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INVARIANT MEANS ON THE CONTINUOUS BOUNDED FUNCTIONS

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ABSTRACT. Let G be a noncompact nondiscrete σ-compact locally compact metric group. A Baire category argument gives measurable sets $\{A_\gamma: \gamma \in \Gamma\}$ of finite measure with $\operatorname{card}(\Gamma) = c$ which are independent on the open sets. One approximates $\{A_\gamma: \gamma \in \Gamma\}$ by arrays of continuous bounded functions with compact support and then scatters these arrays to construct functions $\{f_\gamma: \gamma \in \Gamma\}$ in CB(G) with a certain independence property. If G is also amenable as a discrete group, the existence of these independent functions shows that on CB(G) there are 2^c mutually singular elements of LIM each of which is singular to TLIM.

0. Let G be a nondiscrete σ-compact locally compact group. Fix a left-invariant Haar measure λ on G and let β be the Lebesgue measurable sets. If G is compact, assume $\lambda(G) = 1$. For a continuous function f from G to the real numbers R, $\|f\|_\infty$ denotes the supremum norm of f. Let CB(G) be the Banach space of continuous bounded functions on G in the supremum norm.

The regular action of G on CB(G) is defined by $gf(x) = f(g^{-1}x)$ for all $g, x \in G$ and $f \in$ CB(G). A *right-uniformly continuous bounded function* f is one such that the map $g \to gf$ from G to (CB(G), $\|\cdot\|_\infty$) is continuous. Let $\mathrm{UCB}_r(G)$ denote the subspace of CB(G) consisting of right-uniformly continuous bounded functions; $\mathrm{UCB}_r(G)$ is a closed invariant subspace of CB(G) containing the constants. The space $\mathrm{UCB}_l(G)$ of *left-uniformly continuous bounded function* is defined analogously, and the space of *uniformly continuous bounded functions* on G is $\mathrm{UCB}(G) = \mathrm{UCB}_r(G) \cap \mathrm{UCB}_l(G)$.

Let $S(G)$ be any subspace of $l_\infty(G)$ which contains the constants and is invariant under the regular action of G on $l_\infty(G)$. A *mean* θ on $S(G)$ is a positive linear functional with $\theta(1) = 1$. A *left-invariant mean* is a mean θ with $\theta(gf) = \theta(f)$ for all $g \in G$ and $f \in S(G)$. The *set of left-invariant means on* $S(G)$ *is denoted by* LIM(S). A group G is *amenable as a discrete group* if $\mathrm{LIM}(l_\infty) \neq \varnothing$. A weaker condition is that the group G is an *amenable locally compact group*; that is, $\mathrm{LIM(CB)} \neq \varnothing$. Let $P(G)$ be the positive measurable functions h with $\int h\, d\lambda = 1$. The convolution $*$ of functions f_1 and f_2 on G is defined for all $z \in G$ by

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$$f_1 * f_2(z) = \int f_1(y) f_2(y^{-1}z)\, d\lambda(y).$$

If $h \in P(G)$ and f is a bounded measurable function, then $h * f \in \mathrm{UCB}_r(G)$. A mean θ on $\mathrm{CB}(G)$ is *topologically left-invariant* if $\theta(h * f) = \theta(f)$ for all $h \in P(G)$ and $f \in \mathrm{CB}(G)$. This definition makes sense for other subspaces of $L_\infty(G)$, the bounded measurable functions on G. Say that a subspace S of $L_\infty(G)$ is *admissible* if $1 \in S$ and S is left-invariant by the regular action and convolution by $P(G)$. For S admissible, let $\mathrm{TLIM}(S)$ denote the *topologically left-invariant means* on S. Then $\mathrm{TLIM}(S) \subset \mathrm{LIM}(S)$ for any admissible subspace S. Denote the left-invariant means on $\mathrm{CB}(G)$ by LIM and the *topologically left-invariant means* on $\mathrm{CB}(G)$ by TLIM.

