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SIMPLE LIE ALGEBRAS OF TORAL RANK ONE

BY

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ABSTRACT. Let L be a finite-dimensional simple Lie algebra over an algebraically closed field of characteristic $p > 7$. Let L have Cartan decomposition $L = H + \Sigma_{\gamma\in\Gamma}L_\gamma$. If Γ generates a cyclic group then L is isomorphic to sl(2, F) or to one of the simple Lie algebras of generalized Cartan type $W(1 : \mathbf{n})$ or $H(2 : \mathbf{n} : \Phi)^{(2)}$.

The object of this paper is to classify the simple Lie algebras with "small" Cartan subalgebras. If H is a Cartan subalgebra of L there are two natural ways to measure its size: the dimension of H or the dimension of a maximal torus in a restricted subalgebra generated by H. The first of these is called the *rank*, the second the *toral rank*. Simple Lie algebras of rank one have been investigated in [1] and [8]. Although the general classification problem for such algebras is still open, results are known in many cases. In this paper we investigate the simple Lie algebras of toral rank one and classify such algebras. For restricted Lie algebras this has been done by B. Weisfeiler in unpublished work and J. Schue [10, §3.13] has obtained similar results.

1. Statement of results.

(1.1) Let L be a finite-dimensional Lie algebra over an algebraically closed field F of characteristic $p > 0$. Let P denote the prime field of F. Let R be a restricted Lie algebra containing L. If $L = H + \Sigma_{\gamma\in\Gamma}L_\gamma$ is a Cartan decomposition of L, let $\overline{H}$ denote the restricted subalgebra of R generated by H, T denote the maximal torus of $\overline{H}$, I denote the nil radical of $\overline{H}$, and $\overline{L} = L + \overline{H}$.

(1.2) LEMMA. $\dim_F T/C_T(L) = \dim_P P\Gamma$.

PROOF. Let T^* denote $\{t \in T | t^p = t\}$. Then T^* is a vector space over P and $\dim_P T^*/C_{T^*}(L) = \dim_F T/C_T(L)$. Since $(\gamma, t) \mapsto \gamma(t)$ induces a nondegenerate pairing of $T^*/C_{T^*}(L) \times P\Gamma$ into P, we have the result.

(1.3) DEFINITION. $\dim_F T/C_T(L)$ is called the *toral rank of L with respect to H*.

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By (1.2) this is independent of the choice of the restricted Lie algebra R. This quantity has also been called the *type* of L [10].

Note that the toral rank does depend on the choice of H. For example, Brown [2] has shown that the toral rank of $W(1:\mathbf{n})$ with respect to H is either 1 or n depending on the choice of H.

We now state our main result (which was announced in [12]).

(1.4) THEOREM. *Let L be a finite-dimensional simple Lie algebra over an algebraically closed field of characteristic $p > 7$. If L is of toral rank one with respect to a Cartan subalgebra H then L is isomorphic to* $\mathrm{sl}(2, F)$ *or to one of the simple Lie algebras of generalized Cartan type $W(1:\mathbf{n})$ or $H(2:\mathbf{n}:\Phi)^{(2)}$.*

(For the definition of the Lie algebras of generalized Cartan type see [7] or [11].)

2. **Filtrations of** L. In this section we will relax our hypotheses on L and F somewhat. We assume only that L is a finite-dimensional Lie algebra over an algebraically closed field F of characteristic $p > 0$, that L is of toral rank one with respect to a Cartan subalgebra $H \neq L$, and that $\bar{H} = T + I$, where $T = Fa$ with $a^p = a$.

If $\bar{L}_0$ is a subalgebra of $\bar{L}$ we let $\bar{L}_1 = \{x \in \bar{L}_0 | [\bar{L}, x] \subseteq \bar{L}_0\}$.

