TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 237, March 1978

THE μ-INVARIANT OF 3-MANIFOLDS AND CERTAIN STRUCTURAL PROPERTIES OF THE GROUP OF HOMEOMORPHISMS OF A CLOSED, ORIENTED 2-MANIFOLD

BY

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ABSTRACT. Let $\mathcal{H}(n)$ be the group of orientation-preserving self-homeomorphisms of a closed oriented surface Bd U of genus n, and let $\mathcal{K}(n)$ be the subgroup of those elements which induce the identity on $H_1(\text{Bd } U; \mathbf{Z})$. To each element $h \in \mathcal{H}(n)$ we associate a 3-manifold $M(h)$ which is defined by a Heegaard splitting. It is shown that for each $h \in \mathcal{H}(n)$ there is a representation ρ of $\mathcal{H}(n)$ into $\mathbf{Z}/2\mathbf{Z}$ such that if $k \in \mathcal{H}(n)$, then the μ-invariant $\mu(M(h))$ is equal to the μ-invariant $\mu(M(kh))$ if and only if $k \in$ kernel ρ. Thus, properties of the 4-manifolds which a given 3-manifold bounds are related to group-theoretical structure in the group of homeomorphisms of a 2-manifold. The kernels of the homomorphisms from $\mathcal{H}(n)$ onto $\mathbf{Z}/2\mathbf{Z}$ are studied and are shown to constitute a complete conjugacy class of subgroups of $\mathcal{H}(n)$. The class has nontrivial finite order.

1. Introduction. It is well known that any closed, oriented 3-manifold admits a representation by a Heegaard splitting, i.e. as the union of two cubes-with-handles identified along their boundaries. Since the identification space is uniquely determined by the specification of a homeomorphism from the boundary of one handlebody to the boundary of the other, it is possible to translate many questions about the topology of 3-manifolds into algebraic questions about the group $\mathcal{H}(n)$ of orientation-preserving homeomorphisms of a closed oriented surface of genus n. In particular, one might expect that correspondences would exist between structures in the class of oriented 3-manifolds and structures in the groups $\mathcal{H}(n)$, $n = 0, 1, 2, \ldots$. The purpose of this paper is to exhibit just such a correspondence, as it arises in connection with the study of the μ-invariant of $\mathbf{Z}/2\mathbf{Z}$-homology spheres.

The main results of this paper are contained in Theorems 8, 9, and 11. We review these now. For each genus $n \geqslant 0$, let $U = U(n)$ denote an oriented cube-with-handles of genus n, and let $-U$ denote the same handlebody with its orientation reversed. The group $\mathcal{H}(n)$ is the group of orientation preserving homeomorphism of Bd U onto itself. Let $\mathcal{K}(n)$ denote the subgroup

Received by the editors September 27, 1976.

AMS (MOS) subject classifications (1970). Primary 57A10, 57A05, 55A99.

([1]) The first author acknowledges the partial support of the Alfred P. Sloan Foundation, and also the National Science Foundation under Grant #MPS 71-03442.

consisting of those homeomorphisms that induce the identity on $H_1(\mathrm{Bd}\ U; \mathbf{Z})$. For $h \in \mathcal{H}(n)$, let $M(h)$ denote the 3-manifold defined as the disjoint union of U and $-U$ with the identification $ih(x) = x$ where i is the "identity" function from Bd U to Bd$(-U)$.

If $M(h)$ is a $\mathbf{Z}/2\mathbf{Z}$-homology sphere, there is an invariant $\mu(M(h))$ of the oriented homeomorphism type of $M(h)$. It is due to Eells and Kuiper (see [EK] and [HNK, §7]). The μ-invariant takes the values $j/8, j = 0, 1, \ldots, 7$ in $\mathbf{Q}/\mathbf{Z}$. For $\mathbf{Z}$-homology spheres it is either 0 or $\frac{1}{2}$, and for the 3-sphere it is 0. We will prove the following results about the μ-invariant.

1. If h_1 and h_2 are elements of $\mathcal{H}(n)$, $n > 1$, such that $M(h_2h_1)$ is a $\mathbf{Z}/2\mathbf{Z}$-homology sphere, there is a homomorphism ρ_{h_2,h_1} from $\mathcal{H}(n)$ onto the additive group of order 2, $\{0, \frac{1}{2}\}$ (mod 1), defined by

$$\rho_{h_2,h_1}(k) \equiv \mu(M(h_2kh_1)) - \mu(M(h_2h_1)) \qquad (\mathrm{mod}\ 1), k \in \mathcal{H}(n).$$

2. Let $\mathcal{H}_{h_2,h_1}$ denote ker ρ_{h_2,h_1}. Consider the collection $\mathcal{L}(n)$ of groups $\mathcal{H}_{h_2,h_1}$ as h_2 and h_1 range over all possible elements of $\mathcal{H}(n)$ for which $M(h_2h_1)$ is a $\mathbf{Z}/2\mathbf{Z}$-homology sphere. Then $\mathcal{L}(n)$ is a complete conjugacy class of subgroups of $\mathcal{H}(n)$. The class $\mathcal{L}(n)$ has nontrivial finite order, and bounds are given by $2^n < |\mathcal{L}(n)| < m^2$ where m is the order of the symplectic group $\mathrm{Sp}(2n, \mathbf{Z}/2\mathbf{Z})$.

3. Define $\mathcal{C}(n) = \bigcap \mathcal{H}_{h_2,h_1}$ where the intersection is taken over all possible subgroups $\mathcal{H}_{h_2,h_1}$. Then $\mathcal{C}(n)$ is a normal subgroup of $\mathcal{H}(n)$, and the μ-invariant has an algebraic interpretation (explained in §4) in terms of the sequence of factor groups $(\mathcal{H}(n)/\mathcal{C}(n), n = 0, 1, 2, \ldots)$.

Each of the assertions above clearly implies the corresponding assertion with $\mathcal{H}(n)$ replaced by the quotient group $\mathfrak{M}(n)$ obtained by factoring out homeomorphisms isotopic to the identity. This is so because first, the homeomorphisms factored out belong to $\mathcal{H}(n)$ and second, changing one of the homeomorphisms by an isotopy does not change the oriented homeomorphism type of the manifold defined by it.

Let $\hat{\mathcal{H}}(n)$ denote the subgroup of $\mathcal{H}(n)$ consisting of those homeomorphisms that induce the identity on $H_1(\mathrm{Bd}\ U; \mathbf{Z}/2\mathbf{Z})$. It is natural to conjecture that the "$\mathbf{Z}/2\mathbf{Z}$-regularity" exhibited by $\mathcal{H}(n)$ in effecting changes in the μ-invariant might generalize to a corresponding regularity for $\hat{\mathcal{H}}(n)$, only in this case one might expect to find representations onto $\mathbf{Z}/8\mathbf{Z}$ or $\mathbf{Z}/4\mathbf{Z}$. But it is just not so, not even for $n = 1$! The formulas given by Hirzebruch [HNK, §7] for the μ-invariants of lens spaces reveal quickly that such representations do not exist.

The results here have interest in several directions. First, from a purely group theoretical point of view, they give us a multitude of examples of index-2 subgroups of $\mathcal{H}(n)$. Second, from the point of view of invariants of 3-manifolds, the techniques used to reveal the index-2 subgroups of $\mathcal{H}(n)$

suggest a new general means to obtain invariants for 3-manifolds through representations of subgroups of $\mathcal{H}(n)$. Third, the main results described in 1 above suggest a method for investigating the existence or nonexistence of index-8 homotopy 3-spheres. It will be seen that a necessary and sufficient condition for the existence of index-8 homotopy 3-spheres is that for some n and any $h \in \mathcal{H}(n)$ such that $M(h)$ is the 3-sphere, there exist an element $k \in \mathcal{H}(n)$ such that k does *not* belong to the group $\mathcal{H}_{\mathrm{id},h}$ (which has index 2 in $\mathcal{H}(n)$) and such that $\pi_1(M(kh)) = 1$.

ACKNOWLEDGEMENT. We were led to the result in Theorem 8 when W. B. R. Lickorish suggested to us that the group $\mathcal{H}(n)$ ought to exhibit some kind of $\mathbf{Z}/2\mathbf{Z}$ regularity in effecting changes in the μ-invariant. Conversations with Walter Neumann were helpful in bringing Theorem 8 to its full generality. Walter Neumann also offered helpful suggestions regarding the proof of Theorem 9.

Some of the work done here was done while the second author was a sabbatical visitor at the Science Institute of the University of Iceland. He wishes to thank Halldór Í. Elíasson and the Science Institute for support and hospitality during that visit.

2. Symplectic matrices, Heegaard splittings, map pairs, and triadic 4-manifolds. In this section we introduce notation, set down conventions, and develop the basic tools which will be used in §3 to prove our main results.

