that these values can be extended to give a continuous limit mapping from Δ to M, and that $\{f_m\}$ converges, uniformly over compact subsets, to this limit.

3. **A deformation result.** Our next goal is to study a fixed, compact C^∞ manifold M, with Hermitian metric $|\ |$, which we assume to be endowed with various complex structures. These we take to be smoothly parametrized by the points of a space S; if $s \in S$, we write M_s for the corresponding analytic manifold. Let $D(s) = \sup_{f \in \mathrm{Hol}(\Delta, M_s)} |f'(0)|$. Clearly $D(s) > 0$, $\forall s$. By Lemma 1.1, $D(s) < \infty$ if and only if M_s is hyperbolic.

THEOREM 3.1. *D: $S \to (0, \infty]$ is continuous (in particular, the points of S corresponding to hyperbolic complex structures on M form an open set V).*

PROOF. The proof of upper semicontinuity is by $\bar\partial$-methods. The result was first stated (without proof) by Royden [6]; for proofs the reader is referred to the theses of M. Wright (Stanford) and M. Kalka (New York University). We will now prove lower semicontinuity: Suppose given a sequence $s_i \to s_0$ with $D(s_i) \geqslant c \geqslant 0$, $\forall i$. By definition this means there is a sequence $\{f_i\}$ of maps, $f_i \in \mathrm{Hol}(\Delta, M_{s_i})$, with $|f_i'(0)| \geqslant c$. Applying Lemma 2.1, we obtain a sequence $\tilde f_i \in \mathrm{Hol}(\Delta, M_{s_i})$ with each $\tilde f_i$ satisfying $\sup_{z \in \Delta} |\tilde f_i'(z)|(1 - |z|^2) = |f_i'(0)| = c$. By the discussion at the end of the last section, $\{\tilde f_i\}$ has a subsequence which converges uniformly over compact sets; the limit mapping must be an $\tilde f_0 \in \mathrm{Hol}(\Delta, M_{s_0})$ with $|\tilde f_0'(0)| = c$, and $D(s_0) \geqslant c$. Since c was arbitrary, D is semicontinuous.

COROLLARY 3.2. *If $W \subset S$ is compact, and N is any complex manifold, $\bigcup_{s \in W} \mathrm{Hol}(\Delta, M_s)$ is compact (in the C-O topology).*

PROOF. If K is an arbitrary compact subset of N, Theorem 3.1 gives a uniform bound on $|f'(z)|$ for all f in this family and all $z \in K$. The rest of the proof follows the discussion at the end of §2.

We recall now the infinitesimal Kobayashi metric $K(v)$ discussed in §1, and let $K_s(v)$ be the infinitesimal metric on the tangent bundle of M_s. If we consider all of these metrics to act on $T(M)$, $K_s(v)$ is a continuous function of both s and v: Lower semicontinuity follows directly from Corollary 3.2, while upper semicontinuity was proved in [6]. We recall

$$d_{M_s}(p, q) = \inf \int K_s\left[f_*\left(\frac{\partial}{\partial x}\right)\right],$$

where the infimum is taken over all piecewise smooth curves, parametrized by arc-length, connecting p and q. For such curves, $f_*(\partial/\partial x)$ always lies in the unit sphere bundle $U(M) \subset T(M)$.

COROLLARY 3.3. *$d_{M_s}(p, q)$ is continuous in $M \times M \times V$.*

PROOF. If we restrict s to lie in some compact subset $W \subset V$, then $d_{M_s}(p, q)$ is uniformly continuous in p, q and s when s is held fixed and p and q are allowed to vary. Thus it suffices to fix p and q and show continuity in s. Lower semicontinuity is obvious from the continuity of $K_s(v)$, and will be used in establishing upper semicontinuity. Now let s_0 be an arbitrary point of V, W a relatively compact neighborhood of s_0. Because $K_s(v)$ is continuous, it is bounded from above and below in $U(M) \times W$, say

$$0 < m \leqslant \frac{1}{K_s(v)} \leqslant M < \infty, \qquad v \in U(M), s \in W.$$

