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SPECTRAL THEORY FOR CONTRACTION SEMIGROUPS ON HILBERT SPACE[1]

BY

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ABSTRACT. In this paper we determine the relationship between the spectra of a continuous contraction semigroup on Hilbert space and properties of the resolvent of its infinitesimal generator. The methods rely heavily on dilation theory. In particular, we reduce the general problem to the case that the cogenerator of the semigroup has a characteristic function with unitary boundary values. We then complete the analysis by generalizing the scalar result of J. W. Moeller on compressions of the translation semigroup to the case of infinite multiplicity.

1. Introduction. Let H be a Hilbert space over the field $\mathbf{C}$ of complex numbers and let $\{T_t\}_{t>0}$ be a semigroup of contraction operators on H which is strongly continuous. In the Hille-Phillips [6] terminology such a semigroup is of class C_0. Its infinitesimal generator A is dissipative with respect to the inner product on H and the spectrum of A is contained in the closed left half plane, Π_L. If we let $T = (I + A)(I - A)^{-1}$, T is the so-called cogenerator of the semigroup. In this context T is a contraction and under suitable circumstances T may be modelled as a compression of the shift on H^2 of countable multiplicity. We can then apply the techniques of harmonic analysis and complex function theory to a study of the spectral properties of $\{T_t\}$.

In what follows we use the theory of characteristic operator functions and the lifting theorem of Sz.-Nagy and Foiaş. The reader should consult their excellent monograph [12] for a thorough discussion of these topics. The reader may find our use of characteristic functions to be close to that in [8]. For more on the lifting theorem see [1] and [13].

Nearly the whole of the work below has evolved from papers by Moeller (see [9] and [10]) and by Fuhrmann ([2] and [3]). Their results in combination with dilation theory allow us to add our Theorems 4.4 and 4.5 to the spectral theory of operator semigroups as it was begun by Phillips [11].

2. Compressions of the translation semigroup of infinite multiplicity and the spectra in the punctured disc. In this section we generalize the technique and

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results in [9] to the case of infinite multiplicity. Accordingly, let K be a separable Hilbert space over $\mathbf{C}$. Denote by $\mathcal{L}(K)$ the algebra of bounded operators on K. Let $G: \Pi_R \to \mathcal{L}(K)$ be an inner function in the sense of Lax [7] and Halmos [5]. If $H^2(K)$ is the usual Hardy space of K-valued functions which are analytic in the right half plane, then we denote the inner product on this space by

$$\langle f, g\rangle = \int_{-\infty}^{\infty} (f(iy)\cdot g(iy))\, dy,$$

and write $|f|_2 = (\langle f, f\rangle)^{1/2}$.

The operation of "multiplication" by G determines an isometry on $H^2(K)$. We let $\mathfrak{M} = H^2(K) \ominus GH^2(K)$.

Consider the exponential functions e_t, $t > 0$, defined on Π_R by

$$e_t(w) = e^{-tw}, \qquad w \in \Pi_R.$$

Multiplication by e_t on $H^2(K)$ is unitarily equivalent, via the Laplace-Plancherel transform, to right translation by t on $L^2(K)$ of the positive real line.

If M_{e_t} denotes multiplication by e_t, then $M_{e_t}^*$ leaves $\mathfrak{M}$ invariant. From now on we let E_t denote the restriction of $M_{e_t}^*$ to $\mathfrak{M}$. Thus $\{E_t\}_{t>0}$ is a C_0 contraction semigroup on $\mathfrak{M}$. We will consider below, however, only the details of determining $\sigma(E_1)$. Thus we write $E = E_1$ and consider, following [9], two parts of $\sigma(E)$: (i) the part in D', the punctured unit disc in $\mathbf{C}$, and (ii) the unit circle portion of $\sigma(E)$. We limit the discussion to pointing out the modifications required to adapt Moeller's techniques to the generalization and we therefore consider only (i).

Let $\lambda \in \mathbf{C}$ be chosen with $0 < |\lambda| < 1$.

