

## APPROXIMATION THEOREMS FOR UNIFORMLY CONTINUOUS FUNCTIONS

BY

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**ABSTRACT.** Let  $X$  be a set,  $A$  a family of real-valued functions on  $X$  which contains the constants,  $\mu_A$  the weak uniformity generated by  $A$ , and  $U(\mu_A X)$  the collection of uniformly continuous functions to the real line  $R$ . The problem is how to construct  $U(\mu_A X)$  from  $A$ . The main result here is: *For  $A$  a vector lattice, the collection of suprema of countable, finitely  $A$ -equiuniform, order-one subsets of  $A^+$  is uniformly dense in  $U(\mu_A X)$ .* Two less technical corollaries: *If  $A$  is a vector lattice (resp., vector space), then the collection of functions which are finitely  $A$ -uniform and uniformly locally- $A$  (resp., uniformly locally piecewise- $A$ ) is uniformly dense in  $U(\mu_A X)$ .* Further, *for any  $A$ , a finitely  $A$ -uniform function is just a composition  $F \circ (a_1, \dots, a_p)$  for some  $a_1, \dots, a_p \in A$  and  $F$  uniformly continuous on the range of  $(a_1, \dots, a_p)$  in  $R^p$ .* Thus, such compositions are dense in  $U(\mu_A X)$ . For  $BU(\mu_A X)$ , the compositions with  $F \in BU(R^p)$  are dense ( $B$  denoting bounded functions). So, in a sense, to know  $U(\mu_A X)$  it suffices to know  $A$  and subspaces of the spaces  $R^p$ , and to know  $BU(\mu_A X)$ ,  $A$  and the spaces  $R^p$  suffice.

In case  $A$  is a vector lattice and  $A = BA$  (i.e.,  $A$  consists of bounded functions), the problem of describing  $U(\mu_A X)$  in terms of  $A$  has an elegant solution:  $A$  is uniformly dense in  $U(\mu_A X)$  (i.e., if  $f \in U(\mu_A X)$  and  $\varepsilon > 0$  there is  $a \in A$  with  $|f(x) - a(x)| < \varepsilon$  for all  $x \in X$ ). This can be seen to be essentially equivalent to the usual Stone-Weierstrass Theorem. The result appears to have been first published in 1955 by Maak [M] and Nöbeling and Bauer [NB]; we give a short proof in the course of proving our main theorem.

If  $A$  contains unbounded functions, the situation is more complicated:  $A = U(R)\{N$  is a "uniformly closed" vector lattice, while  $\mu_A$  is (uniformly) discrete. With  $X = R$ , let  $A$  be the piecewise linear functions (finitely many pieces);  $\mu_A$  is the usual uniformity on  $R$  and the closure of  $A$  consists only of continuous eventually linear functions.

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The theorem that we prove shortly derives from the usual Stone-Weierstrass Theorem (or the one mentioned above) and some results of Fenstad [F] and Császár [C] giving conditions sufficient for density and characterizations of the structures  $U(\mu X)$ . See 1.5 below.

1. **The main theorem.** This is 1.3 below, which explicitly, if rather technically, constructs  $U(\mu_A X)$  from  $A$ . We require some preliminaries.

We make the standing assumption that  $A$  is a subset of  $R^X$  which contains the constants. Saying that  $A$  is a vector space or vector lattice refers to the pointwise operations.

$\mu_A$  denotes the weak uniformity generated by  $A$ , that is, the smallest uniformity such that each function in  $A$  is uniformly continuous. If  $\mathfrak{S}(\epsilon)$  denotes the cover of  $R$  consisting of  $\epsilon$ -balls, then  $\{\mathfrak{S}(\epsilon) \mid \epsilon > 0\}$  is a base for the usual uniformity of  $R$  (using the covering description of uniformities [I(2)]). Thus, a subbase for  $\mu_A$  is  $\{f^{-1}\mathfrak{S}(\epsilon) \mid f \in A, \epsilon > 0\}$ .

Thus, a family  $\mathfrak{F} \subset R^X$  is *A-equiform* (i.e., equiform for  $\mu_A$ ) if given  $\epsilon > 0$ , there is  $\mu_A$ -basic cover  $\mathfrak{B}$  such that whenever  $B \in \mathfrak{B}$  and  $x, y \in B$ , then  $|f(x) - f(y)| < \epsilon$  for all  $f \in \mathfrak{F}$ . This means that given  $\epsilon > 0$  there are  $\{a_1, \dots, a_p\} \subset A$  and  $\delta > 0$  such that whenever  $|a_i(x) - a_i(y)| < \delta$  for  $i = 1, \dots, p$ , then  $|f(x) - f(y)| < \epsilon$  for all  $f \in \mathfrak{F}$ —since basic  $\mathfrak{B}$  is of the form  $\bigwedge_{i=1}^p a_i^{-1}\mathfrak{S}(\delta)$ .

In this definition, if the family  $\{a_1, \dots, a_p\}$  may be chosen independently of  $\epsilon$ , we shall call  $\mathfrak{F}$  *finitely-A-equiform*. Evidently, this means that there is finite  $F \subset A$  such that  $\mathfrak{F}$  is *F-equiform*.