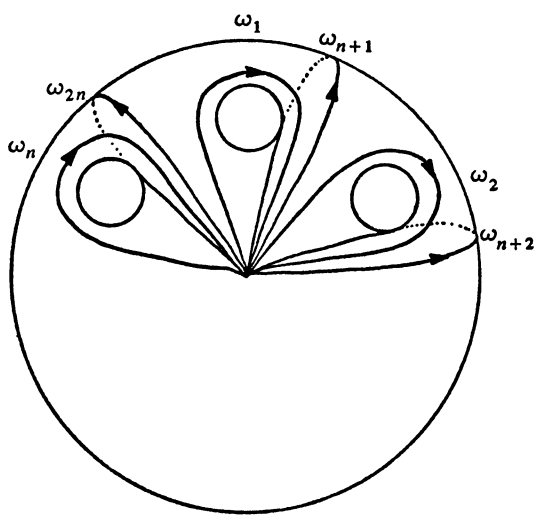

FIGURE 2.1

2.1 NOTATION AND CONVENTIONS. The following symbols and expressions will be used:

$\approx$ a general symbol for equivalence;

$\approx$ equivalence for oriented manifolds, always means an orientation preserving homeomorphism;

$\mathbf{Z}$ the ring of integers;

$\mathbf{Q}$ the field of rational numbers;

Σ the 3-sphere with some fixed orientation;

$U = U(n)$ a fixed, oriented handlebody of genus $n \geqslant 0$ imbedded in a standard manner in Σ (see Figure 2.1);

$\omega_1, \ldots, \omega_{2n}$ standard basis for $\pi_1(\mathrm{Bd}\ U)$ (see Figure 2.1) and also for $H_1(\mathrm{Bd}\ U; \mathbf{Z})$ and $H_1(\mathrm{Bd}\ U; \mathbf{Z}/2\mathbf{Z})$. The context will distinguish these from each other.

$\mathcal{H} = \mathcal{H}(n)$ the group of orientation-preserving self-homeomorphisms of Bd U.

$\tilde{\mathcal{H}} = \tilde{\mathcal{H}}(n)$ the group automorphisms of $H_1(\mathrm{Bd}\ U; \mathbf{Z})$ induced by $\mathcal{H}$.

$\hat{\mathcal{H}} = \hat{\mathcal{H}}(n)$ the group of automorphisms of $H_1(\mathrm{Bd}\ U; \mathbf{Z}/2\mathbf{Z})$ induced by $\mathcal{H}$.

η the natural homomorphism from $\mathcal{H}$ onto $\tilde{\mathcal{H}}$.

ε the natural homomorphism from $\tilde{\mathcal{H}}$ onto $\hat{\mathcal{H}}$.

$\mathcal{K} = \mathcal{K}(n)$ kernel η.

$\mathcal{F} = \mathcal{F}(n)$ the subgroup of homeomorphisms in $\mathcal{H}$ that extend to U.

$\tilde{\mathcal{F}} = \tilde{\mathcal{F}}(n)$ the image of $\mathcal{F}$ under η.

$\hat{\mathcal{F}} = \hat{\mathcal{F}}(n)$ the image of $\mathcal{F}$ under $\varepsilon\eta$.

h, h_1, t, t_1 elements of $\mathcal{H}(n)$.

$f, f_1, \ldots$ elements of $\mathcal{F}(n)$.

$k, k_1, \ldots$ elements of $\mathcal{K}(n)$.

I the $n \times n$ identity matrix.

0 the $n \times n$ zero matrix.

J the $2n \times 2n$ matrix $\left\|\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right\|$.

$F(W)$ the $2n \times 2n$ matrix $\left\|\begin{smallmatrix} I & W \\ 0 & I \end{smallmatrix}\right\|$ over $\mathbf{Z}$ or $\mathbf{Z}/2\mathbf{Z}$ where W is an $n \times n$ symmetric matrix.

$D(U)$ the $2n \times 2n$ matrix $\left\|\begin{smallmatrix} (U')^{-1} & 0 \\ 0 & U \end{smallmatrix}\right\|$ over $\mathbf{Z}$ or $\mathbf{Z}/2\mathbf{Z}$ where U is a unimodular $n \times n$ matrix.

Other symbols will be defined later, as they are introduced. The genus symbol n will sometimes be dropped to simplify notation.

Manifolds and surfaces are compact, oriented, and, unless something is said otherwise, closed. If M is a manifold, then $-M$ denotes the same manifold with opposite orientation. All maps, manifolds, etc. are piecewise linear. Equivalence between manifolds or tuples of manifolds always means an orientation-preserving homeomorphism. If M and M' are manifolds, not necessarily disjoint, then $M + M'$ will denote the disjoint union of M and M'.

2.2 THE SYMPLECTIC GROUP. We have chosen a *standard basis* $\omega_1, \ldots, \omega_{2n}$

for $\pi_1(\mathrm{Bd}\ U)$ as illustrated in Figure 2.1. In terms of this basis, the group $\pi_1(\mathrm{Bd}\ U)$ admits a presentation with the single defining relation

$$\prod_{j=1}^{n} \left(\omega_j \omega_{n+j} \omega_j^{-1} \omega_{n+j}^{-1}\right) = 1.$$

We further regard $H_1(\mathrm{Bd}\ U; \mathbf{Z})$ and $H_1(\mathrm{Bd}\ U; \mathbf{Z}/2\mathbf{Z})$ as the free abelian group on these same generators and its tensor product with $\mathbf{Z}/2\mathbf{Z}$. The context will always distinguish these uses. Let $L(h)$ denote the matrix representing $\eta(h)$, $h \in \mathcal{H}(n)$, with the convention that the (i, j)th entry in $L(h)$ is the coefficient of ω_j in $\eta(h)(\omega_i)$. The elements of $H_1(\mathrm{Bd}\ U)$ can then be regarded as row vectors and the action of $\tilde{\mathcal{H}}$ as right matrix multiplication. Note that in Figure 2.1 orientations are chosen so that the homology intersection form $I(\omega_i, \omega_j)$ is represented by the matrix J. A $2n \times 2n$ matrix L over $\mathbf{Z}$ or $\mathbf{Z}/2\mathbf{Z}$ is *symplectic* if

$$LJL^t = J \tag{1}$$

where equality is taken over $\mathbf{Z}$ or $\mathbf{Z}/2\mathbf{Z}$ as appropriate. The sets of symplectic matrices form the *symplectic groups* $\mathrm{Sp}(2n, \mathbf{Z})$ and $\mathrm{Sp}(2n, \mathbf{Z}/2\mathbf{Z})$. It follows from [Nw, p. 132, Theorem VII.21] that $\mathrm{Sp}(2n, \mathbf{Z}/2\mathbf{Z})$ is the mod 2 reduction of $\mathrm{Sp}(2n, \mathbf{Z})$. Condition (1) is clearly necessary in order that a matrix L correspond to an element of $\tilde{\mathcal{H}}(n)$, for it expresses the condition that an orientation-preserving homeomorphism of Bd U must preserve the homology intersection form. It was proved by Nielsen (see [MKS, Theorem N 13]) that condition (1) is sufficient in order that L represent an element of $\tilde{\mathcal{H}}$. Thus we may identify $\tilde{\mathcal{H}}(n)$ with the symplectic group $\mathrm{Sp}(2n, \mathbf{Z})$ and $\hat{\mathcal{H}}(n)$ with $\mathrm{Sp}(2n, \mathbf{Z}/2\mathbf{Z})$. Beware! The representation η: $\mathcal{H}(n) \to \mathrm{Sp}(2n, \mathbf{Z})$ is an anti-homomorphism; hence the order in compositions must be reversed.

We will have to use certain properties of the groups $\tilde{\mathcal{F}}(n)$ and $\hat{\mathcal{F}}(n)$. These are summarized by the following lemma:

LEMMA 1. (i) $\tilde{\mathcal{F}}(n) = \{\left\|\begin{smallmatrix} R & S \\ 0 & Q \end{smallmatrix}\right\| \in \mathrm{Sp}(2n, \mathbf{Z})\}$, $\hat{\mathcal{F}}(n) = \{\left\|\begin{smallmatrix} R & S \\ 0 & Q \end{smallmatrix}\right\| \in \mathrm{Sp}(2n, \mathbf{Z}/2\mathbf{Z})\}$.

(ii) *For any $f \in \mathcal{F}(n)$, there exist $n \times n$ matrices U, S, S^* with U unimodular and S and S^* symmetric such that $L(f) = F(S)D(U) = D(U)F(S^*)$.*

PROOF. See [Br, EH, §2] for a proof of (i) in the case of $\tilde{\mathcal{F}}(n)$ and for a proof of (ii). The proof of (i) in the case of $\hat{\mathcal{F}}(n)$ goes as follows: One can follow the argument in [Br, EH] to verify that each matrix $\left\|\begin{smallmatrix} R & S \\ 0 & Q \end{smallmatrix}\right\|$ in $\mathrm{Sp}(2n, \mathbf{Z}/2\mathbf{Z})$ has a decomposition $F(S)D(U)$ (over $\mathbf{Z}/2\mathbf{Z}$) as described in (ii). Now S is the mod 2 reduction of a symmetric matrix $\bar{S}$ over $\mathbf{Z}$ whose entries are 0's and 1's. By Lemma VII.8 of [Nw] the matrix U is the mod 2 reduction of a unimodular matrix $\bar{U}$ over $\mathbf{Z}$. But then, $\left\|\begin{smallmatrix} R & S \\ 0 & Q \end{smallmatrix}\right\|$ is the mod 2

reduction of the matrix $\|_0^{\bar{R}}\,{}_{\bar{Q}}^{\bar{S}}\| = F(\bar{S})D(\bar{U})$, and the fact that $\|_0^{R}\,{}_{Q}^{S}\|$ represents an element of $\hat{\mathfrak{F}}(n)$ follows from the special case of (i) quoted from **[Br, EH]**. □

2.3 HEEGAARD SPLITTINGS. A *Heegaard splitting* of genus n of a closed, oriented 3-manifold is a representation of that manifold as $U +_{ih} (-U)$ where $h \in \mathcal{H}(n)$, i: Bd $U \to$ Bd$(-U)$ is the identity map, and the identification is defined by the rule $ih(x) = x$ for all $x \in$ Bd U. This Heegaard splitting is defined uniquely by the surface mapping h, and will accordingly be denoted by the symbol $S(h)$. The manifold which it defines will be denoted by $M(h)$.