The main result of this section is

(2.1) PROPOSITION. *If $\bar{L}$ is as above then there is a maximal subalgebra $\bar{L}_0$ such that $\bar{L} = L + \bar{L}_1$, and either*

(i) $\dim \bar{L}/\bar{L}_0 = 1$,

(ii) $\dim \bar{L}/\bar{L}_0 = 2$ *and* $\bar{L}_0/\bar{L}_1 \cong \mathrm{sl}(\bar{L}/\bar{L}_0)$, *or*

(iii) $\bar{L}_0/\bar{L}_1 \cong W(1:\mathbf{1})$ *and* $\bar{L}/\bar{L}_0$ *is the* $(p-1)$*-dimensional irreducible restricted* $W(1:\mathbf{1})$*-module.*

The proof of this proposition has several steps. The main objects of study will be pairs (M, N) of subspaces of $\bar{L}$ which satisfy:

(2.1.1) $\bar{L} \supseteq M \supseteq N \supseteq \bar{H}$,

(2.1.2) N is a subalgebra of $\bar{L}$,

(2.1.3) M/N is an irreducible N-submodule of $\bar{L}/N$, and

(2.1.4) $[M, I] \subseteq N$.

Given any pair $M \supseteq N$ of subspaces of $\bar{L}$ we define a filtration $\cdots \supseteq N_{-1} \supseteq N_0 \supseteq N_1 \supseteq \cdots$ of the subalgebra generated by M by setting

(2.1.5) $$N_{-1} = M, \qquad N_0 = N,$$

(2.1.6) $$N_{i+1} = \{x \in N_i | [x, N_{-1}] \subseteq N_i\} \quad \text{for } i > 0,$$

and

(2.1.7) $$N_i = [N_{-1}, N_{i+1}] + N_{i+1} \quad \text{for } i < -1.$$

Define $\operatorname{gr}_i N = N_i/N_{i+1}$ so $\Sigma \operatorname{gr}_i N$ is the associated graded algebra of the subalgebra generated by M.

(2.2) LEMMA. *There is a pair* (M, N) *of subspaces of* $\bar{L}$ *satisfying* (2.1.1)–(2.1.4).

PROOF. Since $L \neq H$ we see that $\bar{L}/\bar{H}$ is a nonzero $\bar{H}$-module and hence contains an irreducible submodule $M/\bar{H}$. Since I is a nil ideal of $\bar{H}$ it must annihilate any irreducible $\bar{H}$-module. Hence $[M, I] \subseteq \bar{H}$, so taking $\bar{H} = N$ gives the result.

(2.3) Since $a^p = a$ any subspace S of $\bar{L}$ which is invariant under ad a can be decomposed as $S = \Sigma_{i=0}^{p-1} S^i$, where $S^i = \{x \in S \mid [x, a] = ix\}$. Note that

$$\bar{L}^0 = \bar{H}. \tag{2.3.1}$$

Furthermore, if $\bar{a} = a + N_1 \in \operatorname{gr}_0 N$, then $(\operatorname{ad} \bar{a})^p = \operatorname{ad} \bar{a}$, so $\operatorname{gr}_i N = \Sigma_{j=0}^{p-1}(\operatorname{gr}_i N)^j$, where $(\operatorname{gr}_i N)^j = \{x \in \operatorname{gr}_i N \mid [x, \bar{a}] = jx\}$. Then (2.3.1) gives

$$(\operatorname{gr}_i N)^0 = (0) \quad \text{for } i < 0. \tag{2.3.2}$$

(2.4) LEMMA. *Let* $A \supseteq B \supseteq C \supseteq D$ *be subspaces of* $\bar{L}$. *Assume that*

(2.4.1) *C is a subalgebra,* $C \supseteq \bar{H}$,

(2.4.2) *A, B, and D are stable under* ad C,

(2.4.3) *C acts irreducibly on* A/B, *and*

(2.4.4) $C \underset{\neq}{\supset} D \supseteq I$.

Then $[A, D] \subseteq B$.

PROOF. Since $A^0 \subseteq \bar{L}^0 = \bar{H} \subseteq B$, it is clear that if $x \in D^i$, $i \neq 0$, then ad x induces a nilpotent transformation on A/B. If $\bar{H} \subseteq D$ then (2.4.2) shows $C = D$. Hence $D^0 = I$. Thus for $x \in D^0$, ad x is nilpotent and so induces a nilpotent transformation of A/B. Now Jacobson's version of Engel's theorem for weakly closed systems (Theorem 1′ of [6, p. 34]) applied to the set of linear transformations of A/B induced by $\{\operatorname{ad} x \mid x \in \bigcup_{i=0}^{p-1} D^i\}$ shows that D annihilates a nonzero subspace of A/B. Since D is an ideal of C this subspace is a C-submodule of the irreducible C-module A/B. Hence D annihilates A/B so $[A, D] \subseteq B$.