THEOREM 2.1. *$E - \lambda$ is boundedly invertible iff $G(w_n)$ is boundedly invertible for each w_n of the form*

$$w_n = -\ln\bar{\lambda} + 2\Pi ni, \qquad n \in Z,$$

and

$$\sup_{-\infty<n<\infty} \|G(w_n)^{-1}\| < \infty.$$

PROOF. To show that the stated conditions are implied by $\lambda \notin \sigma(E)$, one may apply Fuhrmann's Corollary 2.1 in [3]. To show that the conditions are sufficient one need only modify Moeller's Lemma 3.2 in [9]. Here one may define a generalized conjugation operation for members of $H^2(K)$. To this end let $\{x_n\}_{n=1}^{\infty}$ be an orthonormal basis for K. Let $f \in H^2(K)$. For $w \in \Pi_R$ we may write

$$f(w) = \sum_{n=1}^{\infty} f_n(w)x_n \quad \text{and} \quad f^*(w) = \sum_{n=1}^{\infty} \overline{f_n(w)}\, x_n,$$

where the bar indicates ordinary complex conjugation. With this change the rest of the proof follows as in [9]. Q.E.D.

For a different approach, using an extension of the Carleson-Newman interpolation theorem on H^∞, the reader is invited to see [4]. The main advantage in using Moeller's attack lies in the sharp estimates it provides on the norm of $(\lambda - E)^{-1}$.

For purposes of later reference we state our version of Moeller's Theorems 3.2 and 3.3.

THEOREM 2.2. *Let λ have unit modulus. Then $\lambda \notin \sigma(E)$ iff there exists a $\mathcal{S} > 0$ and $M > 0$ such that $G(w)^{-1}$ exists and is bounded by M in a $\mathcal{S}$ neighborhood of each point w_n of the form $w_n = -\ln \bar{\lambda} + 2n\Pi i$.*

PROOF. The proof becomes obvious upon generalizing, as above, Moeller's arguments in [9]. Q.E.D.

3. On inverting E_1. Although the invertibility condition in [10] carries over in an obvious way to the case of countable multiplicity, the technique of proof does not. In fact, Moeller's proof may be simplified by taking into account the known structure theory for inner functions. No such comparable theory is available for operator valued functions.

For what follows, it is conveneint to establish some notation. Let $\phi \in H^\infty$ of the right half plane. Multiplication by ϕ determines an operator on $H^2(K)$ which may be denoted by M_ϕ. If $P_{\mathfrak{M}}$ is the orthogonal projection of $H^2(K)$ onto $\mathfrak{M} = H^2 \ominus GH^2$ we will write $\tilde{\phi} = P_{\mathfrak{M}} M_\phi|_{\mathfrak{M}}$.

If T is an operator on a Hilbert space H we will write

$$\gamma(T) = \inf\{\|Tx\|: x \in H \text{ and } \|x\| = 1\}$$

and

$$\nu(T) = \min(\gamma(T), \gamma(T^*)).$$

We now state our modified version of Corollary 2.1 of [3].

PROPOSITION 3.1. *If $\tilde{\phi}$ is invertible in $\mathfrak{L}(\mathfrak{M})$ then there is a $\delta > 0$ such that for all $w \in \Pi_R$,*

$$(*) \qquad |\phi(w)| + \nu(G(w)) > \delta.$$

Furthermore we have the following estimates:

$$\text{(i)} \qquad \left(1 - \gamma(G(w))^2\right)^{1/2} \gamma(\tilde{\phi}) < |\phi(w)| + \|\phi\|_\infty \gamma(G(w)),$$

$$\text{(ii)} \qquad \left(1 - \gamma(G(w)^*)^2\right)^{1/2} \gamma(\tilde{\phi}^*) < |\phi(w)| + \|\phi\|_\infty \gamma(G(w)^*).$$