Two Heegaard splittings $S(h)$ and $S(h')$ of the same genus n are defined to be *equivalent* if there exist elements f_1 and f_2 in $\mathfrak{F}(n)$ such that

$$h' = f_2 h f_1. \tag{2}$$

That is, h and h' must belong to the same double coset of $\mathcal{H}(n)$ modulo $\mathfrak{F}(n)$. Condition (2) states algebraically the geometric condition that there is an equivalence from $M(h)$ to $M(h')$ that restricts to an equivalence from Bd U onto itself. Note that isotopic changes in h do not alter the equivalence class of a Heegaard splitting $S(h)$; hence the splitting $S(h)$ is determined up to equivalence if instead of h one specifies just the induced automorphism h_* on π_1(Bd U). This follows from the fact that each automorphism of π_1(Bd U) is induced by a unique isotopy class of homeomorphisms of Bd U (see **[N1]**). Consider then, for each $n > 1$, the automorphisms $s_* = s_*^{(n)}$ of π_1(Bd U) that is given with respect to the standard basis by:

$$\begin{aligned} s_* = s_*^{(n)}\colon \omega_j &\to \omega_j \omega_{n+j} \omega_j^{-1}, \quad 1 \leqslant j \leqslant n, \\ \omega_j &\to \omega_{j-n}^{-1}, \quad n+1 \leqslant j \leqslant 2n. \end{aligned} \tag{3}$$

Let s be an element of $\mathcal{H}(n)$ that induces the automorphism s_*. Then $S(s)$ is a standard Heegaard splitting of the 3-sphere Σ since each curve ω_{j-n}^{-1}, $n+1 \leqslant j \leqslant 2n$, bounds a disk in $\Sigma - \mathring{U}$.

We now wish to describe, in map language, a method for taking sums of Heegaard splittings which is consistent with the usual notion of connected sums for manifolds. Let $D(n)$ be a disk in Bd $U(n)$. For a pair (m, n) we form the boundary sum $U(m) \#_b U(n)$ and identify it with $U(m+n)$ as follows: Choose an orientation-reversing homeomorphism g: $D(m) \to D(n)$ and take $U(m) \#_b U(n)$ to be the identification space $U(m) +_g U(n)$. Now identify $U(m) \#_b U(n)$ with $U(m+n)$ by some orientation-preserving homeomorphism g'. Given two Heegaard splittings $S(h)$ and $S(h')$ of genus m and n respectively we may, without changing the equivalence classes of $S(h)$ and $S(h')$, isotopically modify h and h' so that they are the identity on $D(m)$ and $D(n)$ respectively. The homeomorphism g' now specifies a unique homeomorphism $h \# h'$ in $\mathcal{H}(m+n)$ corresponding to the homeomorphism

of $\mathrm{Bd}(U(m) \#_b U(n))$ that is defined by restrictions of h and h'. The *sum* $S(h) \# S(h')$ is defined to be the Heegaard splitting $S(h \# h')$. The manifold $M(h \# h')$ then automatically defines the connected sum $M(h) \# M(h')$.

The homeomorphism $h \# h'$ in $S(h \# h')$ depends upon the choices of g, g' and the isotopies used to modify h and h', but the double coset of $h \# h'$ in $\mathcal{H}(m+n)$ modulo $\mathcal{F}(m+n)$ is independent of these choices. Thus the construction above leads to a Heegaard splitting $S(h \# h')$ that is unique up to the equivalence defined in (2).

Specializing the operation above leads to stabilization of a Heegaard splitting by the convention $S(h) \to S(h \# s^{(1)}) = S(h \# s)$. Note that $M(h \# s)$ is homeomorphic to $M(h)$. The sum operation will be used in a second way: to form the sum of a map $h \in \mathcal{H}(n)$ with $\mathrm{id} \in \mathcal{H}(1)$ and so describe a "canonical extension" of an element of $\mathcal{H}(n)$ to an element of $\mathcal{H}(n+1)$.

Our next task will be to interpret certain topological properties of a manifold $M(h)$ that are exhibited in the symplectic groups by means of the matrix $\eta(h) = L(h)$ and its mod 2 reduction. The first result follows from a simple application of the Mayer-Vietoris sequence:

LEMMA 2. *Let $h \in \mathcal{H}(n)$ and let $L(h)$ be given by the symplectic matrix $\|{}^{R}_{P}\,{}^{S}_{Q}\|$ with respect to the standard basis.*

Then $M(h)$ is a **Z***-homology sphere if and only if P is unimodular over* **Z**, *and $M(h)$ is a* $\mathbf{Z}/2\mathbf{Z}$*-homology sphere if and only if P* (mod 2) *is unimodular over* $\mathbf{Z}/2\mathbf{Z}$. □

The second result which we will need is proved in §3 of **[Br, EH]**. We repeat the proof here because it is brief and because it illustrates a technique that will be used repeatedly later.

LEMMA 3. *Let h and h' be elements of $\mathcal{H}(n)$, and suppose that $M(h)$ and $M(h')$ are* $\mathbf{Z}/2\mathbf{Z}$*-homology spheres.*

Then there are elements f_1 and f_2 in $\mathcal{F}(n)$ such that the splitting $S(f_2h'f_1)$ is equivalent to $S(h')$ and $L(f_2h'f_1) \equiv L(h)$ (mod 2).

Moreover, if $M(h)$ and $M(h')$ are both **Z***-homology spheres, then f_1 and f_2 can be chosen so that $L(f_2h'f_1) = L(h)$.*

PROOF. Consider first the special case of the lemma where $h = s$ and where $M(h')$ is a **Z**-homology sphere. Let $L(h') = \|{}^{R}_{P}\,{}^{S}_{Q}\|$. By Lemma 2, the matrix P is unimodular. Also, as a consequence of the symplectic condition (1), the matrices P^tR and QP^t are symmetric. Let L_1 and L_2 be elements of $\mathrm{Sp}(2n, \mathbf{Z})$ defined by

$$L_1 = F(-P^tR)D(-P^{-1}) \quad \text{and} \quad L_2 = F(-P^{-1}Q).$$

Then $L_1L(h')L_2$ has the form $\|\begin{smallmatrix} 0 & X \\ -I & Q \end{smallmatrix}\|$, and (1) now implies that $X = I$. Since, by Lemma 1, L_1 and L_2 belong to $\mathfrak{F}(n)$, we may find elements f_1 and f_2 in $\mathfrak{F}(n)$ such that $L_i = L(f_i)$, $i = 1, 2$. Then we have $L(f_2hf_1) = L(s)$ as promised. The equivalence of $S(h')$ and $S(f_2h'f_1)$ follows from the definition of equivalence (2).

Next we change the special case by allowing h to be any element of $\mathcal{H}(n)$ that defines a $\mathbf{Z}$-homology sphere $M(h)$. From the special case we can find f_3, f_4, f_5, f_6 in $\mathfrak{F}(n)$ such that $L(f_4hf_3) = L(f_6h'f_5) = L(s)$. But then $L(f_4^{-1}f_6h'f_5f_3^{-1}) = L(h)$ so we can take f_1 and f_2 to be $f_5f_3^{-1}$ and $f_4^{-1}f_6$ respectively to obtain the desired equivalence.

The case for $\mathbf{Z}/2\mathbf{Z}$-homology spheres is done in an entirely similar manner. We first locate the appropriate matrices over $\mathbf{Z}/2\mathbf{Z}$, since in this case we only know that det P is odd, and then we use Lemma 1 to lift these matrices back to elements of $\mathfrak{F}(n)$.

2.4 Map Pairs and 4-Manifolds. The map pair theory described below comes from applying the first author's mapping class formalism (see **[Br, EH]**) to rewrite a generalized Heegaard theory developed by the second author (see **[Cr, HS]** and **[Cr, FH]**).

A *map pair* is an element $(h_2, h_3) \in \mathcal{H} \times \mathcal{H}$. Each map pair defines a triple of 3-manifolds $(M(h_3), M(h_2), M(h_3h_2^{-1}))$ which will be called the *fundamental triple* for the map pair (h_2, h_3). We will show, in the next paragraph, how to associate a 4-manifold $N = N(h_2, h_3)$ with the map pair (h_2, h_3) so that Bd N is the disjoint union $-M(h_3) + M(h_2) + M(h_3h_2^{-1})$.