As is well known, if  $\mathfrak{F} \subset R^X$  is  $\mu$ -equiform (for a uniformity  $\mu$  on  $X$ ), and if the pointwise supremum  $\bigvee \mathfrak{F}$  is finite at each point, then  $\bigvee \mathfrak{F} \in U(\mu X)$ . We shall need a generalization:

1.1 LEMMA. *If  $\mathfrak{F}$  is finitely-A-equiform, using finite  $F \subset A$ , with  $\bigvee \mathfrak{F}$  finite at each point, and if  $\mathfrak{S}$  is any collection of subsets of  $\mathfrak{F}$ , then  $\{\bigvee S \mid S \in \mathfrak{S}\}$  is finitely-A-equiform, and using  $F$ .*

This is an immediate consequence of the inequality [F(1)]: for  $\mathfrak{B}, \mathfrak{C} \subset R$ ,  $|\bigvee \mathfrak{B} - \bigvee \mathfrak{C}| \leq \bigvee \{|b - c| \mid b \in \mathfrak{B}, c \in \mathfrak{C}\}$ .

Finally, a family of sets will be said to be *of order one* if each three members have empty intersection. We shall deal with families  $\mathfrak{F} \subset R^X$  for which  $\text{coz } \mathfrak{F}$  is a cover of order one; here,  $\text{coz } \mathfrak{F} = \{\text{coz } f \mid f \in \mathfrak{F}\}$  and  $\text{coz } f = \{x \in X \mid f(x) \neq 0\}$ .

1.2 Notation.  $A_0$  stands for the collection of functions of the form  $\bigvee \mathfrak{F}$ , where

- (a)  $\mathfrak{F} \subset A^+$ ,
- (b)  $\mathfrak{F}$  is countable,
- (c)  $\mathfrak{F}$  is finitely-A-equiform,

(d)  $\mathcal{F}$  is a  $\mu_A$ -uniform cover of order one.

1.3 THEOREM. *If  $A$  is a vector lattice, then  $A_0$  is uniformly dense in  $U(\mu_A X)^+$ .*

The proof of 1.3 is a bit involved. We first sketch it for an almost prototypical special case, and give a corollary.

1.4 Let  $X = [0, +\infty)$ , with  $A$  the vector lattice of continuous piecewise linear functions (with finitely many pieces). Here,  $\mu_A$  is the usual uniformity.

Given  $f \in U(\mu_A X)^+$ , and  $\varepsilon > 0$ , choose  $\delta > 0$  and uniformly continuous  $g$  which is linear on each interval  $[n\delta, (n+1)\delta] \equiv I_n$  and which approximates  $f$  within  $\varepsilon$ . By uniform continuity, a number  $s > 0$  can be chosen so that  $g_n < g$  for each  $n$ , where  $g_n$  is defined like this: let  $L_n(x)$  be linear of slope  $s$  with  $L_n(n\delta) = g(n\delta)$ ,  $M_n(x)$  linear of slope  $-s$  with  $M_n((n+1)\delta) = g((n+1)\delta)$ ; then  $L_n|_{I_n} = g|_{I_n}$ ; for  $x < n\delta$ ,  $g_n = 0 \vee L_n$ ; for  $x > (n+1)\delta$ ,  $g_n = M_n \vee 0$ .

Clearly,  $\bigvee_n g_n = g$ , and  $\{g_n\}$  satisfies 1.2(a), (b), and (c) using  $\{a_1, \dots, a_p\} =$  the singleton  $\{a_1(x) = sx\}$ . But (d) does not yet hold. To achieve (d), define inductively  $K_1 < K_2 < \dots$  in such a way that (d) holds for

$$f_i = \bigvee \{g_n | K_i + 1 \leq n \leq K_{i+1}\}.$$

This is done by making the differences  $|K_{i+1} - K_i|$  grow rapidly. Then  $\bigvee_i f_i = \bigvee_n g_n$ , and  $\{f_i\}$  is finitely  $A$ -equiuniform by 1.1.

1.5 COROLLARY. *Let  $A$  be a vector lattice.*

(a) *If  $A_0 = A^+$ , then  $A$  is uniformly dense in  $U(\mu_A X)$ .*

(b)  *$A = U(\mu_A X)$  iff  $A_0 = A^+$  and  $A$  is "uniformly closed".*

PROOF. (a) Given  $f \in U(\mu_A X)$ , write  $f = f^+ - f^-$ . By 1.3, approximate  $f^+$  and  $f^-$  by  $g$  and  $h$  in  $A_0 = A^+$ . Then  $g - h \in A$ , and approximates  $f$ .

(b) Any  $U(\mu X)$  is closed under taking uniform limits and taking sups of families  $\mathcal{F}$  satisfying 1.2(c). The converse follows from (a).