PROOF. Let $w \in \Pi_R$ and $x \in K$. Let $e_{w,x}$ be the generalized "exponential" function given by

$$e_{w,x}(z) = \frac{\sqrt{\operatorname{Re} w/\Pi}}{w + z}\, x, \qquad z \in \Pi_R.$$

Let $f_{w,x} = P_{\mathfrak{M}} e_{w,x}$ and $g_{w,x} = e_{w,x} - f_{w,x}$. Clearly,

$$f_{w,x}(z) = \frac{\sqrt{\operatorname{Re} w/\Pi}}{w + z}\,(1 - G(z)G(w)^*)x, \qquad z \in \Pi_R.$$

Thus

$$|f_{w,x}|_2^2 = 1 - |g_{w,x}|_2^2 = 1 - \|G(w)^* x\|^2.$$

Since

$$|\tilde{\phi}^* f_{w,x}|_2 > |f_{w,x}|_2 \gamma(\tilde{\phi}^*)$$

we may write

$$\left(1 - \|G(w)^* x\|^2\right)^{1/2} \gamma(\tilde{\phi}^*) < |\tilde{\phi}^* f_{w,x}|_2.$$

As in [2] we have

$$\begin{aligned} |\tilde{\phi}^* f_{w,x}|_2 &< |M_\phi^* e_{w,x}|_2 + |M_\phi^* g_{w,x}|_2 \\ &< |\phi(w)| + \|\phi\|_\infty \|G(w)^* x\|. \end{aligned}$$

Upon combining these last two inequalities one obtains (ii).

We may obtain (i) from (ii) by a direct application of Fuhrmann's Theorem 2.5 in [3]. We leave the details to the reader. Condition (*) now follows easily. Q.E.D.

The reader may note here that the invertibility condition in [10] is equivalent to that given by (*) for the special case of the functions $\phi(z) = e_t(z)$. Furthermore, this condition implies, by Theorem 2.1, that 0 is at worst an isolated point of $\sigma(E)$.

LEMMA 3.2. *Suppose* 0 *is an isolated point of* $\sigma(E)$. *Then we may write* $\mathfrak{M} = \mathfrak{M}_1 + \mathfrak{M}_2$ *where* $\mathfrak{M}_i = H^2 \ominus G_i H^2$, $i = 1, 2$, *and*

(i) $\sigma(E|_{\mathfrak{M}_1}) = \{0\}$,

(ii) $\sigma(E|_{\mathfrak{M}_2}) = \sigma(E) - \{0\}$.

PROOF. If we apply the Riesz decomposition theorem to E we obtain disjoint subspaces $\mathfrak{M}_1$ and $\mathfrak{M}_2$ which reduce any operators in the commutant of E. Thus $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are invariant under $\{E_t\}_{t>0}$ and must have the stated form. Q.E.D.

Note that G_1 must be a nonconstant inner factor of G. We may write $G = G_1 G_2$ for some inner function G_2. Choose $w \in \Pi_R$ and suppose $G(w)^{-1} \in \mathfrak{L}(K)$. Then so are $G_1(w)^{-1}$ and $G_2(w)^{-1}$, and

$$\|G_1(w)^{-1}\| = \|G_2(w)G(w)^{-1}\| \leqslant \|G(w)^{-1}\|.$$

Thus if $G(w)^{-1}$ exists and is bounded in some right half plane $\eta + \Pi_R$, so also is G_1. Using Lemma 3.2 we can show that $G_1(w)$ has a bounded inverse for all $w \in \Pi_R$ and, in fact, $G_1(w)^{-1}$ defines a member of $H^\infty(\mathcal{L}(K))$. Explicitly we have

LEMMA 3.3. *If* 0 *is the only point in* $\sigma(E)$, *then for each* $r > 0$, $G(w)^{-1}$ *is uniformly bounded in the infinite strip* $0 < \operatorname{Re}(w) < r$.