(2.5) COROLLARY. *If the pair* (M, N) *satisfies* (2.1.1)–(2.1.4) *then* $\operatorname{gr}_0 N$ *is either one dimensional or isomorphic to* sl(2, F) *or to the Witt algebra* $W(1 : 1)$.

PROOF. Suppose $J \supseteq N_1$ is a proper ideal of N_0. Since $N_1 \supseteq I$ (by (2.1.4)) we see that the chain of subspaces $N_{-1} \supseteq N_0 \supseteq N_0 \supseteq J$ satisfies (2.4.1)–(2.4.4). Hence $J = N_1$. Thus $\operatorname{gr}_0 N$ contains no proper nonzero ideals, so $\operatorname{gr}_0 N$ is simple or one dimensional. Now $\bar{a}$ spans a one-dimensional Cartan subalgebra of $\operatorname{gr}_0 N$. Since $(\operatorname{ad} \bar{a})^p = \operatorname{ad} \bar{a}$ the characteristic roots of ad $\bar{a}$ are in the prime field. By a result of Kaplansky [8, Theorem 2] we see that if $\mathrm{gr}_0 N$ is simple it is isomorphic to sl(2, F) or to the Witt algebra $W(1:1)$.

(2.6) Before proceeding further with the study of $\bar{L}$ we need to recall some information about irreducible sl(2, F)-modules and about irreducible $W(1:1)$-modules.

Let $\{e, f, h\}$ be a basis for sl(2, F) with $[e, f] = h$, $[e, h] = 2e$, and $[f, h] = -2f$. If S is any sl(2, F)-module, let $S^i = \{s \in S | sh = is\}$.

Let $\{e_i | -1 \leqslant i \leqslant p-2\}$ be a basis for $W(1:1)$ with $[e_i, e_j] = (i-j)e_{i+j}$ if $-1 \leqslant i+j \leqslant p-2$, and $[e_i, e_j] = 0$ otherwise. Note that $\{e_{-1}, e_0, e_1\}$ spans a three-dimensional subalgebra isomorphic to sl(2, F). (One isomorphism is given by $e \mapsto e_{-1}$, $f \mapsto e_1$, $h \mapsto -2e_0$.) If S is any $W(1:1)$-module let $S^i = \{s \in S | s(2e_0) = -is\}$.

By results of Jacobson [5] we know that there are p inequivalent irreducible restricted sl(2, F)-modules, one of each dimension n for $1 \leqslant n \leqslant p$. Furthermore, if V is an irreducible restricted sl(2, F)-module of dimension n, then V has basis $\{v_1, \ldots, v_n\}$ where $v_i e$ spans Fv_{i-1} for $2 \leqslant i \leqslant n$, $v_i f$ spans Fv_{i+1} for $1 \leqslant i \leqslant n-1$, and $v_i h = (n - 2i + 1)v_i$ for $1 \leqslant i \leqslant n$.

By results of Chang [3, Hauptsatz 2′] we know that there are p inequivalent irreducible restricted $W(1:1)$-modules, one each of dimension 1 and $p-1$ and $p-2$ of dimension p. It follows from [3, Satz 5] that if V is an irreducible restricted $W(1:1)$-module then $\dim V^i \leqslant 1$ for all i, $0 \leqslant i \leqslant p-1$. Since the one-dimensional module must be trivial, we see that if $V^0 = (0)$ then $\dim V = p-1$. This module V has basis $\{v_1, \ldots, v_{p-1}\}$ with $v_i e_j = i v_{i+j}$ if $1 \leqslant i+j \leqslant p-1$, and $v_i e_j = 0$ otherwise. This module remains irreducible when viewed as an sl(2, F)-module.

(2.7) Lemma. *Let V be an irreducible restricted* sl(2, F)*-module. Then $V \wedge V$ is generated* (*as an* sl(2, F)*-module*) *by* $(V \wedge V)^0$.