(a) can be viewed as an improvement of a combination of theorems of Fenstad and Császár. [F]  $A$  is dense in  $U(\mu_A X)$  if  $A$  is closed under the taking of suprema of countable, equiuniform, *star-finite* families. (1.2(d) implies  $\mathcal{F}$  is star-two. The result here is a combination of 4.1 of [F(1)] and 4.3 of [F(2)]; see also [H].) [C]  $A$  is dense in  $U(\mu_A X)$  if  $A$  is closed under the taking of pointwise limits of finitely equiuniform sequences. (This is approximately Satz 3 of [C].) Here note that: If  $\{f_n\}$  is a finitely equiuniform sequence, then  $g_n \equiv \bigvee_{k < n} f_k$  defines a sequence  $\{g_n\}$  which converges pointwise to  $\bigvee_n f_n$ , and which is finitely equiuniform by 1.1; thus Császár's condition implies that  $A_0 = A^+$ .

Fenstad and Császár also derive corollaries like (b), which similarly (b) improves.

PROOF OF 1.3. If  $\mathcal{F}$  satisfies 1.2, then  $\mathcal{F}$  is equiuniform for  $\mu_A$ , and  $\bigvee \mathcal{F} \in U(\mu_A X)$ . So  $A_0 \subset U(\mu_A X)$ .

For the density: Let  $\mathcal{S}(c, r)$  be the open interval in  $R$  with center  $c$  and radius  $r$ . Let  $\mathcal{L}(r) = \{S(nr, r) | n \in Z\}$ . Evidently,  $\{\mathcal{L}(r) | r > 0\}$  is a base for the (usual) uniformity of  $R$ , and  $\{a^{-1}\mathcal{L}(r) | a \in A, r > 0\}$  is a subbase for  $\mu_A$ .

Given  $f \in U(\mu_A X)^+$  and  $\epsilon > 0$ , choose a basic cover  $\mathcal{Q} = \bigwedge_{i=1}^p a_i^{-1}\mathcal{L}(\delta) < f^{-1}\mathcal{L}(\epsilon)$ . Members of  $\mathcal{Q}$  will be denoted  $\alpha, \beta, \gamma, \dots$

Given  $\alpha = \bigcap_{i=1}^p a_i^{-1}S(c_i, \delta)$ ,

(1) define  $n(\alpha) \equiv$  least integer  $n$  with  $\alpha \subset f^{-1}S(n\epsilon, \epsilon)$ , and

$$e(\alpha)(x) \equiv 2 - 2 \wedge \frac{1}{\delta} \left[ \bigvee |a_i(x) - c_i| \right] \vee 1.$$

Then

$$\left| \bigvee_{\alpha} \epsilon n(\alpha) e(\alpha)(x) - f(x) \right| < 2\epsilon \quad \text{for all } x \in X.$$

This will follow from

$$\begin{aligned} e(\alpha)(x) &= 1 \quad \text{iff } |a_i(x) - c_i| < \delta/2 \text{ for each } i; \\ (2) \quad e(\alpha)(x) &= 0 \quad \text{iff } |a_i(x) - c_i| > \delta \text{ for some } i; \\ &0 < e(\alpha) < 1, \quad \text{and} \quad \text{coz } e(\alpha) = \alpha. \end{aligned}$$

(3) If  $e(\alpha)(x) \neq 0$ , then  $e(\beta)(x) = 1$  for some  $\beta$  with  $\beta \cap \alpha \neq \emptyset$ .

(4) If  $\beta \cap \alpha \neq \emptyset$  then  $n(\beta) \leq n(\alpha) + 1$ .

PROOFS. (2) is computed straightforwardly. (4) follows from the facts that  $\mathcal{Q} < f^{-1}\mathcal{L}(\epsilon)$  and the latter is star-two.

For (3), if  $e(\alpha)(x) = 1$  then  $\beta = \alpha$  works, so suppose that  $0 < e(\alpha)(x) < 1$ . Then both inequalities in (2) are violated:  $|a_j(x) - c_j| > \delta/2$  for some  $j$ , and  $|a_i(x) - c_i| < \delta$  for each  $i$ . Then let

$$\begin{aligned} K'_j &= K_j \pm \delta, \text{ in such a way that } |a_j(x) - c_j| < \delta/2, \\ K'_i &= K_i \text{ or } K_i \pm \delta, \text{ so that } |a_i(x) - c_i| < \delta/2. \end{aligned}$$

Set  $\beta = \bigcap_i a_i^{-1}S(K'_i, \delta)$ ; then  $\beta \cap \alpha \neq \emptyset$  and  $e(\beta)(x) = 1$ .

PROOF OF (1). Given  $x, x \in f^{-1}S(n\epsilon, \epsilon)$  for at most two consecutive  $n$ 's. Let  $n$  be the first, so that  $n\epsilon - \epsilon < f(x) < (n+1)\epsilon + \epsilon$ .

If  $e(\alpha)(x) \neq 0$  then  $x \in \alpha$  and  $n < n(\alpha) \leq n+1$ . So first:  $\bigvee_{\alpha} \epsilon n(\alpha) e(\alpha)(x) \leq \epsilon(n+1) \cdot 1$ . And second: there is  $\alpha$  with  $x \in \alpha$  so (by (4)) there is  $\beta$  with  $e(\beta)(x) = 1$ . Since  $x \in \beta, n < n(\beta)$ , and

$$\epsilon n < \epsilon n(\beta) e(\beta)(x) \leq \bigvee_{\alpha} \epsilon n(\alpha) e(\alpha)(x).$$

DIGRESSION. (a) If  $A = BA$ , then  $A$  is dense in  $U(\mu_A X)$ . We have

essentially shown this: By writing arbitrary  $f \in U(\mu_A X)$  as  $f = f^+ - f^-$ , we see that it suffices to approximate functions in  $U(\mu_A X)^+$ . Given such  $f$  and  $\epsilon$ , proceed as above. Because each  $a_i$  is bounded,  $\mathcal{Q} = \{\alpha\}$  is finite. Thus  $\bigvee_{\alpha} \epsilon n(\alpha) e(\alpha) \in A$  and (1) applies.