PROOF. Let M be an upper bound for $\|(\lambda - E)^{-1}\|$ with λ in the annulus $e^{-r} \leqslant |\lambda| \leqslant 1$. If we set $\phi_\lambda(w) = \lambda - e^{-w}$ and apply Proposition 3.1 to ϕ_λ and w, where $\phi_\lambda(w) = 0$, we see that

$$\text{(i)} \qquad \left(1 - \gamma(G(w))^2\right)^{1/2}\gamma(\lambda - E) \leqslant \|\phi_\lambda\|_\infty \gamma(G(w))$$

and

$$\text{(ii)} \qquad \left(1 - \gamma(G(w)^*)^2\right)^{1/2}\gamma((\lambda - E)^*) \leqslant \|\phi_\lambda\|_\infty \gamma(G(w)^*).$$

If, in addition, we use the simple estimates

$$\gamma(\lambda - E) \leqslant \|\phi_\lambda\|_\infty = 1 + |\lambda| \quad \text{and}$$

$$\gamma(\lambda - E) \geqslant \|(\lambda - E)^{-1}\|^{-1} \geqslant M^{-1},$$

we obtain, after a computation,

$$\|G(w)^{-1}\| \leqslant 4M \quad \text{for } 0 < \operatorname{Re} w < r.$$

Therefore, $\|G(w)^{-1}\|$ is bounded in the strip. Q.E.D.

We can now state our generalization of Moeller's result.

THEOREM 3.4. E *is boundedly invertible iff* $G(w)^{-1}$ *exists and is bounded in some half plane of the form* $\eta + \Pi_R$, $\eta > 0$.

PROOF. That $0 \notin \sigma(E)$ implies the conditions on G is clear from Proposition 3.1. The converse follows easily from the last two lemmas together with the following basic fact from [5]: if G is an inner function which is invertible in $H^\infty(\mathcal{L}(K))$, then G is a constant. Q.E.D.

4. Properties of the infinitesimal generator which determine the spectra of the semigroup. The idea of relating the spectra of a semigroup to properties of its infinitesimal generator is not new. The main facts known for well-behaved semigroups on a Banach space are to be found in §16.7 of [6]. One has, for example, the spectral inclusion result

$$\exp(t\sigma(A)) \subset \sigma(T_t)$$

for a C_0 semigroup $[T_t]_{t \geqslant 0}$ with infinitesimal generator A. For the point spectrum one has the equality

$$\sigma_p(T_t) \setminus \{0\} = \exp(t\sigma_p(A)),$$

and $0 \in \sigma_p(T_t)$ iff there is a nonzero x in the Banach space such that $(\lambda - A)^{-1}x$ can be extended to an entire function of λ which satisfies the growth condition

$$\|(\sigma + i\tau - A)^{-1}x\| < M \max(1, e^{-\beta\sigma}),$$

where M and β are positive constants depending on x.

There is no such theory for the continuous spectrum. Indeed the available functional calculus for C_0 semigroups appears to be insufficient for an attack on the problem. In the special case of contraction semigroups on a Hilbert space the situation is far more satisfactory, and the answer points to the matter of the growth of the resolvent as the key factor.

As was mentioned in the introduction we can reduce our problem to the case of the model semigroup of the previous two sections. In this context one can obtain a simple representation of the resolvent of the infinitesimal generator. In this discussion it will be most convenient to focus attention on the adjoint semigroup $\{E_t^*\}_{t>0}$. We have, using the notation of §2,

$$E_t^* = P_{\mathfrak{M}} M_{e_t}|_{\mathfrak{M}}, \qquad t > 0.$$

If we define functions α_λ, $\lambda \in \mathbf{C}$, by $\alpha_\lambda(w) = (\lambda + w)^{-1}$ we then have, for $\operatorname{Re}\lambda > 0$,

$$(\lambda - A)^{-1} = \int_0^\infty e^{-\lambda t} E_t^* \, dt = P_{\mathfrak{M}} M_{\alpha_\lambda}|_{\mathfrak{M}}.$$

By a simple device we will be able to extend this representation to points $\lambda \notin (\sigma(A) \cup \Pi_R)$.