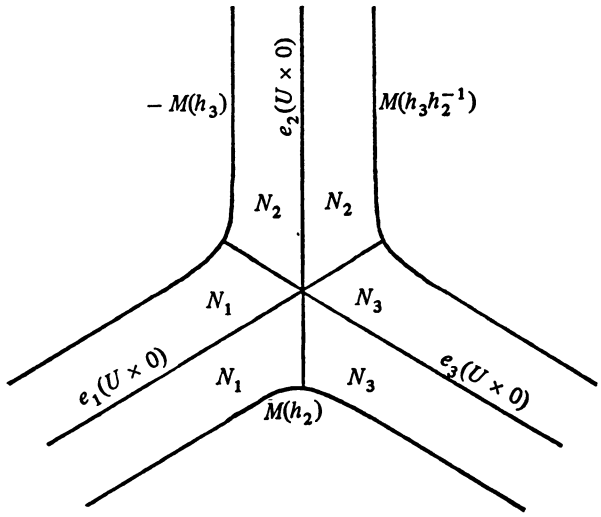

FIGURE 2.2

Form three copies N_1, N_2, N_3 of the 4-manifold $U \times [-1, 1]$, and let

$$e_j\colon U \times [-1, 1] \to N_j, \qquad j = 1, 2, 3,$$

denote the corresponding identity maps. Orient the N_j's so that $U \to U \times 1 \to^{e_j} N_j$ is an orientation-preserving homeomorphism of U into $\operatorname{Bd} N_j$ for $j = 1, 2$ and is an orientation-reversing homeomorphism for $j = 3$. Define an equivalence relation $\sim$ on the points in the 3-manifolds $e_j((\operatorname{Bd} U) \times [-1, 1])$ by $e_1(x, t) = e_2(h_2(x), -t)$, $e_1(x, -t) = e_3(h_3(x), -t)$, $e_2(x, t) = e_3(h_3h_2^{-1}(x), t)$, $x \in \operatorname{Bd} U$, $t \in [0, 1]$. Set $N = (N_1 + N_2 + N_3)/\sim$, the disjoint union of the N_j's modulo $\sim$. See Figure 2.2. Then N is an oriented 4-manifold, and its oriented boundary is naturally identified with $-M(h_3) + M(h_2) + M(h_3h_2^{-1})$ as indicated in Figure 2.2. We say that N is the 4-manifold associated with (h_2, h_3).

For a 4-manifold N let φ_N denote the homology intersection form

$$\varphi_N\colon H_2(N)/\text{Torsion} \times H_2(N)/\text{Torsion} \to \mathbf{Z},$$

and let τ denote the signature of φ_N. Recall that the μ-invariant was described in the introduction. (The reader is directed to **[EK]**, **[HNK]**, **[GA]**, **[Gd]**, **[CS]** for more information about the μ-invariant and its computation.) Two properties of the μ-invariant that we will use here are,

$$\mu(M \# M') \equiv \mu(M) + \mu(M') \pmod 1, \text{ and} \tag{4}$$

$$\mu(-M) \equiv -\mu(M) \pmod 1. \tag{5}$$

For the 4-manifold $N = N(h_2, h_3)$ the relationship between the signature τ of φ_N and the μ-invariants of the boundary components of N is expressed by the following lemma:

LEMMA 4. *Let $N = N(h_2, h_3)$ be the 4-manifold associated with the map pair (h_2, h_3). Suppose that φ_N has even type, i.e. the quadratic values $\varphi_N(\beta, \beta)$ are all even, and suppose that $M(h_3)$, $M(h_2)$, and $M(h_3h_2^{-1})$ are all $\mathbf{Z}/2\mathbf{Z}$-homology spheres.*

Then the signature τ of φ_N satisfies the congruence,

$$-\mu(M(h_3)) + \mu(M(h_2)) + \mu\big(M(h_3h_2^{-1})\big) \equiv -\tau/16 \pmod 1. \tag{6}$$ □

PROOF. It is first necessary to verify that $H_1(N; \mathbf{Z}/2\mathbf{Z}) = 0$. Since each 1-cycle is homologous to a boundary cycle, this follows from the fact that the boundary components of N are $\mathbf{Z}/2\mathbf{Z}$-homology spheres. Now drilling holes in N to reduce the number of boundary components to one does not change either the form φ_N or the fact that $H_1(N)$ has no 2-torsion. The boundary of the drilled 4-manifold is equivalent to the connected sum of the boundary components of N. By the definition of the μ-invariant (see **[EK]** or **[HNK, §7]**) the μ-invariant of the boundary of the drilled manifold is $-\tau/16 \pmod 1$. Now apply equations (4) and (5) to get the congruence (6). □

Two map pairs (h_2, h_3) and (h_2', h_3') are defined to be *equivalent* if there are

elements f_1, f_2, f_3 in the group $\mathfrak{F}$ such that

$$h_j' = f_j h_j f_1, \qquad j = 2, 3. \tag{7}$$

This provides that $h_3'(h_2')^{-1} = f_3 h_3 h_2^{-1} f_2^{-1}$. From condition (2) on the equivalence of Heegaard splittings, we find that equivalent map pairs (h_2, h_3) and (h_2', h_3') define equivalent Heegaard splittings of the corresponding manifolds in the fundamental triples $(M(h_3), M(h_2), M(h_3 h_2^{-1}))$ and $(M(h_3'), M(h_2'), M(h_3'(h_2')^{-1}))$. Moreover, these equivalences are interdependent. Algebraically, this definition of equivalence says that h_j and $h_j', j = 2, 3$, must lie in the same double coset of modulo $\mathfrak{F}$ and there must be a common right coset representative in the two equivalences.

A little reflection should convince the reader that map pairs (h_2, h_3) and (h_2', h_3') are equivalent if and only if the 4-tuples (N, N_1, N_2, N_3) and (N', N_1', N_2', N_3') are equivalent where $N = N(h_2, h_3)$ and $N' = N(h_2', h_3')$. Thus our rather peculiar definition of equivalence in (7) will turn out to be exactly the one that is needed to preserve the signature formula (6) and thus to obtain information about the μ-invariants of the manifolds in a fundamental triple.

An obvious problem arises about how to represent conveniently the bilinear form φ_N and thus calculate its signature and type. With the goal of representing φ_N in mind, we introduce two natural definitions. Let (h_2, h_3) be a map pair. Then $(\eta(h_2), \eta(h_3))$ is defined to be an *abelianized map pair* and $(\varepsilon\eta(h_2), \varepsilon\eta(h_3))$ to be the mod 2 *reduction* of the abelianized map pair. The equivalence relation (7) goes over in a natural way to equivalence relations on abelianized map pairs and their mod 2 reductions: Instead of double cosets of $\mathcal{H}$ mod $\mathfrak{F}$, equivalence classes are represented by double cosets of $\tilde{\mathcal{H}}$ mod $\tilde{\mathfrak{F}}$ and of $\hat{\mathcal{H}}$ mod $\hat{\mathfrak{F}}$.

LEMMA 5. *Let (h_2, h_3) be a map pair with associated 4-manifold N. Suppose that $H_1(M(h_3); \mathbf{Z}) = 0$.*

(i) *The map pair (h_2, h_3) is equivalent to a map pair (h_2', h_3') whose abelianization has the normal form*:

$$(L(h_2'), L(h_3')) = \begin{Vmatrix} R_2 & S_2 \\ P_2 & Q_2 \end{Vmatrix}, \quad \begin{Vmatrix} 0 & I \\ -I & 0 \end{Vmatrix}. \tag{8}$$

(ii) *For the normal form* (8), *the submatrix $P_2^t R_2$ is a symmetric matrix and it represents the bilinear form $\varphi_{N(h_2', h_3')}$ which is equivalent to φ_N.*

(iii) *If further, $H_1(M(h_2); \mathbf{Z}) = 0$, then (h_2, h_3) is equivalent to a map pair (h_2^*, h_3^*) whose abelianization has the normal form*

$$(L(h_2^*), L(h_3^*)) = \begin{Vmatrix} W & I \\ -I & 0 \end{Vmatrix}, \quad \begin{Vmatrix} 0 & I \\ -I & 0 \end{Vmatrix}, \tag{9}$$

where W is a symmetric matrix such that $-W$ represents the bilinear form

$\varphi_N(h_2^*, h_3^*) \approx \varphi_N$. *Moreover,* $H_1(M(h_3, h_2^{-1}); \mathbf{Z}) = 0$ *if and only if* $\det W = \pm 1$.

PROOF. Since by hypothesis $H_1(M(h_3); \mathbf{Z}) = 0$, we may, by Lemma 3, find an equivalent splitting $S(h_3')$ for $M(h_3)$ such that $L(h_3')$ has the normal form given in (8). Now, there exist $f_1, f_3 \in \mathfrak{F}$ such that $h_3' = f_3 h_3 f_1$. Define h_2' to be $h_2 f_1$. Then $(h_2', h_3') \approx (h_2, h_3)$ by condition (7), and $(L(h_2'), L(h_3'))$ has the normal form (8) promised by the lemma. This establishes (i).