Greenleaf [7] describes various other equivalent definitions of an amenable locally compact group. Suppose S is any of the spaces $L_\infty(G)$, $\mathrm{CB}(G)$, $\mathrm{UCB}_r(G)$, or $\mathrm{UCB}(G)$. These spaces are all admissible and $\mathrm{TLIM}(S) \subset \mathrm{LIM}(S)$. The group G is amenable locally compact if and only if $\mathrm{LIM}(S) \neq \varnothing$ and, in this case, the set $\mathrm{TLIM}(S) \neq \varnothing$ also. For S the space $\mathrm{UCB}(G)$ or $\mathrm{UCB}_r(G)$, $\mathrm{LIM}(S) = \mathrm{TLIM}(S)$. Whether this can ever be true for $S = \mathrm{CB}(G)$ when G is noncompact or $S = L_\infty(G)$ remains an open question. If G is also amenable as a discrete group, Granirer [6] and Rudin [13] showed $\mathrm{TLIM}(L_\infty) \neq \mathrm{LIM}(L_\infty)$. Also, if G is noncompact and amenable as a discrete group, then Liu and van Rooij [8] have shown that $\mathrm{LIM} \neq \mathrm{TLIM}$. In the special case of $G = R$, this was proved by Raimi [12]. In this paper, Theorem 1.6 sharpens these results.

If G is a compact group, then λ is the unique element of $\mathrm{LIM(CB)} = \mathrm{TLIM(CB)}$. For G a noncompact σ-compact amenable locally compact group, Chou [2] proved, with $c = \operatorname{card} R$, that $\operatorname{card}(\mathrm{TLIM}(S)) > 2^c$ for S any of the spaces $L_\infty(G)$, $\mathrm{CB}(G)$, $\mathrm{UCB}_r(G)$, or $\mathrm{UCB}(G)$. The method there is different from Chou [1] and applies to the nondiscrete case too. He also describes the structure of $\mathrm{TLIM}(S)$ for admissible subspaces S of $L_\infty(G)$.

A theorem in [10] is that for a nondiscrete σ-compact locally compact group G which is amenable as a discrete group, the cardinality of $\mathrm{LIM}(L_\infty) \setminus \mathrm{TLIM}(L_\infty)$ is no smaller than 2^c. In this paper the technique of [10] is extended to $\mathrm{CB}(G)$ with G noncompact. One assumes that G is a noncompact nondiscrete σ-compact locally compact metric group. Then a large family of independent functions is constructed in $\mathrm{CB}(G)$. These functions can be used when G is amenable as a discrete group to prove that LIM contains 2^c mutually singular elements each of which is singular to TLIM.

1. In this section and the next a group G will mean a σ-compact locally compact Hausdorff group.

1.1. Definition. *A sequence* $\{K_m\}$ *of pairwise disjoint subsets of* G *is scattered* if for all compact sets $C \subset G$, there exists $M > 1$ such that for each

$x \in G$ at most one $Cx \cap K_m \neq \emptyset$ with $m > M$. *An array* $\{K_{mn}: m, n \geq 1\}$ *of pairwise disjoint subsets in G is scattered* if for all compact sets $C \subset G$, there exists $M, N \geq 1$ such that for each $x \in G$ at most one $Cx \cap K_{mn} \neq \emptyset$ with $m > M$ or $n \geq N$.

1.2. LEMMA. *Assume G is a noncompact group. Let* $\{K_m\}$ *be a sequence of compact sets. Then there exist a sequence* $\{g_m\} \subset G$ *and an array* $\{g_{mn}\} \subset G$ *such that* $\{K_m g_m\}$ *is scattered and* $\{K_m g_{mn}\}$ *is scattered.*

PROOF. Let $L_n = \bigcup_{m=1}^n K_m$ for all $n \geq 1$. Let $\{U_n\}$ be an increasing sequence of nonempty open sets with compact closures such that $\bigcup_{n=1}^\infty U_n = G$. Fix any $g_1 \in G$ and suppose $g_1, \ldots, g_n \in G$ have been chosen for $n \geq 1$. Choose g_{n+1} not in any of the sets $L_{n+1}^{-1} U_n U_n^{-1} L_{n+1} g_j$ where $j = 1, \ldots, n$. This is possible because G is noncompact. The resulting sequence $\{K_m g_m\}$ has the property that for each $x \in G$ at most one $K_n g_n \cap U_m x \neq \emptyset$ with $n > m$. If not, then for some n and j with $n \geq j > m$, $K_{n+1} g_{n+1} \cap U_m x \neq \emptyset$ and $K_j g_j \cap U_m x \neq \emptyset$. Hence, $L_{n+1} g_{n+1} \cap U_n x \neq \emptyset$ and $L_{n+1} g_j \cap U_n x \neq \emptyset$ for $n \geq j$. But then $g_{n+1} \in L_{n+1}^{-1} U_n U_n^{-1} L_{n+1} g_j$, which contradicts the choice of g_{n+1}. Thus, $K_n g_n \cap U_m x \neq \emptyset$ at most once with $n > m$. But for any compact set $C \subset G$, there exists $M \geq 1$ with $C \subset U_M$. Also, the construction guarantees that $K_m g_m \cap K_n g_n = \emptyset$ unless $m = n$. These facts show $\{K_m g_m\}$ is a scattered sequence.