Proof. Assume V has dimension n and let W denote the sl(2, F)-submodule of $V \wedge V$ generated by $(V \wedge V)^0$. Let $\{v_1, \ldots, v_n\}$ be a basis for V as in (2.6). Set $v_0 = v_{n+1} = 0$. Clearly $v_1 \wedge v_n \in (V \wedge V)^0 \subseteq W$. Assume that for some s, $1 \leqslant s \leqslant n-1$,

$$\{v_j \wedge v_k | 1 \leqslant j < k \leqslant n, k - j \geqslant s\} \subseteq W.$$

Suppose $1 \leqslant j < k \leqslant n$ and $k - j = s - 1$. Then either $j \neq 1$ or $k \neq n$. Hence (by (2.6)) $v_j \wedge v_k$ belongs to the F-span of

$$\{(v_{j-1} \wedge v_k)f, (v_j \wedge v_{k+1})e, (v_{j-1} \wedge v_{k+1})\} \subseteq Wf + We + W = W.$$

Thus

$$\{v_j \wedge v_k | 1 \leqslant j < k \leqslant n, k - j \geqslant s - 1\} \subseteq W$$

and so, by induction, $V \wedge V = W$.

(2.8) LEMMA. *If the pair* (M, N) *satisfies* (2.1.1)–(2.1.4) *then* M *is a subalgebra of* $\bar{L}$. *Furthermore*:

(i) *if* $\dim \mathrm{gr}_0 N = 1$, *then* $\dim \mathrm{gr}_{-1} N = 1$;

(ii) *if* $\mathrm{gr}_0 N \cong \mathrm{sl}(2, F)$ *and* $N_1 \neq N_2$ *then* $\mathrm{gr}_{-1} N$ *is the 2-dimensional irreducible* $\mathrm{sl}(2, F)$-*module*;

(iii) *if* $\mathrm{gr}_0 N \cong W(1:1)$ *then* $\mathrm{gr}_{-1} N$ *is the* $(p-1)$-*dimensional irreducible* $W(1:1)$-*module*.

PROOF. Define a homomorphism

$$\Phi\colon \mathrm{gr}_{-1} N \wedge \mathrm{gr}_{-1} N \to \mathrm{gr}_{-2} N$$

by

$$((x + N_0) \wedge (y + N_0))\Phi = [x, y] + N_{-1} \quad \text{for } x, y \in N_{-1}.$$

By (2.1.7) we see that Φ is surjective.

By Corollary 2.5 either $\dim \mathrm{gr}_0 N = 1$, or else $\mathrm{gr}_0 N \cong \mathrm{sl}(2, F)$ or $W(1:1)$. We will consider these cases separately. By (2.3.2), the representation of $\mathrm{gr}_0 N$ on $\mathrm{gr}_{-1} N$ is restricted.

If $\dim \mathrm{gr}_0 N = 1$ then (since $\mathrm{gr}_{-1} N$ is an irreducible $\mathrm{gr}_0 N$-module) $\dim \mathrm{gr}_{-1} N = 1$. Hence, $\mathrm{gr}_{-1} N \wedge \mathrm{gr}_{-1} N = (0)$, so, since Φ is surjective, $\mathrm{gr}_{-2} N = (0)$. Thus $N_{-2} = N_{-1}$, so M is a subalgebra.

If $\mathrm{gr}_0 N \cong \mathrm{sl}(2, F)$ then (2.3.2) shows that $(\mathrm{gr}_{-1} N \wedge \mathrm{gr}_{-1} N)^0 \subseteq \ker \Phi$. But Lemma 2.7 shows that $(\mathrm{gr}_{-1} N \wedge \mathrm{gr}_{-1} N)^0$ generates $\mathrm{gr}_{-1} N \wedge \mathrm{gr}_{-1} N$. Since Φ is surjective, we have $\mathrm{gr}_{-2} N = (0)$, so $N_{-2} = N_{-1}$ and M is a subalgebra. Also, by (2.3.2), $(\mathrm{gr}_{-1} N)^0 = (0)$ so $\dim \mathrm{gr}_{-1} N$ is even. If $\dim \mathrm{gr}_{-1} N > 2$ then $\dim \mathrm{gr}_{-1} N > \dim \mathrm{gr}_0 N$. If $N_1 \neq N_2$ then $\mathrm{gr}_1 N \neq (0)$ so we can find $x \in \mathrm{gr}_{-1} N, y \in \mathrm{gr}_1 N$ such that $[x, y] = \bar{a}$. Since $\dim \mathrm{gr}_{-1} N > \dim \mathrm{gr}_0 N$, we can find $z \in \mathrm{gr}_{-1} N$ such that $[z, y] = 0$. Then

$$\begin{aligned} 0 &= [[x, z], y] = [[x, y], z] + [x, [z, y]] \\ &= [\bar{a}, z] \neq 0. \end{aligned}$$

This contradiction shows that $\dim \mathrm{gr}_{-1} N = 2$ if $N_1 \neq N_2$.