The matrix $P_2' R_2$ is symmetric because $L(h_2')$ is symplectic, and hence satisfies (1). The 4-manifolds $N = N(h_2, h_3)$ and $N' = N(h_2', h_3')$ are homeomorphic under an orientation-preserving homeomorphism, so $\varphi_N \approx \varphi_{N'}$. Thus to establish (ii) it is enough to show that $P_2' R_2$ represents $\varphi_{N'}$. We first identify $\varphi_{N'}$ with a linking form for $M(h_3')$. Let θ: Bd $U \to$ Int U be a homeomorphism that translates points of Bd U along the fibers of a collar on Bd U in U. Let B denote the subgroup of $H_1(\text{Bd } U; \mathbf{Z})$ generated by the set $\{h_{2*}^{-1}(\omega_{i+n}): i \leqslant n\}$. Now regard U as the first handlebody in $M(h_3') = U +_{ih'} (-U)$ and consider the linking form

$$L: B \times B \to \mathbf{Z}, \qquad L(\beta, \beta') = \operatorname{lk}\big(\theta_*(\beta), \beta'\big),$$

where lk denotes linking number over $\mathbf{Z}$ in $M(h_3')$. Linking numbers are well defined in $M(h_3')$ since $H_1(M(h_3'); \mathbf{Z}) = 0$. Moreover the bilinear form L is symmetric because linking numbers are symmetric in 3-manifolds. In the next paragraph we relate L with $\varphi_{N'}$ by identifying a handle decomposition for N.

Consider the submanifold $(N_1' + N_3')/\sim$ in N'. It is homeomorphic, by an orientation-preserving homeomorphism, to $M(h_3') \times [0, 1]$ where $M(h_3') \times [0, 1]$ is oriented so that $\text{Bd}(M(h_3') \times [0, 1]) = M(h_3') \times 1 \cup -M(h_3') \times 0$. This homeomorphism sends the component of Bd N' identified with $-M(h_3')$ to $-M(h_3') \times 0$. The homeomorphism may be presumed to preserve the sides of $M(h_3)$ in the correspondence and to send $(\text{Bd } U) \times 0$ in $(N_1' + N_3')/\sim$ to $(\text{Bd } U) \times 1$ in $M(h_3') \times [0, 1]$ so that $(x, 0) \to (x, 1)$. Choose a complete system of meridianal disks $D_1, \ldots, D_n$ in U and thicken these disks slightly to disjoint 3-balls $C_1, \ldots, C_n$ that are regular neighborhoods of the disks D_i in U. Let C_0 denote the 3-ball $(U \setminus \bigcup C_i)$. Now regard the disks $D_1, \ldots, D_n$ and the 3-balls $C_0, \ldots, C_n$ as sitting in the 0-section of the 4-manifold N_2'. Then $N_2' = \bigcup\{C_i \times [-1, 1]/0 \leqslant i \leqslant n\}$, and N' has a handle decomposition as the sum of $M(h_3) \times [0, 1]$ (identified with $(N_1' + N_3')/\sim$), n 2-handles, $C_1 \times [-1, 1], \ldots, C_n \times [-1, 1]$, and one 3-handle, $C_0 \times [-1, 1]$.

Each of the disks D_i is an attaching disk for the 2-handle $C_i \times [-1, 1]$. Its boundary is attached to $M(h_3') \times 1$ as $h_2^{-1}(\text{Bd } D_i) \times 1$, where $h_2^{-1}(\text{Bd } D_i)$ is considered to be a subset of the first side of $M(h_3')$. Associate with each $h_2^{-1}(\text{Bd } D_i)$ the homology class $\beta_i \in H_1(\text{Bd } U; \mathbf{Z})$ of a 1-cycle that is represented by some orientation of $h_2^{-1}(\text{Bd } D_i)$. Now $\{\beta_1, \ldots, \beta_n\}$ is a free basis

for the subgroup B of $H_1(\mathrm{Bd}\, U; \mathbf{Z})$. Also $H_2(N; \mathbf{Z})$ is free on n generators, and a free basis for $H_2(N; \mathbf{Z})$ can be constructed by orienting each D_i and capping it off with a relative 2-cycle in $M(h_3') \times [0, 1]$. By choosing the orientations properly we can identify the intersection numbers of pairs of these 2-cycles with the values of L on the corresponding pairs of elements in $\{\beta_1, \ldots, \beta_n\}$. Thus L is equivalent to $\varphi_{N'}$.

To complete the verification of (ii) we show that the matrix $P_2^t R_2$ represents the linking form L. Now $L(h_2^{-1})$ is given by

$$\begin{Vmatrix} R_2 & S_2 \\ P_2 & Q_2 \end{Vmatrix}^{-1} = \begin{Vmatrix} Q_2^t & -S_2^t \\ -P_2^t & R_2^t \end{Vmatrix}$$

so B is freely generated by the elements,

$$\beta_j = h_{2*}^{-1}(\omega_{n+j}) = \sum_{i=1}^{n} -p_{ij}\omega_i + \sum_{i=1}^{n} r_{ik}\omega_{i+n}.$$

The matrix $\|L(\beta_j, \beta_k)\|$ represents L, and the (j, k)th entry of this matrix is given by,

$$\begin{aligned} \mathrm{lk}\big(\theta_*(\beta_j), \beta_k\big) &= \mathrm{lk}\Big(\theta_*\Big(\sum -p_{ij}\omega_i + \sum r_{ij}\omega_{i+n}\Big), \sum -p_{ik}\omega_i + \sum r_{ik}\omega_{i+n}\Big) \\ &= -\mathrm{lk}\Big(\theta_*\Big(\sum p_{ij}w_i\Big), \sum r_{ik}\omega_{i+n}\Big) = -\sum p_{ij}\mathrm{lk}\Big(\theta_*(\omega_i), \sum r_{ik}\omega_{i+n}\Big) \\ &= -\sum p_{ij}r_{ik}\mathrm{lk}\big(\theta_*(\omega_i), \omega_{i+n}\big) = -\sum p_{ij}r_{ik}(-1) = \sum_{i=1}^{n} p_{ij}r_{ik}. \end{aligned}$$

But $\sum_{i=1}^{n} p_{ij}r_{ik}$ is the (j, k)th entry in the matrix $P_2^t R_2$, so $P_2^t R_2$ represents L, and hence φ_N, and the verification of (ii) is now complete.

Consider (iii). Suppose that $H_1(M(h_2); \mathbf{Z}) = 0$. Then by Lemma 2, the matrix P_2 in $L(h_2')$ is unimodular. We may then find unimodular matrices V_1, V_2 such that $V_1 P_2 V_2 = -I$. From equation (1) it follows that $V_1 Q_2 (V_2^t)^{-1}$ is symmetric. We may thus define the abelian map pair equivalence:

$$D(V_1)L(h_2)D\big((V_2^t)^{-1}\big)F\big(V_1 Q_2 (V_2^t)^{-1}\big) = \begin{Vmatrix} W & S_2 \\ -I & 0 \end{Vmatrix}.$$

$$D(V_1)L(h_2)D(V_1^t) = \begin{Vmatrix} 0 & I \\ -I & 0 \end{Vmatrix}.$$

The symplectic condition (1) now implies that W is symmetric and that $S_2 = I$, and we have the desired normal form at the abelian level. Since η: $\mathcal{F} \to \tilde{\mathcal{F}}$ is surjective, we can lift the matrices $D(-)$ and $F(-)$ to elements of $\mathcal{F}$ and so convert the equivalence to an equivalence $(h_2, h_3) \approx (h_2, h_3)$ so that $(L(h_2'), L(h_3'))$ has the normal form just described.

By (7), $S(h_3 h_2^{-1}) \approx S(h_3'(h_2')^{-1})$. Moreover

$$L\left(h_3'(h_2')^{-1}\right) = L(h_2')^{-1}L(h_3') = \begin{Vmatrix} 0 & -I \\ I & 0 \end{Vmatrix} \begin{Vmatrix} W & I \\ -I & 0 \end{Vmatrix} = \begin{Vmatrix} I & 0 \\ W & I \end{Vmatrix}.$$

By part (ii) of this lemma, $-W = (-I)^t W$ represents the form $\varphi_{N(h_2^*, h_3^*)} \approx \varphi_N$. It follows from Lemma 2 that

$$H_1\left(M\left(h_3 h_2^{-1}\right); \mathbf{Z}\right) = H_1\left(M\left(h_3^*(h_2^*)^{-1}\right); \mathbf{Z}\right) = 0$$

if and only if $\det W = \pm 1$. This completes the proof of the lemma. □

Lemma 5 tells us how to obtain information about φ_N from the abelianization of a map pair. Our next lemma completes the picture by relating equivalence of abelianized map pairs to equivalence of the associated bilinear forms.