To scatter $\{K_m\}$ in an array of right translates, arrange the compact sets $\{K_m\}$ in an array with the mth-column constantly K_m. Let $\{\mathcal{K}_j\}$ be the enumeration of the array by Cantor diagonalization. We choose $\{\mathcal{K}_j g_j\}$ scattered and enumerate $g_j = g_{mn}$ when $\mathcal{K}_j$ is chosen from the mth-column and nth-row. This forces $\mathcal{K}_j = K_m$ if $j \equiv (m, n)$. Because the condition $m > M$ or $n > N$ can be made stronger than $(m, n) = j > J$, $\{K_m g_{mn}\}$ is scattered. □

REMARK. The definition of a scattered sequence or array used here is scattering with respect to $\{Cx: x \in G\}$ for every compact C. One can prove if G is noncompact and $\{K_m\}$ is a sequence of compact sets, then there exists $\{K_m g_m\}$ scattered with respect to $\{Cx: x \in G\}$ and $\{xC: x \in G\}$ simultaneously for every compact C. This is useful for constructing functions with properties as in 1.3 and 1.4 simultaneously from the left and the right. To avoid technical complications, everything here is being done to best describe properties of LIM and TLIM.

1.3. LEMMA. *Assume G is a noncompact group. Suppose* $f \in \mathrm{CB}(G)$ *and* $|f| < \sum_{n=1}^\infty \chi_{U_n}$ *where* $\{U_n\}$ *is a scattered sequence of open sets with compact closures. Then if* $\sup_n \lambda(U_n^{-1}) < \infty$, $\theta(|f|) = 0$ *for all* $\theta \in$ TLIM.

PROOF. Suppose C is compact and $\lambda(C) > 0$. Let $\eta = \chi_C/\lambda(C) \in P(G)$.

Choose $m \geq 1$ such that for each $x \in G$ at most one $C^{-1}x \cap U_n \neq \emptyset$ with $n \geq m$. Let f_m be $|f|$ restricted to $\bigcup_{n=m}^{\infty} U_n$. Because $\{U_n\}$ is scattered, at most a finite number of $\{U_n\}$ meet each compact set C and so $f_m \in \mathrm{CB}(G)$. Because G is noncompact and $|f| - f_m$ is compactly supported, for any $\theta \in \mathrm{LIM}$, $\theta(|f|) = \theta(f_m)$. Hence, $\theta(|f|) = \theta(\eta * f_m)$ if $\theta \in \mathrm{TLIM}$; we claim that $|\eta * f_m| \leq \sup_n \lambda(U_n^{-1})/\lambda(C)$. It follows that $\theta(|f|) = 0$ for all $\theta \in$ TLIM. But for any $x \in G$,

$$\eta * f_m(x) = \lambda(C)^{-1} \int \chi_{x^{-1}C}(y) f_m(y^{-1})\, d\lambda(y).$$

Yet at most one $C^{-1}x \cap U_n \neq \emptyset$ for $n \geq m$, and so at most one $x^{-1}C \cap U_n^{-1} \neq \emptyset$ for $n \geq m$. Since $|f_m| \leq \sum_{n=m}^{\infty} \chi_{U_n}$, for each $x \in G$, there is $n(x) \geq m$ with

$$|\eta * f_m(x)| \leq \lambda(C)^{-1} \sum_{n=m}^{\infty} \int \chi_{x^{-1}C \cap U_n^{-1}}(y)\, d\lambda(y)$$

$$\leq \lambda(C)^{-1} \int \chi_{x^{-1}C \cap U_{n(x)}^{-1}}(y)\, d\lambda(y) \leq \lambda(U_{n(x)}^{-1})/\lambda(C).$$

Thus, for any $x \in G$, $|\eta * f_m(x)| \leq \sup_n \lambda(U_n^{-1})/\lambda(C)$. □

1.4. Definition. An element $f \in \mathrm{CB}(G)$ such that $\theta(|f|) = 0$ for all $\theta \in$ TLIM will be called *topologically null*.