Finally, if $\mathrm{gr}_0 N \cong W(1:1)$ then, since $(\mathrm{gr}_{-1} N)^0 = (0)$ by (2.3.2), we see from (2.6) that $\mathrm{gr}_{-1} N$ is the $(p-1)$-dimensional irreducible $W(1:1)$-module. Then $\mathrm{gr}_{-1} N$ is an irreducible $\mathrm{sl}(2, F)$-module, so by Lemma 2.7, $(\mathrm{gr}_{-1} N \wedge \mathrm{gr}_{-1} N)^0$ generates $\mathrm{gr}_{-1} N \wedge \mathrm{gr}_{-1} N$. As above this implies $\mathrm{gr}_{-2} N = (0)$ and so M is a subalgebra.

(2.9) LEMMA. *If the pair* (M, N) *satisfies* (2.1.1)–(2.1.4) *and the pair* (Q, M) *satisfies* (2.1.1)–(2.1.3), *then either*

(2.9.1) $$M_1 \not\subseteq N_0$$

or

$$I \subsetneq \bigcap_i N_i. \tag{2.9.2}$$

PROOF. Suppose $z_1, \ldots, z_{p-1} \in M^j$ where $j \neq 0$. Since, by (2.3.1), $Q^0 = \overline{H} \subseteq M$, we have $Q(\operatorname{ad} z_1) \cdots (\operatorname{ad} z_{p-1}) \subseteq M$. In particular, if $x, y \in M^j$, $j \neq 0$, then $Q(\operatorname{ad}(x(\operatorname{ad} y)^r)) \subseteq M$ for all $r > p - 2$. Hence $x(\operatorname{ad} y)^r \in M_1$. Thus to establish (2.9.1) it is enough to find $x, y \in M^j, j \neq 0$, such that

$$x(\operatorname{ad} y)^r \not\subseteq N_0 \quad \text{for some } r > p - 2. \tag{2.9.3}$$

Now by Corollary 2.5 we know either $\operatorname{gr}_0 N$ is one dimensional or else $\operatorname{gr}_0 N \cong \operatorname{sl}(2, F)$ or $W(1:1)$. We will consider these three cases separately. In each case we will show that if (2.9.2) does not hold then there are $x, y \in M^j$, $j \neq 0$, such that (2.9.3) holds.

If $\dim \operatorname{gr}_0 N = 1$ then, by Lemma 2.8(i), $\dim \operatorname{gr}_{-1} N = 1$. Let $y \in N_{-1}$, $y \not\subseteq N_0$. Since $\overline{L^0} = \overline{H}$ we can assume $y \in M^j$ for some $j \neq 0$. If (2.9.2) does not hold then we can find $z \in I$ such that $z \in N_k$, $z \not\subseteq N_{k+1}$ for some $k > 1$. Since $N_{-1} = Fy + N_0$ we see that for $0 < t < k + 1$, $z(\operatorname{ad} y)^t \not\subseteq N_{k+1-t}$. Thus $z(\operatorname{ad} y)^{k+1} \not\subseteq N_0$. Since $z \in I \subseteq M^0$ we have $z(\operatorname{ad} y)^{k+1} \in M^{(k+1)j}$. But since $\dim \operatorname{gr}_{-1} N = 1$ we also have $z(\operatorname{ad} y)^{k+1} \in M^j$. Thus $k \equiv 0 \pmod p$ so $k > p$. Set $x = z(\operatorname{ad} y)^{k+1-p}$. Then $x, y \in M^j$ and $x(\operatorname{ad} y)^p = z(\operatorname{ad} y)^{k+1} \not\subseteq N_0$, so (2.9.3) holds, as required.