LEMMA 6. *Let (h_2, h_3) and (h_2', h_3') be map pairs with associated 4-manifolds N and N'. Suppose that $M(h_2)$ and $M(h_3')$ are* **Z**-*homology spheres.*

(i) *If the abelianized map pairs $(\eta(h_2), \eta(h_3))$ and $(\eta(h_2'), \eta(h_3'))$ are equivalent, then the forms φ_N and φ_N' are equivalent. Moreover, if $M(h_2)$, $M(h_3)$, $M(h_2')$, $M(h_3')$ are all* **Z**-*homology spheres then the converse holds: The equivalence of φ_N and $\varphi_{N'}$ implies the equivalence of $(\eta(h_2), \eta(h_3))$ and $(\eta(h_2'), \eta(h_3'))$.*

(ii) *If the* mod 2 *reductions $(\varepsilon\eta(h_2), \varepsilon\eta(h_3))$ and $(\varepsilon\eta(h_2'), \varepsilon\eta(h_3'))$ are equivalent, then the* mod 2 *reductions of φ_N and $\varphi_{N'}$ are equivalent and thus φ_N and $\varphi_{N'}$ are either both of even type or both of odd type.* □

PROOF. Suppose first that $(\eta(h_2), \eta(h_3))$ and $(\eta(h_2'), \eta(h_3'))$ are equivalent. By Lemma 5 we may suppose that $(L(h_2'), L(h_3'))$ has the normal form (8), so that $\varphi_{N'}$ is represented by $P_2^t R_2$. The equivalence of $(\eta(h_2), \eta(h_3))$ and $(\eta(h_2'), \eta(h_3'))$ implies the existence of elements $f_j, j = 1, 2, 3$ in $\mathcal{F}$ such that

$$L(h_j') = L(f_1)L(h_j)L(f_j), \quad j = 2, 3.$$

But then $(f_2 h_2 f_1, f_3 h_3 f_1)$ is equivalent to (h_2, h_3), so by part (ii) of Lemma 5, $P_2^t R_2$ also represents φ_N and we conclude that φ_N and $\varphi_{N'}$ are equivalent.

Suppose that $M(h_3)$ and $M(h_3')$ are **Z**-homology spheres. Then by part (ii) of Lemma 5 we may suppose that $(L(h_2), L(h_3))$ has the normal form (9), and $(L(h_2'), L(h_3'))$ has this same normal form with W' in place of W. Then $-W$ and $-W'$ represent the bilinear forms φ_N and $\varphi_{N'}$. If φ_N and $\varphi_{N'}$ are equivalent, then for some unimodular matrix U we have $W' = UWU^t$. But then

$$\left(L(h_2'), L(h_3')\right) = \left(D(U^t)L(h_2)D(U), D(U^t)L(h_3)D(U)\right)$$

so $(L(h_2), L(h_3))$ and $(L(h_2'), L(h_3'))$ are equivalent as desired.

The proof of (iii) is similar and is obtained by reducing mod 2 in Lemma 5 and the first two parts of the lemma. The final assertion in (iii) follows from the fact that when φ_N and $\varphi_{N'}$ are equivalent, φ_N has an odd quadratic value

$\varphi_N(\beta, \beta)$ for some β if and only if $\varphi_{N'}$ has an odd quadratic value $\varphi_N(\beta', \beta')$ for some β'. $\square$

2.5 A TECHNICAL LEMMA ABOUT MAP PAIRS. The lemma proved below translates into map pair language a weak version of the fact that 3-manifolds bound parallelizable 4-manifolds.

LEMMA 7. *Let $S(h)$ be a Heegaard splitting of even genus for a $\mathbf{Z}/2\mathbf{Z}$-homology sphere $M(h)$.*

Then there exists a map pair (h, h') with the following properties:

(i) *The bilinear form φ_N for the 4-manifold $N = N(h, h')$ has even type.*

(ii) *In the fundamental triple $(M(h'), M(h), M(h'h^{-1}))$ for (h, h') the manifolds are all three $\mathbf{Z}/2\mathbf{Z}$-homology spheres and in addition $M(h'h^{-1})$ is the 3-sphere.*

PROOF. We will prove the lemma by first constructing a map pair (h_0, h_1) with a nice fundamental triple so that $\varphi_{N(h_0,h_1)}$ has even type and so that $L(h_0) \equiv L(h) \pmod 2$. We will then make some substitutions to obtain the desired map pair (h, h').

Let W be the $n \times n$ matrix which is the direct sum of $n/2$ copies of the 2×2 matrix $\left\|\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right\|$. Consider the symplectic matrices

$$(L_3, L_4) = \left(\left\| \begin{matrix} W & I \\ -I & 0 \end{matrix} \right\|, \left\| \begin{matrix} 0 & I \\ -I & 0 \end{matrix} \right\| \right), \qquad L_5 = L_3^{-1}L_4 = \left\| \begin{matrix} I & 0 \\ -W & I \end{matrix} \right\|.$$

Choose maps h_3, h_5 such that $L_3 = L(h_3)$ and $L_5 = L(h_5)$. Since W is unimodular, Lemma 2 shows that $M(h_3)$ and $M(h_5)$ are $\mathbf{Z}$-homology spheres. From Lemma 3 we may further assume that the lift h_5 has been chosen so that $M(h_5)$ is the 3-sphere. Next, define a lift h_4 of L_4 by $h_4 = h_5h_3$. Then $L(h_4) = L_3L_5 = L_4$ so (by a second application of Lemma 2) the manifold $M(h_4)$ is also a $\mathbf{Z}$-homology sphere.

By Lemma 5, the bilinear form $\varphi_{N(h_3,h_4)}$ is represented by the matrix $-W$ and since W has even diagonal entries this form has even type.

By Lemma 3, there are elements f_1, f_2 in $\mathcal{F}(n)$ such that $L(f_2h_3f_1) \equiv L(h) \pmod 2$. Define $h_0 = f_2h_3f_1$ and $h_1 = h_4f_1$. Then (h_0, h_1) and (h_3, h_4) are equivalent map pairs so the manifolds in the fundamental triples are equivalent and the associated bilinear forms are equivalent (Lemma 6). Define $h' = h_1h_0^{-1}h$. Then, since $L(h_0) \equiv L(h) \pmod 2$, it follows that $L(h') \equiv L(h_1) \pmod 2$.

The mod 2 reductions of $(L(h), L(h'))$ and $(L(h_3), L(h_4))$ are now seen to be equivalent. Therefore, by Lemma 6, the bilinear form φ_N for $N = N(h, h')$ has even type. Now $L(h') \equiv L(h_1) \pmod 2$, so by Lemma 2 and our previous observations, $M(h')$ is a $\mathbf{Z}/2\mathbf{Z}$-homology sphere. Finally, $h'h^{-1} = h_1h_0^{-1}$, and since $S(h_1h_0^{-1}) \approx S(h_4h_3^{-1})$ we find that $M(h'h^{-1}) \cong M(h_4h_3^{-1}) \cong M(h_5)$.

But h_5 was chosen so that $M(h_5)$ is the 3-sphere; thus $M(h'h^{-1})$ is the 3-sphere. □

3. Representations of $\mathcal{H}$ onto $\mathbf{Z}/2\mathbf{Z}$. This section contains the main results of the paper. It studies how the μ-invariant for 3-manifolds implies group theoretic properties of the class of groups $\{\mathcal{H}(n), n = 0, 1, 2, \ldots\}$. In §4 the reverse problem will be studied: how topological invariants of 3-manifolds can be obtained from group theoretic properties of the class $\{\mathcal{H}(n)\}$.

Consider a Heegaard splitting $S(h_2h_1)$ of a $\mathbf{Z}/2\mathbf{Z}$-homology sphere $M(h_2h_1)$ where h_1 and h_2 are elements of $\mathcal{H} = \mathcal{H}(n)$. From Lemma 2 we know that if $S(h')$ is another Heegaard splitting of genus n such that $L(h') = L(h_2h_1)$, then $M(h')$ is also a $\mathbf{Z}/2\mathbf{Z}$-homology sphere. But $L(h') = L(h_2h_1)$ if and only if $h' = h_2kh_1$ for some $k \in \mathcal{H}(n)$. Thus, for $k \in \mathcal{H}(n)$, we can ask about the change in the μ-invariant, $\mu(M(h_2kh_1)) - \mu(M(h_2h_1))$. We will show in Theorem 8 that this change is quite regular and that it leads to representations of $\mathcal{H}(n)$ onto $\mathbf{Z}/2\mathbf{Z}$ for $n \geqslant 2$.

3.1 MAIN THEOREM. A finite sequence $\langle h_p, \ldots, h_1\rangle$ of maps in $\mathcal{H}(n)$ will be said to be *admissible* if $M(h_p \cdots h_1)$ is a $\mathbf{Z}/2\mathbf{Z}$-homology sphere. We will be interested chiefly in the case $p = 2$ and we have chosen the term "sequence" to diminish confusion with map pairs. Note that if $\langle h_2, h_1\rangle$ is an admissible sequence, then $\langle h_2, k, h_1\rangle$ is also an admissible sequence for each $k \in \mathcal{H}(n)$ (Lemma 2).