The previous lemma gives a criterion for $f \in \mathrm{CB}(G)$ to be topologically null. The next lemma will show that one can construct a topologically null function $f \in \mathrm{CB}(G)$ with $0 \leq f \leq 1$ such that f has a certain permanence property. For $f \in \mathrm{CB}(G)$, f is *permanently near one* if for $g_1, \ldots, g_m \in G$ and $\delta > 0$, there is $x \in G$ with $|1 - g_i f(x)| < \delta$ for all $i = 1, \ldots, m$. For any $f \in \mathrm{CB}(G)$, $1 - f$ generates an invariant closed ideal under pointwise operations which is proper if f is permanently near one. For this reason, if G is amenable as a discrete group and f is permanently near one, there exists $\theta \in \mathrm{LIM}$ with $\theta(f) = 1$. To show that θ is singular to TLIM, one constructs a function f which is topologically null and yet is permanently near one.

1.5. Proposition. *Assume G is a noncompact nondiscrete group. Then there exists an $f \in \mathrm{CB}(G)$ with $0 \leq f \leq 1$ such that f is topologically null and permanently near one.*

Proof. Let $\{U_m\}$ be an increasing sequence of nonempty open sets with compact closures $\bar{U}_m$ such that $G = \bigcup_{m=1}^{\infty} U_m$. Because G is nondiscrete and σ-compact, there is an open dense set V in G with $\lambda(V^{-1}) < 1$. Let $V_m = U_m \cap V$. Then V_m is open dense in U_m and $\lambda(V_m^{-1}) < \lambda(V^{-1}) < 1$ for all $m \geq 1$. For each $m \geq 1$, choose a sequence of continuous functions $\{f(m, n): n \geq 1\}$ with compact support contained in V_m such that (1) $0 \leq f(m, n) \leq 1$ for all $m, n \geq 1$; and (2) $f(m, n) \to^n \chi_{V_m}$ pointwise a.e. $[\lambda]$. Choose an array $\{g_{mn}\}$

with $\{\overline{U}_m g_{mn}\}$ scattered and define $f \in \mathrm{CB}(G)$ by $f(x) = \Sigma\{f(m, n)(xg_{mn}^{-1}): m, n \geq 1\}$ for all $x \in G$. The function $f \in \mathrm{CB}(G)$ and $0 \leq f \leq 1$ because for all $m, n \geq 1$, $\{x: f(m, n)(xg_{mn}^{-1}) \neq 0\} \subset V_m g_{mn}$ and $\{\overline{U}_m g_{mn}\}$ is scattered.

Because $|f| \leq \Sigma_{m,n}\chi_{V_m g_{mn}}$ and $\sup_{m,n}\lambda((V_m g_{mn})^{-1}) \leq \sup_m \lambda(V_m^{-1}) < 1$, an obvious variation on Lemma 1.3 shows f is topologically null. To see f is permanently near one, fix $g_1, \ldots, g_k \in G$. Because $\{U_m\}$ is increasing and covers G, there exists M with $g_i^{-1} \in U_M$ for all $i = 1, \ldots, k$; so $e \in \bigcap_{i=1}^k g_i U_M$. Because V_M is open dense in U_M, $\bigcap_{i=1}^k g_i V_M \neq \varnothing$ and has positive measure; choose $v \in \bigcap_{i=1}^k g_i V_M$ such that for all $i = 1, \ldots, k$, $g_i f(M, n)(v) \to^n \chi_{g_i V_M}(v) = 1$. Now fix $N \geq 1$ and let $x_N = v g_{MN}$. For each $i = 1, \ldots, k$,

$$g_i f(x_N) = \sum_{m,n} f(m, n)\big(g_i^{-1} x_N g_{mn}^{-1}\big) = \sum_{m,n} f(m, n)\big(g_i^{-1} v g_{MN} g_{mn}^{-1}\big).$$

But $g_i^{-1} v \in V_M \subset \overline{U}_M$ and the support of $f(m, n)$ is in V_m. Hence, if $f(m, n)(g_i^{-1} v g_{MN} g_{mn}^{-1}) \neq 0$ then $\underline{V}_m \cap V_M g_{MN} g_{mn}^{-1} \neq \varnothing$ and, therefore, $V_m g_{mn} \cap V_M g_{MN} \neq \varnothing$. Because the $\{\overline{U}_m g_{mn}\}$ is scattered, we must have $(m, n) = (M, N)$ if $V_m g_{mn} \cap V_M g_{MN} \neq \varnothing$. This shows that for all $i = 1, \ldots, k$ and $N \geq 1$, $g_i f(x_N) = f(M, N)(g_i^{-1} v)$. Since $f(M, n)(g_i^{-1} v) \to^n 1$ for all $i = 1, \ldots, k$, given any $\delta > 0$ there exists $N \geq 1$ such that $g_i f(x_N) \geq 1 - \delta$ for all $i = 1, \ldots, k$. □

By dividing the rows of the array which defines f into unbounded disjoint subsets, one can construct $f_1, f_2 \in \mathrm{CB}(G)$ with $0 \leq f_i \leq f$ for $i = 1, 2$ and $f = f_1 + f_2$ such that both f_1 and f_2 are permanently near one. If it is done correctly, this process can be repeated inductively. From this one can get improved versions of the following theorem. In §2, a different technique will give much better results.