If $\operatorname{gr}_0 N \cong \operatorname{sl}(2, F)$ and (2.9.2) does not hold, then $\operatorname{gr}_2 N \neq (0)$. (For $N_{-1} = N_{-1}^{-1} + N_{-1}^{1} + N_0$ and $N_0 = N_0^{-2} + N_0^0 + N_0^2 + N_1$. Since $I \subseteq N_1$ we have

$$[I, N_{-1}] \subseteq N_0^{-1} + N_0^1 + N_1 \subseteq N_1.$$

Thus $I \subseteq N_2$. But $I \not\subseteq \bigcap_i N_i$ so $N_2 \neq N_3$.) Thus by [9, Chapter III, Theorem 2] the associated graded algebra $\Sigma \operatorname{gr}_i N$ contains a graded subalgebra isomorphic to $H(2:1)^{(2)}$. In this algebra (using the notation of [11, (1.8)])

$$\mathfrak{D}_H(x_1^{p-1})(\operatorname{ad} \mathfrak{D}_H(x_2))^{p-2} \not\subseteq H(2:1)_0.$$

Since $\mathfrak{D}_H(x_1^{p-1})$ and $\mathfrak{D}_H(x_2)$ belong to the same nonzero root space, taking $x, y \in N_{-1}$ such that $x + N_{p-2} = \mathfrak{D}_H(x_1^{p-1})$ and $y + N_0 = \mathfrak{D}_H(x_2)$ satisfies (2.9.3).

Finally, if $\operatorname{gr}_0 N \cong W(1:1)$ we have seen that $\operatorname{gr}_{-1} N$ is a $(p-1)$-dimensional irreducible module for the isomorphic copy of $\operatorname{sl}(2, F)$ spanned by $\{e_{-1}, e_0, e_1\} \subseteq W(1:1)$. Let $\{v_1, \ldots, v_{p-1}\}$ be a basis for $\operatorname{gr}_{-1} N$ (as in (2.6)). Then $v_{p-1}(\operatorname{ad} e_{-1})^{p-2}$ spans Fv_1 and v_{p-1} and e_{-1} belong to the same nonzero root space. Thus choosing $x, y \in N_{-1}$ such that $x + N_0 = v_{p-1}$, $y + N_1 = e_{-1}$, we have $x(\operatorname{ad} y)^{p-2} \not\subseteq N_0$, so (2.9.3) is satisfied.

(2.10) COROLLARY. *If the pair* (M, N) *satisfies* (2.1.1)–(2.1.4) *and the pair* (Q, M) *satisfies* (2.1.1)–(2.1.3), *then* (Q, M) *satisfies* (2.1.4).

PROOF. If (2.9.2) does not hold then (2.9.1) holds and so, since M_1 is an ideal in $M_0 = M = N_{-1}$ and N_0 acts irreducibly on M/N_0, we have $M = M_1 + N_0$. Thus since Q/M is an irreducible M-module it is also an irreducible N_0-module. Then applying Lemma 2.4 to the chain of subspaces $Q \supseteq M \supseteq N_0 \supseteq N_1$ gives the result. If (2.9.2) holds, then applying Lemma 2.4 to the chain of subspaces $Q \supseteq M \supseteq M \supseteq \bigcap_i N_i$ gives the result.

(2.11) We can now prove Proposition 2.1. Let (P, Q) be a pair satisfying (2.1.1)–(2.1.4) and such that if (M, N) satisfies (2.1.1)–(2.1.4) then $\dim Q > \dim N$. Then Corollary 2.10 shows that $P = \bar{L}$ (for otherwise there is some pair (R, P) satisfying (2.1.1)–(2.1.4) and $P \supsetneq Q$). Since Q acts irreducibly on $\bar{L}/Q$ we see that Q is a maximal subalgebra.

As $H \not\subseteq I$ and $\dim \bar{H}/I = 1$ we have $\bar{H} = H + I$. Thus

$$\bar{L} = L + \bar{H} = L + H + I = L + I \subseteq L + Q_1.$$

Thus if $\bar{L}_0 = Q$ we have $\bar{L} = L + \bar{L}_1$, and Corollary 2.5 and Lemma 2.8 give the result unless $Q_0/Q_1 \cong \mathrm{sl}(2, F)$ and $Q_1 = Q_2$.