THEOREM 8. *For an admissible sequence $\langle h_2, h_1\rangle$ in $\mathcal{H}(n)$, $n \geqslant 2$, let ρ_{h_2,h_1} be the set function from $\mathcal{H}(n)$ to $\mathbf{Q}/\mathbf{Z}$ defined by the rule,*

$$\rho_{h_2,h_1}(k) \equiv \mu(M(h_2kh_1)) - \mu(M(h_2h_1)) \pmod 1, k \in \mathcal{H}(n). \tag{10}$$

These set functions have the following properties:

(i) *Each function ρ_{h_2,h_1} is a homomorphism from $\mathcal{H}(n)$ onto the additive group $\{0, \frac{1}{2}\}$ (mod 1) $\approx \mathbf{Z}/2\mathbf{Z}$.*

(ii) *If $\langle h_2, h_1\rangle$ and $\langle t_2, t_1\rangle$ are admissible sequences with $L(t_i) \equiv L(h_i)$ (mod 2), $i = 1, 2$, then $\rho_{h_2,h_1} = \rho_{t_2,t_1}$.*

(iii) *Let $\mathcal{H}_{h_2,h_1}$ denote kernel ρ_{h_2,h_1}. Let $\mathcal{L}(n)$ denote the collection of groups $\{\mathcal{H}_{h_2,h_1}\}$ as $\langle h_2, h_1\rangle$ ranges over all admissible sequences of length 2 in $\mathcal{H}(n)$. Then $\mathcal{L}(n)$ is a complete conjugacy class(²) of subgroups of $\mathcal{H}(n)$.*

PROOF. The proof of Theorem 8 is long and is divided into several parts. We will first consider the special case where $h_2 = \mathrm{id}$ and $t_2 = \mathrm{id}$. We will establish (i) and (ii), and instead of (iii) we will show that the groups $\mathcal{H}_{\mathrm{id},h}$ are conjugates of each other by elements of $\mathcal{F}(n)$. With these results we will then establish the theorem in the general case.

(²) Finite upper and lower bounds on the order of $\mathcal{L}(n)$ will be given in Theorem 9.

Special case ($h_2 = \mathrm{id}$, $t_2 = \mathrm{id}$). The key to dealing with this case is to first establish that (ii) holds for the special set functions. The trick used in the proof of Lemma 7 to turn one nice map pair into another one will be instrumental in establishing (ii). To simplify notation in this case we will replace h_1 and t_1 by h and t.

Consider then elements h and t in $\mathcal{H}(n)$ such that $L(h) \equiv L(t) \pmod 2$ and such that $M(h)$ is a $\mathbf{Z}/2\mathbf{Z}$-homology sphere. Let $k \in \mathcal{H}(n)$. If n is odd, we immediately replace, h, t, k, by $h \# s^{(1)}$, $t \# s^{(1)}$, and $k \# \mathrm{id}$ as indicated in §2.3. Because stabilization by $s^{(1)}$ corresponds to the formation of the connected sum of a manifold with Σ, we have $M(h \# s^{(1)}) \cong M(h)$ and $M((k \# \mathrm{id})(h \# s^{(1)})) \cong M((kh) \# s^{(1)}) \cong M(kh)$. Similar equivalences hold for t. Thus the values of $\rho_{\mathrm{id},h}(k)$ and $\rho_{\mathrm{id},t}(k)$ remain unchanged, and we may assume that n is even.

By Lemma 7 there is a map pair (h, h') such that $M(h')$ is a $\mathbf{Z}/2\mathbf{Z}$-homology 3-sphere, $M(h'h^{-1})$ is the 3-sphere, and the bilinear form $\varphi_{N(h,h')}$ has even type. At this point we apply the trick from Lemma 7 and define $t' = h'h^{-1}t$ to obtain another map pair (t, t'). Note that $t't^{-1} = h'h^{-1}$. Because $L(h) \equiv L(t) \pmod 2$, it follows that $L(h') \equiv L(t') \pmod 2$. Thus from Lemma 2, $M(t')$ is a $\mathbf{Z}/2\mathbf{Z}$-homology sphere. By definition $M(t't^{-1})$ is the 3-sphere, and by Lemma 6, the form $\varphi_{N(t,t')}$ has even type.

Consider the modified map pairs (kh, h') and (kt, t') obtained by replacing h and t by kh and kt. The abelianizations of the new map pairs are identical with the old ones so by another application of Lemma 6, we find that the bilinear forms $\varphi_{N(h,h')}$ and $\varphi_{N(kh,h')}$ are equivalent as are the forms $\varphi_{N(t,t')}$ and $\varphi_{N(kt,t')}$. In particular, then, the signatures must coincide for $\varphi_{N(h,h')}$ and $\varphi_{N(kh,h')}$ and similarly for $\varphi_{N(t,t')}$ and $\varphi_{N(kt,t')}$. The signature formula (6) of Lemma 4 then gives the congruences:

$$\begin{aligned} &-\mu(M(h)) + \mu(M(h')) + \mu(M(h'h^{-1})) \\ &\qquad \equiv -\mu(M(kh)) + \mu(M(h')) + \mu(M(h'h^{-1}k^{-1})) \pmod 1 \end{aligned} \tag{11}$$

and

$$\begin{aligned} &-\mu(M(t)) + \mu(M(t')) + \mu(M(t't^{-1})) \\ &\qquad \equiv -\mu(M(kt)) + \mu(M(t')) + \mu(M(t't^{-1}k^{-1})) \pmod 1. \end{aligned} \tag{12}$$

Because $M(h'h^{-1})$ is the 3-sphere, whose μ-invariant is 0, and because $t't^{-1}$ is defined to be equal to $h'h^{-1}$, (11) and (12) simplify to,

$$\mu(M(kh)) - \mu(M(h)) \equiv \mu(M(h'h^{-1}k^{-1})) \pmod 1 \tag{13}$$

and

$$\mu(M(kt)) - \mu(M(t)) \equiv \mu(M(h'h^{-1}k^{-1})) \pmod 1. \tag{14}$$

Comparison of the two congruences now reveals that $\rho_{\mathrm{id},h}(k) = \rho_{\mathrm{id},t}(k)$. Assertion (ii) has now been verified.

Because $M(h'h^{-1})$ is the 3-sphere, Lemma 2 shows that $M(h'h^{-1}k^{-1})$ is a $\mathbf{Z}$-homology sphere and thus $\mu(M(h'h^{-1}k^{-1}))$ is either 0 or $\frac{1}{2}$. The image of the set function $\rho_{\mathrm{id},h}$ is therefore contained in $\{0, \frac{1}{2}\}$ (mod 1), and the second part of assertion (i) has been verified.

To show that $\rho_{\mathrm{id},h}$ is a homomorphism we must show that $\rho_{\mathrm{id},h}$ converts products in $\mathcal{K}$ into sums in $\{0, \frac{1}{2}\}$ (mod 1). Let k_1 and k_2 be elements in $\mathcal{K}$. Then

$$\begin{aligned}\rho_{\mathrm{id},h}(k_2k_1) &\equiv \mu(M(k_2k_1h)) - \mu(M(k_1h)) \\ &\quad + \mu(M(k_1h)) - \mu(M(h)) \qquad (\mathrm{mod}\ 1) \\ &\equiv \rho_{\mathrm{id},k_1h}(k_2) + \rho_{\mathrm{id},h}(k_1).\end{aligned}$$

But $L(k_1h) = L(h)$ so by (ii) we have $\rho_{\mathrm{id},k_1h}(k_2) = \rho_{\mathrm{id},h}(k_2)$; thus $\rho_{\mathrm{id},h}(k_2k_1) = \rho_{\mathrm{id},h}(k_2) + \rho_{\mathrm{id},h}(k_1)$ as required.

To complete the special case, it remains to establish surjectivity of $\rho_{\mathrm{id},h}$ and to establish the conjugacy of the subgroups $\mathcal{K}_{\mathrm{id},h}$ in $\mathcal{K}$ by elements of $\mathcal{F}$. Given that $\langle \mathrm{id}, h\rangle$ is admissible, there are, by Lemma 3, elements f_1 and f_2 in $\mathcal{F}$ such that $L(h) \equiv L(f_2sf_1)$ (mod 2). Thus $\rho_{\mathrm{id},h} = \rho_{\mathrm{id},f_2sf_1}$. By making use of the equivalence condition (2) we can rewrite (10) as

$$\begin{aligned}\rho_{\mathrm{id},f_2sf_1}(k) &= \mu\big(M(f_2^{-1}kf_2sf_1)\big) - \mu(M(f_2sf_1)) \\ &= \mu\big(M(f_2^{-1}kf_2s)\big) - \mu(M(s)).\end{aligned} \tag{15}$$

Because $\mathcal{K}$ is a normal subgroup of $\mathcal{K}$, it follows that

$$\rho_{\mathrm{id},f_2sf_1}(k) = \rho_{\mathrm{id},s}(f_2^{-1}kf_2). \tag{16}$$

This shows that $\rho_{\mathrm{id},f_2sf_1}$ is the composition of $\rho_{\mathrm{id},s}$ with the restriction to $\mathcal{K}$ of the inner automorphism of $\mathcal{K}$ which sends each element x to $f_2^{-1}xf_2$. The relation between the subgroups $\mathcal{K}_{\mathrm{id},f_2sf_1}$ and $\mathcal{K}_{\mathrm{id},s}$ is now given by

$$\mathcal{K}_{\mathrm{id},f_2sf_1} = f_2\mathcal{K}_{\mathrm{id},s}f_2^{-1}. \tag{17}$$

The groups $\mathcal{K}_{\mathrm{id},h}$ are thus all conjugates of $\mathcal{K}_{\mathrm{id},s}$ by elements of $\mathcal{F}$. To establish that $\rho_{\mathrm{id},h}$ is in general surjective it is now sufficient to establish that $\rho_{\mathrm{id},s}$ is surjective.