1.6. THEOREM. *Assume G is a noncompact nondiscrete group which is amenable as a discrete group. Then there exists $f \in \mathrm{CB}(G)$ with $0 \leq f \leq 1$ and $\theta \in \mathrm{LIM}$ such that f is topologically null and $\theta(f) = 1$.*

PROOF. Take $f \in \mathrm{CB}(G)$ with $0 \leq f \leq 1$ and f both topologically null and permanently near one. Let S be the invariant subspace of $\mathrm{CB}(G)$ generated by 1 and $1 - f$. Every $s \in S$ is of the form $c + \Sigma_{i=1}^m a_i g_i(1 - f)$ for some $c, a_1, \ldots, a_m \in R$ and $g_1, \ldots, g_m \in G$. Define $\theta(s) = c$ if s is written in the above form. Because f is permanently near one, θ is a well-defined invariant mean on S. Since S contains the constants and G is amenable as a discrete group, there exists $\psi \in \mathrm{LIM}$ with $\psi = \theta$ on S. For this ψ, $\psi(1 - f) = 0$ and f is topologically null. □

This theorem shows for a nondiscrete noncompact σ-compact locally compact group what is shown in Liu and van Rooij [8]. The result is stronger in the following sense. Let $\theta_1, \theta_2 \in \mathrm{LIM}$ be *mutually singular* if there exists $f \in \mathrm{CB}(G)$ with $0 \leq f \leq 1$ such that $\theta_1(f) = 0$ and $\theta_2(f) = 1$. Theorem 1.6

gives a $\theta \in$ LIM which is mutually singular with any $\psi \in$ TLIM.

It is not clear that the discrete amenability assumption is necessary here. Chou [2] speculates that if G is noncompact and nondiscrete then LIM $\neq$ TLIM. However, Liu and van Rooij [8] also use the hypothesis that G is amenable as a discrete group. In §2, some justification for using this hypothesis in general will be given. Also unanswered under the same hypotheses as Theorem 1.6 are these two questions. If $\theta \in$ LIM(UCB), does there exist $\psi \in \text{LIM} \setminus \text{TLIM}$ such that $\psi = \theta$ on UCB(G)? If $\theta \in$ LIM(UCB), does there exist $\psi \in$ LIM with $\psi = \theta$ on UCB(G) and ψ mutually singular with all of TLIM? See [11].

2. The following proposition is in [10], [11].

2.1. Proposition. *Assume G is a nondiscrete metric group. Then there exists a continuum of measurable sets $\{A_\gamma: \gamma \in \Gamma\}$ of finite measure such that for all distinct $\gamma_1, \dots, \gamma_m \in \Gamma$, for all choices $x_{i1}, \dots, x_{in}, y_{i1}, \dots, y_{in} \in G$ distinct for fixed $i = 1, \dots, m$, and for all nonempty open sets $V \subset G$,*

$$\lambda\left[V \cap \left[\bigcap_{i=1}^{m}\left(\bigcap_{j=1}^{n} x_{ij}A_{\gamma_i} \cap \bigcap_{j=1}^{n} y_{ij}A_{\gamma_i}^c\right)\right]\right] > 0.$$

In this section the scattering lemma will be used to construct a family $\{f_\gamma: \gamma \in \Gamma\} \subset \mathrm{CB}(G)$ with a similar independence property. Notice that in the following theorem the property that f and $\{f_\gamma: \gamma \in \Gamma\}$ have is that for all choices as below the functions $\{g_i f\}$, $\{x_{ij}f_{\gamma_i}\}$, and $\{1 - y_{ij}f_{\gamma_i}\}$ are simultaneously arbitrarily close to one.