(b) In the general case, the approximation in (1) is only preliminary. Clearly,  $\{e(\alpha) | \alpha \in \mathcal{Q}\}$  is finitely equiuniform (using  $\{a_1, \dots, a_p\}$ ), but not so for  $\{\epsilon n(\alpha) e(\alpha) | \alpha \in \mathcal{Q}\}$  which, roughly speaking, grow from 0 to  $\epsilon n(\alpha)$  nonuniformly in  $\alpha$ ; also,  $e(\beta) \wedge e(\alpha) \neq 0$  for possibly  $3^p - 1$  other  $\beta$ 's because  $\text{coz } e(\alpha) = \alpha$  (from (2)), and  $\mathcal{Q}$  has the same "starring" properties as the cover  $\bigwedge_{i=1}^p \pi_i^{-1} \mathcal{L}(\alpha)$  of  $R^p$ .

So we shall, first, spread out the support of each  $\epsilon n(\alpha) e(\alpha)$  so as to achieve equiuniformity, and then do some collecting together as in 1.4 to make the supports of order one.

Given  $\alpha$ , let  $\alpha_0 = \alpha$ ,  $\alpha_1 = \bigcup \{\beta \in \mathcal{Q} | \beta \cap \alpha \neq \emptyset\}$ , and  $\dots$   $\alpha_i = \bigcup \{\beta \in \mathcal{Q} | \beta \cap \alpha_{i-1} \neq \emptyset\}$ . As noted earlier, the "starring" properties of  $\mathcal{Q}$  are those of  $\bigwedge_{i=1}^p \pi_i^{-1} \mathcal{L}(\delta)$  in  $R^p$ : so each  $\alpha$  meets  $< 3^p$  members of  $\mathcal{Q}$ ; note that  $3^p$  is the volume of a cube in  $R^p$  of side 3.  $|\{\beta | \beta \cap \alpha_1 \neq \emptyset\}| <$  the volume of the cube in  $R^p$  of side 5, and, in general,  $|\{\beta | \beta \cap \alpha_{i-1} \neq \emptyset\}| < (i + 2)^p$ , which is the volume of the cube of side  $(i + 2)$ .

Let  $e(\alpha_i) \equiv \bigvee \{e(\beta) | \beta \subset \alpha_i\}$ . There are  $< (i + 2)^p$  such  $\beta$ 's, so  $e(\alpha_i) \in A$ . Moreover,  $\{e(\alpha_i) | \alpha \in \mathcal{Q}, i = 0, 1, 2, \dots\}$  is finitely  $A$ -equiuniform by 1.1.

(5) Let  $f(\alpha) \equiv 2\epsilon \sum \{e(\alpha_i) | 0 < i < n(\alpha)/2\}$ ; then  $\{f(\alpha) | \alpha \in \mathcal{Q}\}$  is finitely  $A$ -equiuniform, and  $|\bigvee_{\alpha} f(\alpha)(x) - f(x)| < 3\epsilon$  for each  $x \in X$ .

Note that  $f(\alpha)$  is approximately  $\epsilon n(\alpha) e(\alpha)$  on the set  $\alpha$ , but the jump in  $\epsilon n(\alpha) e(\alpha)$  has been spread over

$$\text{coz } f(\alpha) = \bigcup \left\{ \text{coz } e(\alpha_i) | 0 < i < \frac{n(\alpha)}{2} \right\} = \bigcup \left\{ \alpha_i | 0 < i < \frac{n(\alpha)}{2} \right\}.$$

To prove (5), observe that

$$(6) \quad e(\alpha_{i+1}) | \alpha_i \equiv 1; \quad \beta \subset \alpha_i \text{ implies } n(\beta) < n(\alpha) + i.$$

PROOF. The first is implied by (3), and the second follows by iterating (4).

PROOF OF (5). Given  $\epsilon_0 > 0$ , choose  $\delta > 0$  so that whenever  $|a_i(x) - a_i(y)| < \delta$  for  $i = 1, \dots, p$ , then (i) whenever  $x \in \alpha$  then  $y \in \alpha_1$ , and if  $y \in \alpha$  then  $x \in \alpha_1$ , and (ii)  $|e(\alpha_i)(x) - e(\alpha_i)(y)| < \epsilon_0$  for every  $\alpha_i$ . Suppose  $x$  and  $y$  are "this close".