Let λ be chosen in $\mathbf{C} - \Pi_R$ in such a way that $G(-\lambda)$ and $G(-\lambda)^{-1}$ are well defined. Consider the modified function $\tilde{\alpha}_\lambda$ defined by

$$\tilde{\alpha}_\lambda(w) = \frac{1 - G(w)G(-\lambda)^{-1}}{\lambda + w}, \qquad w \in \Pi_R.$$

Such an $\tilde{\alpha}_\lambda$ defines a member of $H^\infty(\mathcal{L}(K))$ and, therefore, a multiplication operator $M_{\tilde{\alpha}_\lambda}$ on $H^2(K)$. Furthermore, a simple calculation shows that $M_{\tilde{\alpha}_\lambda}$ leaves GH^2 invariant. This leads to the following lemma.

LEMMA 4.1. *Let $\lambda \in \mathbf{C} \setminus \Pi_R$. If $-\lambda$ is a point of analyticity of G and G^{-1}, then $\lambda \notin \sigma(A)$ and $(\lambda - A)^{-1} = P_{\mathfrak{M}} M_{\tilde{\alpha}_\lambda}|_{\mathfrak{M}}$.*

PROOF. Let $\mu \in \Pi_R$. We easily verify the resolvent equation $P_{\mathfrak{M}} M_{\tilde{\alpha}_\lambda} P_{\mathfrak{M}} M_{\alpha_\mu}|_{\mathfrak{M}} = (\mu - \lambda)^{-1}[P_{\mathfrak{M}} M_{\tilde{\alpha}_\lambda}|_{\mathfrak{M}} - P_{\mathfrak{M}} M_{\alpha_\mu}|_{\mathfrak{M}}]$ which implies the result. Q.E.D.

We now state and prove our theorem characterizing the invertibility of $\{E_t^*\}$.

THEOREM 4.2. *E_t^* is boundedly invertible iff there exists M and $\gamma > 0$ such that $-\lambda \notin \sigma(A)$ for all* $\operatorname{Re}\lambda > \gamma$ *and*

$$\|(\lambda + A)^{-1}\| \leq M(\operatorname{Re}\lambda)^{-1}.$$

PROOF. If $0 \notin \sigma(E_t^*)$ then $\{(E_t^*)^{-1}\}_{t\geq 0}$ is a C_0 semigroup. The generalized Hille-Yosida theorem may therefore be applied to yield our condition.

If, conversely, M and $\gamma > 0$ satisfy the hypotheses we show that the conditions of Theorem 3.4 are satisfied. Accordingly let $\lambda \in \gamma + \Pi_R$ so that $-\lambda \notin \sigma(A)$. Since $(\lambda + A)^{-1}$ commutes with our semigroup $\{E_t^*\}$, the Sz.-Nagy-Foiaş lifting theorem (see [12, Chapter VI]) may be applied to interpolate $(\lambda + A)^{-1}$. Thus there exists $F_\lambda \in H^\infty(\mathcal{L}(K))$ leaving GH^2 invariant such that

$$(\lambda + A)^{-1} = P_{\mathfrak{M}} M_{F_\lambda}|_{\mathfrak{M}} \quad \text{and} \quad \|(\lambda + A)^{-1}\| = \|F_\lambda\|_\infty.$$

Let $\mu \in \Pi_R$. Then $(\mu - A)^{-1} = P_{\mathfrak{M}} M_{\alpha_\mu}|_{\mathfrak{M}}$. Since $(\lambda + A)^{-1}$ and $(\mu - A)^{-1}$ satisfy the resolvent equation there must exist $U_{\lambda,\mu} \in H^\infty(\mathcal{L}(K))$ satisfying the following identity for $w \in \Pi_R$:

$$F_\lambda(w) + \frac{1}{\mu + w} - \frac{\mu + \lambda}{\mu + w} F_\lambda(w) = G(w) U_{\lambda,\mu}(w).$$