2.2. Theorem. *Assume G is a noncompact nondiscrete metric group. There exists $f \in \mathrm{CB}(G)$ with $0 \leq f \leq 1$ which is topologically null and permanently near one and there exists a continuum of functions $\{f_\gamma: \gamma \in \Gamma\} \subset \mathrm{CB}(G)$ with $0 \leq f_\gamma \leq 1$ for all $\gamma \in \Gamma$ such that for all distinct $\gamma_1, \dots, \gamma_m \in \Gamma$, for all choices $x_{i1}, \dots, x_{in}, y_{i1}, \dots, y_{in} \in G$ distinct for fixed $i = 1, \dots, m$, and for all $g_1, \dots, g_m \in G$,*

$$\left\| \prod_{i=1}^{m} g_i f \left[\prod_{i=1}^{m}\left(\prod_{j=1}^{n} x_{ij}f_{\gamma_i} \prod_{j=1}^{n} (1 - y_{ij}f_{\gamma_i})\right)\right]\right\|_\infty = 1.$$

Proof. Choose $\{A_\gamma: \gamma \in \Gamma\}$ as in 2.1. Let $\{U_m\}$ be an increasing sequence of nonempty open sets with compact closures $\overline{U}_m$ such that $G = \bigcup_{m=1}^{\infty} U_m$. Choose $\{g_{mn}\}$ with $\{\overline{U}_m g_{mn}\}$ scattered. Let V be an open dense symmetric set in G with $\lambda(V) < 1$. For each $m \geq 1$, let $A_\gamma(M) = A_\gamma \cap U_m$ and $V(m) = V \cap U_m$. For $m \geq 1$, choose $\{f(m, n): n \geq 1\}$ in CB(G) with compact support contained in $V(m)$ such that (1) $0 \leq f(m, n) \leq 1$ for all $m, n \geq 1$; and (2)

$f(m, n) \to^n \chi_{V(m)}$ pointwise a.e. [λ]. For each $\gamma \in \Gamma$ and $m \geq 1$, choose $\{f_\gamma(m, n): n \geq 1\}$ in CB(G) with compact support contained in U_m such that (1) $0 \leq f_\gamma(m, n) \leq 1$ for all $\gamma \in \Gamma$ and $m, n \geq 1$; and (2) $f_\gamma(m, n) \to^n \chi_{A_\gamma(m)}$ pointwise a.e. [λ]. For each $x \in G$, define f and f_γ by

$$f(x) = \sum_{m,n} f(m, n)(xg_{mn}^{-1}) \quad \text{and} \quad f_\gamma(x) = \sum_{m,n} f_\gamma(m, n)(xg_{mn}^{-1}).$$

Because $\{\overline{U}_m g_{mn}\}$ is scattered and $\{x: f_\gamma(m, n)(xg_{mn}^{-1}) \neq 0\} \subset U_m g_{mn}$, f_γ is a well-defined continuous bounded function. The same is true of f. By construction, $0 \leq f_\gamma \leq 1$ and $0 \leq f \leq 1$. Also, f is topologically null and permanently near one just as in the proof of Proposition 1.5.

Take $\gamma_1, \ldots, \gamma_m \in \Gamma$. Choose $n \geq 1$ and distinct elements $x_{i1}, \ldots, x_{in}$, $y_{i1}, \ldots, y_{in} \in G$ for each $i = 1, \ldots, m$. Let $g_1, \ldots, g_p \in G$ contain $\{y_{ij}\}$. The independence of $\{A_\gamma: \gamma \in \Gamma\}$ implies

$$\lambda\left[\bigcap_{i=1}^{p} g_i V \cap \left[\bigcap_{i=1}^{m}\left(\bigcap_{j=1}^{n} x_{ij}A_{\gamma_i} \cap \bigcap_{j=1}^{n} y_{ij}A_{\gamma_i}^c\right)\right]\right] > 0.$$

Each of the sets V and A_γ is of finite measure. Let $A \triangle B = A \setminus B \cup B \setminus A$. As $m \to \infty$ both $\lambda(V(m) \triangle V) \to 0$ and $\lambda(A_\gamma(m) \triangle A_\gamma) \to 0$ for all γ. Using the left-invariance of λ and the facts $A^c \triangle B^c = A \triangle B$ and $(\bigcap A_i) \triangle (\bigcap B_i) \subset \bigcup (A_i \triangle B_i)$, one can see that for a sufficiently large $M \geq 1$,

$$\lambda\left[\bigcap_{i=1}^{p} g_i V(M) \cap \left[\bigcap_{i=1}^{m}\left(\bigcap_{j=1}^{n} x_{ij}A_{\gamma_i}(M) \cap \bigcap_{j=1}^{n} y_{ij}A_{\gamma_i}^c(M)\right)\right]\right] > 0.$$

Hence this intersection contains an element v such that for all i and j the following hold:

(a) $$g_i f(M, n)(v) \xrightarrow{n} \chi_{g_i V(M)}(v) = 1,$$

(b) $$x_{ij} f_{\gamma_i}(M, n)(v) \xrightarrow{n} \chi_{x_{ij}A_{\gamma_i}(M)}(v) = 1,$$