Given  $\alpha$ : if  $f(\alpha)(x) = f(\alpha)(y) = 0$ , there is nothing to prove. Suppose, say, that  $f(\alpha)(x) \neq 0$ , and let  $i$  be the least integer  $< n(\alpha)/2$  with  $e(\alpha_i)(x) \neq 0$ . Then  $e(\alpha_{i+1})(x) = 1$ . We then have: If  $j < i$ ,  $e(\alpha_j)(x) = 0$ ; so if  $j < i - 1$ ,  $e(\alpha_j)(y) = 0$ . If  $j > i$ ,  $e(\alpha_j)(x) = 1$ ; so if  $j > i + 1$ ,  $e(\alpha_j)(y) = 1$ . Thus,

$$\begin{aligned}
 |f(\alpha)(x) - f(\alpha)(y)| &= 2\epsilon \left| \sum_j \{e(\alpha_j)(x) - e(\alpha_j)(y)\} \right| \\
 &= 2\epsilon \left| \sum \{e(\alpha_j)(x) - e(\alpha_j)(y) \mid j = i - 1, i, i + 1\} \right| \\
 &< 2\epsilon \cdot 3 \cdot \epsilon_0.
 \end{aligned}$$

For present purposes,  $\epsilon$  is fixed; so  $\{f(\alpha) \mid \alpha \in A\}$  is finitely  $A$ -equiuniform.

Next, we shall show that  $|\bigvee_\alpha f(\alpha)(x) - \bigvee_\alpha \epsilon n(\alpha)e(\alpha)(x)| < \epsilon$  for all  $x \in X$ . Then (1) will apply.

We always have  $e(\alpha) < e(\alpha_i)$ , and the number of terms in  $f(\alpha)$  is at least  $(n(\alpha) - 1)/2$ . So

$$\epsilon(n(\alpha) - 1)e(\alpha) < 2\epsilon \left( \frac{n(\alpha) - 1}{2} \right) e(\alpha) < 2\epsilon \sum e(\alpha_i) = f(\alpha).$$

Thus for any  $\alpha$ ,  $\epsilon n(\alpha)e(\alpha) < f(\alpha) + \epsilon$ , and thus  $\bigvee_\alpha \epsilon n(\alpha)e(\alpha) < \bigvee_\alpha f(\alpha) + \epsilon$ .

Now, given  $\alpha$  and  $x \in X$ , we show there is  $\beta$  with  $f(\alpha)(x) < \epsilon n(\beta)e(\beta) + \epsilon$ . Then  $\bigvee_\alpha f(\alpha) < \bigvee_\alpha \epsilon n(\alpha)e(\alpha) + \epsilon$  follows. If  $f(\alpha)(x) = 0$ , then take  $\beta = \alpha$ . Otherwise, there is least  $i < n(\alpha)/2$  with  $e(\alpha_i)(x) \neq 0$ , and

$$\begin{aligned}
 f(\alpha)(x) &= 2\epsilon \sum \{e(\alpha_j)(x) \mid i < j < n(\alpha)/2\} \\
 &< 2\epsilon(n(\alpha)/2 - i) = \epsilon(n(\alpha) - 2i).
 \end{aligned}$$

Choose  $\beta \subset \alpha_{i+1}$  with  $e(\beta)(x) = 1$ , so  $\alpha \subset \beta_{i+1}$  as well, and  $n(\alpha) < n(\beta) + (i + 1)$ . Then

$$n(\alpha) - 2i < n(\alpha) - i < n(\beta) + 1.$$

Thus  $f(\alpha)(x) < \epsilon(n(\beta) + 1) = \epsilon n(\beta)e(\beta)(x) + \epsilon$ .

We now complete the construction by making the supports of order one.

$\mathcal{E}$  breaks into equivalence classes of the relation  $\beta \sim \alpha$  if  $\beta \subset \alpha_i$  for some positive integer  $i$ . Let  $\mathcal{R}$  consist of one representative from each class.

Given  $\alpha \in \mathcal{R}$  and  $\beta \sim \alpha$ , let  $d(\beta) =$  the least  $i$  with  $\beta \subset \alpha_i$ . We shall determine a sequence  $K_1 < K_2 < \dots$  of positive integers (perhaps finite, and depending on  $\alpha$ ) so that with

$$(7) \quad f_i(\alpha) \equiv \bigvee \{f(\beta) \mid K_i + 1 < d(\beta) < K_{i+1}\},$$

the family  $\{\text{coz } f_i(\alpha) \mid i = 1, 2, \dots\}$  is of order one. Note that according to the remarks preceding (5),

$$|\{\beta \mid \beta \subset \alpha_{K_{i+1}}\}| < (K_{i+1} + 2)^p, \text{ so that each } f_i(\alpha) \in A.$$

To define  $\{K_i\}$  precisely, we use

$$(8) \text{ If } f(\beta) \wedge f(\alpha) \neq 0, \text{ then } \beta \subset \alpha_{2n(\alpha)+3}.$$

PROOF. If  $f(\beta) \wedge f(\alpha) \neq 0$ , then there are  $\gamma$  and  $i < n(\alpha)/2, j < n(\beta)/2$

with  $\gamma \subset \alpha_{i+1} \cap \beta_{j+1}$ . Then  $\beta \subset \gamma_{j+1} \subset (\alpha_{i+1})_{j+1} = \alpha_{i+j+2}$ , and thus

$$n(\beta) \leq n(\alpha) + i + j + 2 \leq n(\alpha) + n(\alpha)/2 + n(\beta)/2 + 2$$

(using (6)). Solving for  $n(\beta)$ ,  $n(\beta) \leq 3n(\alpha) + 1$ . Thus

$$i + j + 2 \leq n(\alpha)/2 + (3n(\alpha)/2 + \frac{1}{2}) + 2 \leq 2n(\alpha) + 3,$$

and  $\beta \subset \alpha_{2n(\alpha)+3}$ .