Letting $w = \lambda$, we immediately obtain from this equation

$$\frac{1}{\mu + \lambda} = G(\lambda) U_{\lambda,\mu}(\lambda)$$

which implies the existence of a right inverse for $G(\lambda)$. By a routine application of Fuhrmann's Theorem 2.5 in [3] we find that $G(\lambda)$ must also have a left inverse. Thus we see $G(\lambda)^{-1} = (\lambda + \mu) U_{\lambda,\mu}(\lambda)$ and we have the estimate

$$\|G(\lambda)^{-1}\| \leq |\lambda + \mu| \, \|U_{\lambda,\mu}\|_\infty.$$

If we now set $\mu = \bar{\lambda}$ we obtain, for $w \in \Pi_R$,

$$B_\lambda(w) F_\lambda(w) + (\bar{\lambda} + w)^{-1} = G(w) U_{\lambda\bar{\lambda}}(w),$$

where B_λ is the Blaschke factor $B_\lambda(w) = (w - \lambda)/(w + \bar{\lambda})$; to estimate $\| U_{\lambda\bar{\lambda}}\|_\infty$ we multiply both sides on the right by G^* and obtain

$$\begin{aligned} \|G(\lambda)^{-1}\| &\leq 2 + 2\operatorname{Re}\lambda \|F_\lambda\|_\infty = 2 + 2\operatorname{Re}\lambda \|(\lambda + A)^{-1}\| \\ &\leq 2(1 + M). \end{aligned}$$

This last inequality concludes the analysis. Q.E.D.

We now proceed to translate the conditions of the theorems in §2.

THEOREM 4.3. *Let* $0 < |\lambda| < 1$. *Then $\lambda \notin \sigma(E_t^*)$ iff there exists $M > 0$ such that if w_0 is a zero of $e_t - \lambda$, then $w_0 \notin \sigma(A)$ and $\|(w_0 + A)^{-1}\| \leq M$.*

Proof. We will discuss only the case $|\lambda| = 1$ since the case $0 < |\lambda| < 1$ is a simplified version of this. Accordingly, suppose $\lambda \notin \sigma(E_t^*)$. By Theorem 2.2 there exist a $\delta > 0$ and $M > 0$ such that $G(w)^{-1}$ exists and is norm bounded by M in a δ neighborhood of every zero, w_0 of $e_t - \lambda$. If we denote the δ neighborhood of a particular w_0 by $N_\delta(w_0)$, then the domains of definition of G and G^{-1} may be extended over $N_\delta(w_0)$ in such a way that $\sup_{w \in N_\delta(w_0)} \|G(w)\| \leq M$. If we set

$$F_0(w) = \left(1 - G(w)G(w_0)^{-1}\right)/(w_0 - w),$$

then by Lemma 4.1 we see that $-w_0 \notin \sigma(A)$ and

$$(w_0 + A)^{-1} = P_{\mathfrak{M}} M_{F_0}|_{\mathfrak{M}}.$$

Thus $\|(w_0 + A)^{-1}\| \leq \|F_0\|_\infty$. To estimate the latter we employ the maximum modulus theorem, applied to the restriction of F_0 to $N_\delta(w_0)$, and the relation $G(w_0)^{-1} = G(w_0)^*$. We then have a crude estimate,

$$\|F_0\|_\infty \leq \delta^{-1}(1 + M).$$

Suppose now that there is an $M > 0$ which bounds the norms $\|(w_0 + A)^{-1}\|$ when $-w_0$ is a zero of $e_t - \lambda$. Let $\varepsilon = (2M)^{-1}$, and choose w in $N_\varepsilon(w_0)$. Then $(w + A)^{-1}$ exists as a member of $\mathcal{L}(\mathfrak{M})$ since we may write

$$(w + A)^{-1} = (w_0 + A)^{-1} \sum_{k=0}^{\infty} (w_0 - w)^k (w_0 + A)^{-k},$$

and we have the estimate for $w \in N_\varepsilon(w_0)$, namely $\|(w + A)^{-1}\| \leq 2M$. Reasoning now as in Theorem 4.2 we see that the hypotheses of Theorem 2.2 are satisfied. Q.E.D.