Thus the curves which give the infimal line integrals $\int K_s$ are eventually bounded in arc-length by $Md_{M_s}(p,q) + \varepsilon < Md_{M_{s_0}}(p,q) + 2\varepsilon$, for any given $\varepsilon > 0$ and an appropriate choice of W. By compactness of $U(M)$, W may further be chosen so that $|K_{s_0}(v) - K_s(v)| < \varepsilon$, $v \in U(M)$, $s \in W$. This means

$$d_{M_{s_0}}(p,q) \leqslant \int_\alpha K_{s_0} \leqslant \int_\alpha K_s + \varepsilon \operatorname{length}(\alpha)$$

or, combining with our earlier estimate on length(α),

$$(1 - \varepsilon M)d_{M_{s_0}}(p,q) \leqslant d_{M_s}(p,q) + 2\varepsilon^2 + \varepsilon$$

for appropriate choices of the curve α, which implies the desired semicontinuity.

4. **Hyperbolicity and complex lines.** As discussed in §1, no hyperbolic manifold can contain a complex line. That the converse is false is shown by an example of D. Eisenman and L. Taylor [4, p. 130]:

Let M be the region in $\mathbf{C}^2$ given by $\{(z,w)\colon |z| < 1, |zw| < 1 \text{ and } |w| < 1 \text{ if } z = 0\}$. Let π_1 and π_2 be the projections on z and w, respectively. If $f\colon \mathbf{C} \to M$, $\pi_1 \circ f$ is a bounded entire function, hence constant; now the same argument may be applied to show that $\pi_2 \circ f$ is constant, so M contains no nontrivial complex lines. However, M is not hyperbolic, because the points $(0,0)$ and $(0,w_0)$ are at Kobayashi distance zero: Consider the connecting path given by $f_0(z) = (z,0)$, $f_1(x) = (1/n, nz)$ and $f_2(z) = (1/n + 1/2z, w_0)$ ($p_0 = 1/n, p_1 = w_0/n, p_2 = -2/n$) and let n go to infinity.

Under the added assumption of compactness however, the implication may be reversed. In fact, the proof gives a bit more.

THEOREM 4.1. *Let M be a compact, nonhyperbolic complex manifold. Then M contains a nontrivial complex line $f\colon \mathbf{C} \to M$ of order $\leqslant 2$ (that is to say, the area of $f(\Delta_r)$, measured with respect to $|\ |$, is dominated by $O(r^2)$).*

PROOF. By Lemma 1.1, there is a sequence of maps $f_i \in \operatorname{Hol}(\Delta, M)$ with $|f_i'(0)| \nearrow \infty$. We may equally well take f_i to be in $\operatorname{Hol}(\Delta_{r_i}, M)$ with $|f_i'(0)|$

$= 1$ and $r_i \nearrow \infty$; applying Lemma 2.1 gives a sequence of maps

$$\tilde{f}_i \in \operatorname{Hol}(\Delta_{r_i}, M)$$

satisfying

$$\sup_{z \in \Delta_n} |\tilde{f}_i'(z)| \left(\frac{r_i^2 - |z|^2}{r_i^2} \right) = |\tilde{f}_i'(0)| = 1.$$

Some subsequence of $\{f_i\}$ converges on Δ to a limit mapping f; a further refinement gives a subsequence convergent on Δ_2; continuing in this way allows us to extend f analytically to all of $\mathbf{C}$. f cannot be a constant mapping, since $|f'(0)| = \operatorname{Lim}_{i\to\infty} |f_i'(z_0)| = 1$. Finally, for any $z_0 \in \mathbf{C}$,

$$|f'(z_0)| = \operatorname*{Lim}_{i\to\infty} |\tilde{f}_i'(z_0)| \leqslant \operatorname*{Lim}_{i\to\infty} \frac{r_i^2}{r_i^2 - |z_0|^2} = 1;$$

therefore the area of $f(\Delta_r)$ is bounded by r^2 and f has order < 2.