Now, what is required of  $\{K_i\}$  is that  $f(\beta) \wedge f(\gamma) = 0$  whenever  $d(\beta) > K_{i+1}$  and  $d(\gamma) \leq K_i$ .  $d(\gamma) \leq K_i$  implies  $n(\gamma) \leq n(\alpha) + K_i$  (by (6)), so if  $f(\beta) \wedge f(\gamma) \neq 0$  then (using (8)),

$$\beta \subset \gamma_{2n(\gamma)+3} \subset (\alpha_{K_i})_{2(n(\alpha)+K_i)+3} = \alpha_{2n(\alpha)+2K_i+3}.$$

So take  $K_1 = 1$ , and inductively,  $K_{i+1} = 2n(\alpha) + 2K_i + 3$ . Then  $\{\text{coz } f_i(\alpha) \mid i = 1, 2, \dots\}$  is of order one.

Let  $\mathcal{F} \equiv \{f_i(\alpha) \mid \alpha \in \mathcal{R}; i = 1, 2, \dots\}$ . If  $\alpha \neq \beta$  in  $\mathcal{R}$ , then  $f_i(\alpha) \wedge f_j(\beta) = 0$ ; so  $\text{coz } \mathcal{F}$  is also of order one. To see that it is a  $\mu_A$ -uniform cover: Fix  $\alpha$ . Then

$$\{\text{coz } f_i(\alpha)\}_i > \{\text{coz } f(\beta) \mid \beta \sim \alpha\} > \{\text{coz } e(\beta) \mid \beta \sim \alpha\} > \{\beta \mid \beta \sim \alpha\}.$$

Thus  $\text{coz } \mathcal{F}$  is refined by the  $\mu_A$ -uniform cover  $\mathcal{Q}$  consisting of all  $\alpha$ 's.

By (5) and 1.1,  $\mathcal{F}$  is finitely  $A$ -equiuniform. For each  $\beta \in \mathcal{Q}$ ,  $f(\beta)$  is a term in some  $f_i(\alpha)$ , so that  $\bigvee \mathcal{F} = \bigvee \{f(\beta) \mid \beta \in \mathcal{Q}\}$ ; by (5)  $\bigvee \mathcal{F}$  approximates  $f$ .

**2. Locally- $A$  functions.** We present a less technical, perhaps more memorable, corollary of 1.3, isolating a class of functions dense in  $U(\mu_A X)$  with a considerably simpler description.

**2.1 DEFINITION.**  $A_1$  stands for the collection of functions  $g \in R^X$  which are

(a) finitely  $A$ -uniform:  $g \in U(\mu_F X)$  for some finite  $F \subset A$ ; and

(b) uniformly locally- $A$ : there is a  $\mu_A$ -uniform cover on each member of which  $g$  agrees with some function in  $A$ .

There are other ways to put (b). For example, each of the following is equivalent to "g is uniformly locally- $A$ ": (b') There are  $a_1, \dots, a_p \in A$  and  $\delta > 0$  such that if  $S$  is a subset of  $X$  for which each  $\text{osc}_S a_i < \delta$ , then there is  $a_S \in A$  with  $g|_S = a_S|_S$ . (b'') There is a sequence  $\{a_n\} \subset A$  such that  $\{x \in X \mid g(x) = a_n(x)\}_n$  is a  $\mu_A$ -uniform cover.

§4 will say more about the finitely  $A$ -uniform functions.

**2.2 THEOREM.** *If  $A$  is a vector lattice, then  $A_1$  is uniformly dense in  $U(\mu_A X)$ .*

**PROOF.** First,  $A_0 \subset A_1$ : Let  $\mathcal{F}$  be as in 1.2. By 1.1,  $\bigvee \mathcal{F}$  is finitely  $A$ -uniform. By 1.2(d),  $\text{coz } \mathcal{F}$  is  $\mu_A$ -uniform, and on the member  $\text{coz } g$ ,  $\bigvee \mathcal{F} = \bigvee \{h \in \mathcal{F} \mid \text{coz } h \cap \text{coz } g \neq \emptyset\}$ ; the latter function is in  $A$  because of the "order one" condition.

By 1.3,  $A_0 - A_0$  is dense in  $U(\mu_A X)$ ; of course,  $A_0 - A_0 \subset A_1 - A_1$ . But

$A_1 - A_1 = A_1$ , as is readily checked. (In fact,  $A_1$  is a vector lattice, because  $A$  is.)

**2.3 REMARK.** If in 2.2 (or 1.3) we assume that  $A$  is only a vector space, then the conclusion fails: Consider the vector space  $A$  of linear functions on  $R$ , for which  $\mu_A$  is the usual uniformity. Any uniformly piecewise- $A$  function is locally linear, and an easy chaining argument shows that a locally linear function is in fact linear. Thus,  $A_1 = A_0 = A$ , and is not dense in  $U(\mu_A R) = U(R)$ .