If $Q_0/Q_1 \cong \mathrm{sl}(2, F)$ and $Q_1 = Q_2$, then $Q_1 = \bigcap_i Q_i \supseteq I$ is an ideal in $\bar{L}$. Now $\bar{L}/Q_1$ has Cartan subalgebra $(\bar{H} + Q_1)/Q_1$ spanned by the element $a + Q_1$ which satisfies $(a + Q_1)^p = a + Q_1$. Thus if $\bar{L}/Q_1$ is simple, Kaplansky's theorem [8, Theorem 2] shows $\bar{L}/Q_1 \cong \mathrm{sl}(2, F)$ or $W(1 : \mathbf{1})$. In either case $\bar{L}/Q_1$ contains a subalgebra of codimension 1. Then take $\bar{L}_0$ to be the preimage in $\bar{L}$ of this subalgebra. Since $\bar{L}/Q_1$ is simple and $\bar{L}/L$ is abelian, $(L + Q_1)/Q_1 = \bar{L}/Q_1$. Thus $\bar{L} = L + Q_1 = L + \bar{L}_1$ so (2.1)(i) is satisfied.

Finally, if $\bar{L}/Q_1$ is not simple it contains a nonzero proper ideal U/Q_1 (for $\dim \bar{L}/Q_1 > \dim Q_0/Q_1 > 1$) where U is an ideal in $\bar{L}$. Now Q_0/Q_1 is simple so $(U \cap Q_0)/Q_1 = Q_0/Q_1$ or (0). Since Q_0 acts irreducibly on $\bar{L}/Q_0$ we see that $(U \cap Q_0) \supseteq Q_0$ is impossible so $(U \cap Q_0)/Q_1 = (0)$. Now $U + Q_0$ is a subalgebra of $\bar{L}$ properly containing Q_0, hence $U + Q_0 = \bar{L}$. If $V \supseteq \bar{H} + Q_1$ is a subalgebra of codimension 1 in Q_0 (such subalgebras exist since $Q_0/Q_1 \cong \mathrm{sl}(2, F)$), then $U + V$ is a subalgebra of codimension 1 in $\bar{L}$. Taking $\bar{L}_0 = U + V$ gives a subalgebra satisfying (2.1)(i). This completes the proof of Proposition 2.1.

3. Proof of Theorem 1.4.

(3.1) Let L satisfy the hypotheses of Theorem 1.4. Then [13, Theorem 2.1] shows that $\bar{H} = T + I$. As L has toral rank one with respect to H we have $\dim T = 1$, so $T = Fa$ for some a with $a^p = a$. Thus the hypotheses of Proposition 2.1 are satisfied.

Now let $\bar{L}_0$ be the subalgebra given by Proposition 2.1. Since $\bar{L} = L + \bar{L}_1$, [11, Lemma 2.2] shows that $L_0 = \bar{L}_0 \cap L$ is a maximal subalgebra of L and

$$L_1 = \{x \in L | [x, L] \subseteq L_0\} = \bar{L}_1 \cap L.$$

Hence the graded algebras $(\bar{L}/\bar{L}_0) + (\bar{L}_0/\bar{L}_1)$ and $(L/L_0) + (L_0/L_1)$ are isomorphic.

Now by the classification of the simple Lie algebras of generalized Cartan type ([7] or [11]), if L_0/L_1 is one dimensional or sl(2, F), then L is one of the algebras listed in Theorem 1.4.

(3.2) Thus we may assume that L/L_0 is $(p-1)$-dimensional and that L_0/L_1 is isomorphic to $W(1:1)$. Let $\{e_{-1}, \ldots, e_{p-2}\}$ be a basis for $W(1:1)$ as described in (2.6). Let $e_0 = r + L_1$ where $r \in L_0$.

(3.3) Lemma. *$G = \{x \in L | x(\operatorname{ad} r)^s = 0$ for some $s\}$ is a Cartan subalgebra in L and L has toral rank one with respect to G.*

Proof. Clearly $G = Fr + G \cap L_1$ is nilpotent. By [4, Lemma 15.2.B] it is self-normalizing, thus is a Cartan subalgebra. Since $(\operatorname{ad} r)^p - (\operatorname{ad} r)$ is nilpotent, as is ad x for every $x \in G \cap L_1$, L has toral rank one with respect to G.