This result allows us to construct an important example (supplied by Mark Green), which will be presented in detail in a forthcoming joint paper. The example is of a smooth family of nonsingular hypersurfaces in P^3, parametrized by the punctured disc, all of which are hyperbolic. However, the manifold corresponding to the puncture is the Fermat variety, which is not hyperbolic. This example resolves two open questions: It gives a family of compact, simply connected hyperbolic manifolds; and it shows that the limit of an analytic family of diffeomorphic, compact hyperbolic manifolds need not be hyperbolic, even if it is nonsingular.

5. The automorphism group.

THEOREM 5.1. *Let $\Psi \xrightarrow{\pi} S$ be a family of diffeomorphic, compact manifolds smoothly parametrized by S.*[2] *Then the natural mapping of $\bigcup_{X,Y\in\Psi} \operatorname{Isom}(X, Y)$ to $S \times S$ is proper.*

PROOF. It suffices to prove that if $W \subset S$ is compact, then $\bigcup_{X,Y\in\pi^{-1}(W)} \operatorname{Isom}(X, Y)$ is compact. Suppose that $X_n, Y_n \to X_0, Y_0 \in \pi^{-1}(W)$, and suppose given a sequence of isomorphisms $f_n: X_n \to Y_n$ with inverses $f_n^{-1}: Y_n \to X_n$. We wish to construct a limit mapping $f_0: X_0 \to Y_0$. Cover X_0 with a finite number of open sets analytically equivalent to balls, say $B_0, \ldots, B_N$. By smoothness of the parametrization, we may assume that uniformly small deformations $B_0^{(n)}, \ldots, B_N^{(n)}$ of $B_0, \ldots, B_N$ cover X_n, for all large n. The proof of Corollary 3.2 now shows that some subsequence of $\{f_n\}$ converges uniformly to a limit mapping $f_0 \in \operatorname{Hol}(X_0, Y_0)$. Reversing the roles

[2] The totality of possible complex structures on a C^∞ manifold M, considered as integrable almost-complex structures, form an analytic space in a natural way–see [2].

of X_n and Y_n, we may take a refinement $\{\tilde{f}_n\}$ of this subsequence such that $\{f_n^{-1}\}$ converges uniformly to $g_0 \in \mathrm{Hol}(Y_0, X_0)$. Then, by uniform convergence we must have

$$\begin{aligned} f_0 \circ g_0 &= (\mathrm{Lim} f_n) \circ (\mathrm{Lim} f_n^{-1}) = \mathrm{Lim}(f_n \circ f_n^{-1}) = \mathrm{Id} \\ &= \mathrm{Lim}(f_n^{-1} \circ f_n) = (\mathrm{Lim} f_n^{-1}) \circ (\mathrm{Lim} f_n) = g_0 \circ f_0 \end{aligned}$$

and f_0 is an isomorphism, with inverse g_0.

COROLLARY 5.2 ([4, P. 70]). *If M is a compact hyperbolic manifold,* $\mathrm{Aut}(M)$ *is a finite group.*

PROOF. By a theorem of Bochner and Montgomery [1], the automorphism group of any compact complex manifold is a complex Lie group. Theorem 5.1 shows that $\mathrm{Aut}(M)$ is compact; we next show that it contains no one-parameter subgroups: Suppose $\mathbf{C} \to \mathrm{Aut}(M)$ nontrivially, say $z \to A_z$, and let $p \in M$ be a point not fixed by the action of this subgroup. Then $z \to A_z(p)$ is a complex line through p; this contradicts our assumption that M is hyperbolic. $\mathrm{Aut}(M)$ is thus a compact complex Lie group with no one-parameter subgroups, so it must be a finite group.

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