(c) $$1 - y_{ij} f_{\gamma_i}(M, n)(v) \xrightarrow{n} \chi_{y_{ij}A_{\gamma_i}^c(M)}(v) = 1.$$

For all i and j, $\overline{U_M}$ contains $g_i^{-1}v$, $x_{ij}^{-1}v$, and $y_{ij}^{-1}v$. Fix $N \geq 1$ and let $x_N = vg_{MN}$. Because $\{\overline{U}_m g_{mn}\}$ is scattered, for any i,

$$g_i f(x_N) = \sum_{m,n} f(m, n)(g_i^{-1} v g_{MN} g_{mn}^{-1}) = f(M, N)(g_i^{-1} v).$$

Similarly for any i and j,

$$x_{ij} f_{\gamma_i}(x_N) = \sum_{m,n} f_{\gamma_i}(m, n)(x_{ij}^{-1} v g_{MN} g_{mn}^{-1}) = f_{\gamma_i}(M, N)(x_{ij}^{-1} v)$$

and

$$y_{ij}f_{\gamma_i}(x_N) = f_{\gamma_i}(M, N)\big(y_{ij}^{-1}v\big).$$

But then (a), (b), (c) say that for all $\delta > 0$, there exists N sufficiently large such that each $g_i f(x_N) > 1 - \delta$, each $x_{ij}f_{\gamma_i}(x_N) > 1 - \delta$, and each $1 - y_{ij}f_{\gamma_i}(x_N) > 1 - \delta$. □

Suppose D *denotes the maximal ideal space of* CB(G) as a Banach algebra under pointwise multiplication. Then D consists of the real-valued homomorphisms of CB(G) and is a w^*-compact subset of the unit ball of the dual of CB(G). For each $g \in G$, the point evaluation functional $e_g \in D$ and $\{e_g: g \in G\}$ is w^*-dense in D.

Assume f and $\{f_\gamma: \gamma \in \Gamma\}$ are as in Theorem 2.2. Fix a continuum $\{F_\gamma: \gamma \in \Gamma\}$ of arbitrary subsets of G. Take tuples $x_{i1}, \ldots, x_{in}, y_{i1}, \ldots, y_{in}$ as in the theorem but consistently take $x_{ij} \in F_{\gamma_i}$ and $y_{ij} \in F_{\gamma_i}^c$. The property of f and $\{f_\gamma: \gamma \in \Gamma\}$ shows that one can choose a suitable w^*-limit $\theta \in D$ of point evaluations such that $\theta(gf) = 1$ for all $g \in G$ and $\theta(gf_\gamma) = \chi_{F_\gamma}(g)$ for all $\gamma \in \Gamma$ and $g \in G$.

Take any $\theta \in D$. Define a function R_θ: CB(G) $\to l_\infty(G)$ by $R_\theta(F)(g) = \theta(g^{-1}F)$ for all $F \in$ CB(G). Then R_θ is a ring homomorphism which commutes with the left-regular actions by G. Also, $R_\theta(1) = 1$ and $R_\theta(f) > 0$ if $f > 0$. It is easy to see that R_θ has a closed range. The above comments give the following propositions which are stated without proof. The details are similar to those in [10].

2.3. Proposition. *Assume G is a noncompact nondiscrete metric group. Then for any subspace S of $l_\infty(G)$ with* card(S) $< c$, *there exists $\theta \in D$ such that the homomorphism R_θ*: CB(G) $\to l_\infty(G)$ *has* image(R_θ) $\supset S$.

2.4. Proposition. *Assume G is a noncompact nondiscrete metric group and H is a countable subgroup of G. Then there is a closed H-invariant ideal I in* CB(G) *such that* CB(G)$/I$ *is naturally isometric to $l_\infty(H)$ by a ring isometry which commutes with the action of H. It follows that the Stone-Čech compactification of H as a discrete group embeds into D by a homeomorphism commuting with the action of H.*

One of the important consequences of Theorem 2.2 is the following.

2.5. Theorem. *Assume G is a noncompact nondiscrete group which is amenable as a discrete group. Then there are at least 2^c mutually singular elements of* LIM *all of which are singular to* TLIM.