(3.4) In view of (3.3) we may assume that $H = G$, that $\bar{L}/\bar{L}_0$ has basis $\{v_1, \ldots, v_{p-1}\}$ as in (2.6) where $v_i = w_i + \bar{L}_0$, $w_i \in \bar{L}$, that $\bar{L}_0/\bar{L}_1$ has basis $\{e_{-1}, \ldots, e_{p-2}\}$ where $e_i = u_i + \bar{L}_0$, that u_0 spans T and $u_0^p = u_0$, and hence that $[u_i, u_0] = iu_i$, $[w_i u_0] = iw_i$ for all i.

It is then immediate that

$$[w_{p-1}, u_i] \in u_{i-1}F + \bar{L}_1 \quad \text{for all } i > 1. \tag{3.4.1}$$

In particular,

$$[w_{p-1}, u_{p-3}] \in bu_{p-4} + \bar{L}_1 \quad \text{for some } b \in F.$$

Replacing w_{p-1} by $w_{p-1} - (b/2)u_{-1}$ shows that we may assume

$$[w_{p-1}, u_{p-3}] \in \bar{L}_1. \tag{3.4.2}$$

(3.5) Lemma. *Let $\{w_1, \ldots, w_{p-1}; u_{-1}, \ldots, u_{p-2}\}$ be as above. Let $K = \bar{L}_1 + Fu_0 + \cdots + Fu_{p-2} + Fw_{p-1}$. Then*

(3.5.1) *K is a subalgebra,*

(3.5.2) *$K \supseteq [\bar{L}, I]$, and*

(3.5.3) *$\bar{L}_1 + Fu_{p-3} + Fu_{p-2}$ is an ideal in K.*

Proof. Note that

$$\left[\left[\bar{L}_1, w_j\right]w_i\right] \subseteq \left[\bar{L}_1, \bar{L}_{-1}\right] + \left[\left[\bar{L}_1, w_i\right]w_j\right] \subseteq \bar{L}_0 + \left[\bar{L}_0, w_j\right]$$

$$\subseteq Fw_{j-1} + \cdots + Fw_{p-1} + \bar{L}_0.$$

If $j > 3$, using this with $i = 1, 2$ gives

$$\left[\bar{L}_1, w_j\right] \subseteq Fu_{j-2} + \cdots + Fu_{p-2} + \bar{L}_1 \subseteq K.$$

Furthermore, if $x \in \bar{L}_1$ then

$$[x, w_2] = ([[x, w_3]u_{-1}] - [[x, u_{-1}]w_3])/3$$
$$\in Fu_0 + \cdots + Fu_{p-2} + \bar{L}_1 \subseteq K.$$

This, together with (3.4.1)–(3.4.2), proves (3.5.1) and (3.5.3). To prove (3.5.2) we need $[I, w_1] \subseteq K$. Since $[I, u_0] = (0)$ we see $[I, w_1] \subseteq Fu_1 + \bar{L}_1 \subseteq K$ as required.

(3.6) If $\bar{L}$ is as above then $\bar{L}$ contains a maximal subalgebra M such that $(\bar{L}, M)$ satisfies (2.1.1)–(2.1.4) and M_0/M_1 is one dimensional or isomorphic to sl(2, F).

PROOF. From (3.5.2) we see that some pair (P, K) satisfies (2.1.1)–(2.1.4) (where K is as in (3.5)). By Corollary 2.5 K_0/K_1 contains no nontrivial proper ideals. But if $P = \bar{L}$ then $(K_1 + \bar{L}_1 + Fu_{p-3} + Fu_{p-2})/K_1$ is a nontrivial proper ideal of K_0/K_1. Thus $P \neq \bar{L}$. Hence, by Corollary 2.10 we can find $M \supseteq K_0$ such that $(\bar{L}, M)$ satisfies (2.1.1)–(2.1.4) (and hence M is a maximal subalgebra). Since dim $M >$ dim K, case (iii) of Proposition 2.1 is excluded, so M_0/M_1 is one dimensional or isomorphic to sl(2, F).

In view of (3.1) this completes the proof of the theorem.

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