The extension of these results to contraction semigroups is now a routine application of the theory in [12]. Suppose, in fact, that $\{T_t\}$ is such a contraction semigroup. The modified semigroup $\{T_t^{(\varepsilon)}\}_{t>0}$ defined by $T_t^{(\varepsilon)} = e^{-\varepsilon t}T_t$, with $\varepsilon > 0$, surely satisfies the conditions

(1) $\|T_t^{(\varepsilon)}x\| \to 0$ as $t \to \infty$ for each $x \in H$, and

(2) $\|T_t^{(\varepsilon)*}x\| \to 0$ as $t \to \infty$ for each $x \in H$.

Such a semigroup is in the Sz.-Nagy-Foiaş class C_{00}. The cogenerator of the semigroup is also of class C_{00} as a consequence of [12, Proposition 9.1, Chapter III]. As is proved in Chapter VI of [12], all such semigroups have models of the type discussed above. The infinitesimal generator is clearly $A - \varepsilon$, where A is the generator of $\{T_t\}$. We thus arrive at the following general results.

Theorem 4.4. *Let $\{T_t\}$ be a strongly continuous semigroup of contractions on the Hilbert space H, and let A be the infinitesimal generator. For $t > 0$, $0 \notin \sigma(T_t)$ iff there exist M and $\gamma > 0$ such that for all* $\operatorname{Re}\lambda \geq \gamma$, $-\lambda \notin \sigma(A)$ *and* $\|(\lambda + A)^{-1}\| \leq M(\operatorname{Re}\lambda)^{-1}$.

THEOREM 4.5. *For $\{T_t\}$ and A as above, let $0 < |\lambda|$. Then $\lambda \notin \sigma(T_t)$ iff there is an $M > 0$ such that if w_0 is a zero of $e_t - \lambda$, then $-w_0 \notin \sigma(A)$ and $\|(w_0 + A)^{-1}\| < M$.*

In the present stage of our theorem we can only offer rather crude estimates for $\|(T_t - \lambda)^{-1}\|$ in terms of $\|(w + A)^{-1}\|$. We submit the following estimates taken from the above proofs together with the theory in [10]: We first let

$$M_\lambda = -\sup_{-\infty < n < \infty} \|(\ln \lambda + 2\Pi ni - A)^{-1}\|.$$

Then we obtain

$$\|(T_t - \lambda)^{-1}\| < \frac{1}{1 - |\lambda|} + 2\frac{1 - \ln|\lambda| M_\lambda}{|\lambda|(1 - |\lambda|)(1 - |\lambda|^2)^{1/2}}$$

whenever $0 < |\lambda| < 1$. This is easily modified when $|\lambda| = 1$. When $\lambda = 0$ we have the special estimate

$$\|T_t^{-1}\| < \sqrt{1 + e^{2\gamma}/\delta^2 (1 - e^{-2\gamma})^2}$$

where

$$\frac{1}{\delta} = \sup_{\mathrm{Re}(w) > \gamma} \left(2 + 2\,\mathrm{Re}(w)\|(w + A)^{-1}\|\right).$$

To estimate $\|(w + A)^{-1}\|$ in terms of $\|(T_1 - \lambda)^{-1}\|$, where $\lambda = e^{-w}$, we may employ classical techniques. Consider the bounded operator B on the space H given by $B = \lambda \int_0^1 e^{wt} T_t \, dt$. A simple calculation suffices to verify that for x in the domain of A we have $B(w + A)x = (T_1 - \lambda)x$. From this we have $(w + A)^{-1} = (T_1 - \lambda)^{-1} B$ and the estimate

$$\|(w + A)^{-1}\| < \left(\frac{1 - e^{-\mathrm{Re}\, w}}{\mathrm{Re}\, w}\right)\|(T_1 - \lambda)^{-1}\|.$$

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