Proof. In this case card(G) $> c$. So by the results of [3] (see also[11]) there are subsets $\{F_\gamma: \gamma \in \Gamma\}$ in G with card(Γ) = c such that for any $\xi: \Gamma \to \{0, 1\}$, there is some $m_\xi \in \mathrm{LIM}(l_\infty)$ with $m_\xi(\chi_{F_\gamma}) = \xi(\gamma)$ for all $\gamma \in \Gamma$. Assume G is metric. Then by Theorem 2.2, there exist f which is topologically null and $\{f_\gamma: \gamma \in \Gamma\} \subset \mathrm{CB}(G)$ with $0 < f < 1$ and $0 < f_\gamma < 1$ for all γ such that for some $\theta \in D$,

(1) $\theta(gf) = 1$ for all $g \in G$.

(2) $\theta(gf_\gamma) = \chi_{F_\gamma^{-1}}(g)$ for all $g \in G$, $\gamma \in \Gamma$.

Let R_θ: $\mathrm{CB}(G) \to l_\infty(G)$ be the associated homomorphism defined by $R_\theta(F)(g) = \theta(g^{-1}F)$ for $F \in \mathrm{CB}(G)$. Then $R_\theta(f) = 1$ and $R_\theta(f_\gamma) = \chi_{F_\gamma}$ for all $\gamma \in \Gamma$. Fix $\xi: \Gamma \to \{0, 1\}$ and let $\psi_\xi \in \mathrm{LIM}$ be defined by $\psi_\xi(F) = m_\xi(R_\theta(F))$ for $F \in \mathrm{CB}(G)$. Because $\psi_\xi(f_\gamma) = \xi(\gamma)$ for all $\gamma \in \Gamma$, this gives 2^c mutually singular elements of LIM. Because each $\psi_\xi(f) = 1$, the invariant means ψ_ξ are mutually singular to any element of TLIM.

If G is not metric, choose a compact normal subgroup N of G such that G/N is nondiscrete and metric. Let $\pi: G \to G/N$ be the canonical projection. Then π induces π^*: $\mathrm{CB}(G/N) \to \mathrm{CB}(G)$ by $\pi^*(F) = F \circ \pi$ for $F \in \mathrm{CB}(G/N)$. The map π^* is a ring isometry into CB(G) which commutes with the action by G such that $\pi^*(1) = 1$ and $\pi^*(F) > 0$ if $F > 0$. Theorem 2.2 applied in G/N shows there is a topologically null function f and functions $\{f_\gamma: \gamma \in \Gamma\} \subset \mathrm{CB}(G/N)$ with the independence property. Choose $\{\psi_\xi: \xi: \Gamma \to \{0, 1\}\} \subset \mathrm{LIM}(\mathrm{CB}(G/N))$ as before. Then $\{\psi_\xi \circ \pi^{*-1}: \xi\}$ contains 2^c mutually singular left-invariant means on $\pi^*(\mathrm{CB}(B/N))$ such that each $\psi_\xi \circ \pi^{*-1}(\pi^*(f)) = 1$. By inspecting the construction of f, one can see that $\pi^*(f)$ is topologically null in CB(G). To finish the proof, use the fact that G is amenable as a discrete group to get extensions $\varphi_\xi \in \mathrm{LIM}(\mathrm{CB}(G))$ of each $\psi_\xi \circ \pi^{*-1}$. Then $\{\phi_\xi: \xi: \Gamma \to \{0, 1\}\}$ contains 2^c mutually singular elements of LIM each of which is singular to TLIM. □

The cardinality result here is strong enough to prove the following. Let $\mathcal{C}(G)$ = closed span of $\{\varphi * f - f: \varphi \in P(G), f \in \mathrm{CB}(G)\}$. Let $\mathfrak{T}(G)$ = closed span of $\{gf - f: g \in G, f \in \mathrm{CB}(G)\}$. Then $\mathfrak{T}(G) \subset \mathcal{C}(G)$ and $\mathcal{C}(G) = \mathfrak{T}(G)$ if and only if TLIM = LIM.

2.6. Corollary. *Assume G is a noncompact nondiscrete group which is amenable as a discrete group. Then the space $\mathcal{C}(G)/\mathfrak{T}(G)$ is not separable.*

Again, the necessity of the assumption that G is amenable as a discrete group in 2.5–2.6 is not clear. But Theorem 2.2 does say that if G is a noncompact nondiscrete group, then G is not amenable as a discrete group if and only if there is a closed left-invariant ideal $I \subset \mathrm{CB}(G)$ such that no $\theta \in \mathrm{LIM}$ satisfies $\theta(I) = \{0\}$. In other words, if G is a noncompact nondiscrete group, then G is amenable as a discrete group if and only if Image R_θ is a subspace of $l_\infty(G)$ with a left-invariant mean for all $\theta \in D$.

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