The next two sections present theorems in which the hypotheses on  $A$  are weakened.

**3. Piecewise- $A$  functions.** We prove, again as a corollary of 1.3, a theorem like 2.2 but applied to a vector space; what is needed is a condition (3.1(b)) more permissive than 2.1(b).

**3.1 DEFINITION.** A function  $g \in R^X$  is piecewise- $A$  if there is finite  $F \subset A$  such that at each point of  $X$ ,  $g$  agrees with some function in  $F$ .

$g$  is uniformly locally piecewise- $A$  if there is a  $\mu_A$ -uniform cover  $\mathcal{Q}$  such that  $g$  agrees with a piecewise- $A$  function on each member of  $\mathcal{Q}$ .

$A_2$  (respectively,  $A_3$ ) stands for the collection of functions in  $R^X$  which are

(a) finitely  $A$ -uniform, and

(b) piecewise- $A$  (respectively, uniformly locally piecewise- $A$ ).

**3.2 THEOREM.** *If  $A$  is a vector space, then  $A_3$  is uniformly dense in  $U(\mu_A X)$ .*

The following prepares for application of 1.3:

**3.3 PROPOSITION.** *Let  $A$  be a vector space. Then*

(a)  $A_2$  and  $A_3$  are vector lattices;

(b)  $\mu_{A_2} = \mu_{A_3} = \mu_A$ ;

(c) a family  $\mathcal{F}$  is finitely- $A$ -equiuniform if (and only if)  $\mathcal{F}$  is finitely- $A_2$ -equiuniform;

(d)  $(A_2)_0 \subset A_3$ .

**PROOF OF 3.3.** (a) It is easy to see that  $A_2$  and  $A_3$  are vector spaces, because  $A$  is. They are lattices because at each point,  $a \vee b$  and  $a \wedge b$  agree with either  $a$  or  $b$ .

(b) is implied by 3.1(a).

(c) The "only if" part is obvious. Conversely, let  $\mathcal{F}$  be  $F$ -equiuniform with finite  $F \subset A_2$ . For  $f \in F$ , there is finite  $G_f \subset A$  such that at each point of  $X$ ,  $f$  agrees with some function in  $G_f$ . With  $H = \cup \{G_f | f \in F\}$ ,  $\mathcal{F}$  is  $H$ -equiuniform, hence finitely- $A$ -equiuniform.

(d) Let  $\mathcal{F}$  be a countable subset of  $A_2^+$  which is finitely- $A_2$ -equiuniform, with  $\text{coz } \mathcal{F}$  a  $\mu_{A_2}$ -uniform cover of order one. We show  $\bigvee \mathcal{F} \in A_3$ : By (c) and 1.1,  $\bigvee \mathcal{F}$  is finitely- $A$ -uniform. By (b),  $\text{coz } \mathcal{F}$  is  $\mu_A$ -uniform. Let  $g \in \mathcal{F}$ , and



let  $F = \{h \in \mathcal{F} \mid \text{coz } h \cap \text{coz } g \neq \emptyset\}$ . By "order-one",  $F$  has at most three elements, so  $\bigvee F \in A_2$  by (a). But on the set  $\text{coz } g$ ,  $\bigvee \mathcal{F} = \bigvee F$ . So  $\bigvee \mathcal{F} \in A_3$ .

PROOF OF 3.2. Using 3.3(a) and (b), then 1.3,  $(A_2)_0$  is dense in  $U(\mu_A X)^+$ . Using 3.3(a), (b) and (d),  $A_3$  is dense in  $U(\mu_A X)$ .

4. **Compositions.** We derive some corollaries concerning functions of the form  $G \circ (a_1, \dots, a_p)$ , where  $(a_1, \dots, a_p): X \rightarrow R^p$  is the evaluation (or parametric) map defined by  $(a_1, \dots, a_p)(x) = (a_1(x), \dots, a_p(x))$ , and  $G$  is a function defined at least on the range  $(a_1, \dots, a_p)(X)$ . The essential observation is this:

4.1. PROPOSITION. *Let  $A \subset R^X$ , and  $g \in R^X$ . Then  $g$  is finitely- $A$ -uniform (2.1) iff  $g = G \circ (a_1, \dots, a_p)$  for some  $a_1, \dots, a_p \in A$  and  $G \in U((a_1, \dots, a_p)(X))$ .*

PROOF. If  $g = G \circ (a_1, \dots, a_p)$ , with  $G$  uniformly continuous, then certainly  $g \in U(\mu_F X)$  for  $F = \{a_1, \dots, a_p\}$ .

Let  $g \in U(\mu_F X)$ , for  $F = \{a_1, \dots, a_p\} \subset A$ . Clearly, if  $a_i(x) = a_i(y)$  for  $i = 1, \dots, p$ , then  $g(x) = g(y)$ . Thus, defining  $G: (a_1, \dots, a_p)(X) \rightarrow R$  by  $G((a_1(x), \dots, a_p(x))) = g(x)$  makes sense. Let  $\epsilon > 0$ . Since  $g \in U(\mu_F X)$ , there is  $\delta > 0$  with  $g^{-1}\mathfrak{S}(\epsilon) > \bigwedge a_i^{-1}\mathfrak{S}(\delta)$ . Now,

$$\begin{aligned} G^{-1}\mathfrak{S}(\epsilon) &= (a_1, \dots, a_p)g^{-1}\mathfrak{S}(\epsilon) > (a_1, \dots, a_p) \bigwedge a_i^{-1}\mathfrak{S}(\delta) \\ &= (\bigwedge \pi_i^{-1}\mathfrak{S}(\delta)) \cap (a_1, \dots, a_p)(X). \end{aligned}$$

This last is a uniform cover, so is  $G^{-1}\mathfrak{S}(\epsilon)$ , and  $G$  is uniformly continuous.

Given  $A \subset R^X$ , let  $\text{comp } A$  denote the class of functions  $g$  described in 3.1. Note that, here, we are not assuming  $A$  to have any algebraic properties.

4.2 THEOREM. *For any  $A \subset R^X$ ,  $\text{comp } A$  is uniformly dense in  $U(\mu_A X)$ .*

PROOF.  $\text{comp } A$  is itself a vector lattice: for example,  $a_1 + a_2 = G \circ (a_1, a_2)$ , where  $G(x, y) = x + y$ ; the other operations go similarly.

It is clear that a finitely  $\text{comp } A$ -uniform function is finitely- $A$ -uniform, hence by 4.1, in  $\text{comp } A$ . This shows that  $(\text{comp } A)_1 = \text{comp } A$ , so by 2.2,  $\text{comp } A$  is dense in  $U(\mu_{\text{comp } A} X)$ . But, of course,  $\mu_{\text{comp } A} = \mu_A$ .

In a sense, 4.2 reduces the problem of describing  $U(\mu_A X)$  to the problem for subsets of  $R^p$  ( $p = 1, 2, \dots$ ). One feels that one knows more about the functions in  $U(R^p)$  than about those in  $U(S)$  for  $S \subset R^p$ . So cases in which one can reduce to this may be worth considering.

4.3 COROLLARY.  $\{\{G \circ (a_1, \dots, a_p) \mid p \in N; a_1, \dots, a_p \in A; G \in BU(R^p)\} \text{ is dense in } BU(\mu_A X)\}$ .

PROOF. If  $f \in U(\mu_A X)$ , then by 4.2,  $f$  is approximable by a composition  $G \circ (a_1, \dots, a_p)$  with  $G \in U((a_1, \dots, a_p)(X))$ . If  $f$  is bounded,  $G$  is also bounded and thus extends over all of  $R^p$  by the Katětov Theorem [K].

4.4 COROLLARY.  $U(\mu_A X) = \{G \circ (a_n) | \{a_n\} \subset A; G \in U((a_n)(X))\}$  and  $BU(\mu_A X) = \{G \circ (a_n) | \{a_n\} \subset A; G \in BU(R^{k_0})\}$  (where the sets  $\{a_n\}$  are countable).

PROOF. The inclusions  $\supset$  are automatic.

If  $f \in U(\mu_A X)$ , then by 4.2,  $f$  is the uniform limit of a sequence  $\{f_n\}$ , where  $f_n = F_n \circ (a_1^n, \dots, a_{p_n}^n)$ . It is easily arranged inductively that if  $m < n$ , then  $p_m < p_n$  and for  $i < p_m$ ,  $a_i^m = a_i^n$ . We thus write  $f_n = F_n \circ (a_1, \dots, a_{p_n})$ . Now let  $G_n \in U((a_n)(X))$  be defined by  $G_n((a_n(x))) = F_n((a_1(x), \dots, a_{p_n}(x)))$ . Since  $g_n \rightarrow f$  uniformly,  $G_n$  converges uniformly to some  $G \in U((a_n)(X))$ . That  $f = G \circ (a_n)$  follows.

The proof for  $BU(\mu_A X)$  then uses the Katětov Theorem, as in 4.3.

Finally, we derive a theorem of Isbell [I(1)].

To say that  $A$  has *continuous composition* is to say that if  $p \in N$ ,  $a_1, \dots, a_p \in A$  and  $G \in C(R^p)$ , then  $G \circ (a_1, \dots, a_p) \in A$ . Assuming only this about  $A$ , it follows easily that  $A$  is a vector lattice and ring, as in the proof of 4.2.

4.5 COROLLARY. If  $A$  has *continuous composition*, then  $A$  is dense in  $U(\mu_A X)$ .

PROOF. Given  $f \in U(\mu_A X)$ , approximate within  $\varepsilon$  by  $G \circ (a_1, \dots, a_p)$ , using 4.2. Extend  $G$  to a uniformly continuous function  $G_1$  on the closure of  $(a_1, \dots, a_p)(X)$ , and then continuously over  $R^p$ , to  $G_2$ , by the Tietze Extension Theorem. Then  $G \circ (a_1, \dots, a_p) = G_2 \circ (a_1, \dots, a_p) \in A$ .

This proof uses 1.3, of course, Isbell's proof is rather simple.

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