We first show that for each value of $n \geqslant 2$, there is an element $t \in \mathcal{K}$ such that $M(t)$ is a $\mathbf{Z}$-homology sphere with $\mu(M(t)) = \frac{1}{2}$. We first establish this for $n = 2$. By **[Sf, TR]** the spherical dodecahedral space is the 2-fold covering over Σ branched over the torus knot of type (3, 5). By **[Pc]** this space is defined by a Heegaard splitting $S(t)$ of genus 2. By **[HNK]**, $\mu(M(t)) = \frac{1}{2}$. Since $M(t)$ is a $\mathbf{Z}$-homology sphere, this takes care of the case $n = 2$. For $n > 2$, stabilize $S(t)$ to $S(t \# s^{(1)} \# \cdots \# s^{(1)})$ to obtain the desired t.

Continuing the proof of surjectivity, we observe next that by Lemma 3, there is an element $t' \in \mathcal{H}(n)$ with $S(t') \approx S(t)$ such that $L(t') = L(s)$. Thus there is some $k \in \mathcal{K}$ such that $t' = ks$. By (2), we have $M(t') \cong M(t)$ so $\rho_{\mathrm{id},s}(k) \equiv \mu(M(ks)) - \mu(M(s)) \equiv \mu(M(t)) - 0 \equiv \frac{1}{2}$ (mod 1), and we conclude that $\rho_{\mathrm{id},s}$ is surjective.

This completes the special case of the theorem.

General case. (i) Consider the general set function ρ_{h_2,h_1}. Rewriting (10) we obtain

$$\begin{aligned} \rho_{h_2,h_1}(k) &\equiv \mu\big(M(h_2kh_2^{-1})h_2h_1\big) - \mu(M(h_2h_1)) \qquad (\mathrm{mod}\ 1) \\ &\equiv \rho_{\mathrm{id},h_2h_1}\big(h_2kh_2^{-1}\big). \end{aligned} \tag{18}$$

Thus ρ_{h_2,h_1} is the composition of $\rho_{\mathrm{id},h_2h_1}$ with the restriction of the inner automorphism: $x \to h_2xh_2^{-1}$. The normality of $\mathcal{K}$ in $\mathcal{H}$ implies as in the special case that ρ_{h_2,h_1} is a homomorphism and that, since $\rho_{\mathrm{id},h_2h_1}$ is surjective, ρ_{h_2,h_1} is surjective.

(ii) The verification of (ii) will be based on several inversion formulas which we now give. First, for any $h \in \mathcal{H}(n)$, the manifold $M(h^{-1})$ is equivalent to $-M(h)$; that is, the obvious homeomorphism from $M(h^{-1})$ to $M(h)$ is orientation reversing. By (5) we have,

$$\mu\big(M(h^{-1})\big) \equiv \mu(-M(h)) \equiv -\mu(M(h)) \qquad (\mathrm{mod}\ 1) \tag{19}$$

and

$$\begin{aligned} \rho_{h,\mathrm{id}}(k) &\equiv \mu(M(hk)) - \mu(M(h)) \\ &\equiv -\mu\big(M(k^{-1}h^{-1})\big) + \mu\big(M(h^{-1})\big) \\ &\equiv -\rho_{\mathrm{id},h^{-1}}(k) \qquad (\mathrm{mod}\ 1). \end{aligned} \tag{20}$$

The last equality follows from the fact that squares map to 0 under ρ. Combining (18) and (20), we obtain the more general inversion formula,

$$\begin{aligned} \rho_{h_2,h_1}(k) &\equiv \rho_{\mathrm{id},h_2h_1}\big(h_2kh_2^{-1}\big) \\ &\equiv -\rho_{h_1^{-1}h_2^{-1},\mathrm{id}}\big(h_2kh_2^{-1}\big) \\ &\equiv -\rho_{h_1^{-1},h_2^{-1}}(k) \qquad (\mathrm{mod}\ 1). \end{aligned} \tag{21}$$

Let $\langle h_2, h_1\rangle$ and $\langle t_2, t_1\rangle$ be admissible sequences with $L(h_i) \equiv L(t_i)$ (mod 2), $i = 1, 2$. Note first that $L(h_2h_1) \equiv L(h_2t_1)$ (mod 2) so by (18) and the special case, we find that $\rho_{h_2,h_1} = \rho_{h_2,t_1}$. By (21) we have

$$\rho_{h_2,t_1} = -\rho_{t_1^{-1},h_2^{-1}} \quad \text{and} \quad \rho_{t_2,t_1} = -\rho_{t_1^{-1},t_2^{-1}}.$$

The same application of the special case as the one we just made now shows that $\rho_{t_1^{-1},h_2^{-1}} = \rho_{t_1^{-1},t_2^{-1}}$ and by transitivity we have that $\rho_{h_2,h_1} = \rho_{t_2,t_1}$.

(iii) Observe that equation (18) implies that

$$\mathcal{K}_{h_2,h_1} = h_2^{-1}(\mathcal{K}_{\mathrm{id},h})h_2, \qquad h = h_2h_1. \tag{22}$$

Note that in (22) the map h_2 can be an arbitrary element of $\mathcal{H}$; however $h = h_2h_1$ is subject to the restriction that $M(h)$ be a $\mathbf{Z}/2\mathbf{Z}$-homology sphere. From (17) and (22) it now follows that the collection of groups $\mathfrak{L}(n)$ is given by,

$$\mathfrak{L}(n) = \left\{(h_2^{-1}f_2)(\mathcal{K}_{\mathrm{id},s})(f_2^{-1}h_2)/h_2 \in \mathcal{H}, f_2 \in \mathcal{F}\right\} \tag{23}$$

and this is clearly a complete conjugacy class of subgroups of $\mathcal{H}$.

This completes the proof of Theorem 8. □

3.2 *Bounds on the order of* $\mathfrak{L}(n)$. Theorem 8 will allow us to compute, in Theorem 9, upper and lower bounds on the order of $\mathfrak{L}(n)$. Before giving the proof of Theorem 9 we review in the next three paragraphs some material on twist maps, knot surgeries, and the Arf invariant of a knot. This is for the purpose of obtaining a lower bound for $|\mathfrak{L}(n)|$.

Let R be a simple closed curve in Bd U that separates Bd U, and let t be a twist map of Bd U about R. For some annulus $A \subseteq$ Bd U and $R \subset$ Bd A the map severs Bd U along R, then twists A holding (Bd A) $\setminus R$ fixed so that R is rotated a full revolution, and finally reattaches the two sides of R in Bd U by the identity map. Note that since R separates Bd U we have $t \in \mathcal{K}(n)$. Let h be an arbitrary element in $\mathcal{F}(n)$. Then t transforms $M(h)$ to a new 3-manifold $M(th)$. Lickorish observed **[LK]** that up to an orientation-preserving homeomorphism, $M(th)$ results from $M(h)$ by a knot surgery. If $W(R)$ is a tubular neighborhood of R with $W(R) \cap$ Bd U an annulus, then this surgery has the following description: Let R' result from one of the two curves in $W(R) \cap$ Bd U by twisting once about a meridian in Bd $W(R)$. (The direction of the twist is not important to us here.) Remove $W(R)$ from $M(h)$ and reattach Bd $W(R)$ to Bd $(M(h) \setminus \mathrm{Int}\, W(R))$ so that a meridian in Bd $W(R)$ is taken to the curve R'.

Results due to Gonzalez-Acuna **[GA**, Theorem 4] and Gordon **[Gd**, Theorem 2] show that if $M(h)$ is a $\mathbf{Z}$-homology sphere, then the manifold $M(th)$, obtained by the knot surgery described above, is a $\mathbf{Z}/2\mathbf{Z}$-homology sphere and its μ-invariant is given by

$$\mu(M(th)) \equiv \mu(M(h)) + a_{M(n)}(R) \qquad (\mathrm{mod}\ 1) \tag{24}$$

where $a_{M(n)}(R)$ denotes the Arf invariant of the knot R (to be described below).

If R is any knot in a $\mathbf{Z}$-homology sphere $M(h)$, then the *Arf invariant* of R in $M(h)$ is defined as follows (see **[Sf, TR]**): The curve R bounds an orientable surface Q in $M(h)$. Let Q^+ and Q^- be disjoint, parallel copies of Q slightly to either side of Q. There is a quadratic linking form χ_Q: $H_1(Q\colon \mathbf{Z}/2\mathbf{Z}) \to \mathbf{Z}/2\mathbf{Z}$ given by $\chi_Q(\beta) = \mathrm{lk}(\beta^+, \beta^-)$ (mod 2) where lk

denotes linking number and β^+ and β^- are the parallel copies of β in Q^+ and Q^-. Let $\gamma_1, \ldots, \gamma_p, \alpha_1, \ldots, \alpha_p$ be a symplectic basis for $H_1(Q; \mathbf{Z}/2\mathbf{Z})$; i.e. $I(\gamma_i, \gamma_j) = I(\alpha_i, \alpha_j) = 0 \pmod 2$ and $I(\gamma_i, \alpha_j) = \delta_{ij} \pmod 2$ where I denotes intersection number. Then the Arf invariant $a_{M(n)}(R)$ of R is given by

$$a_{M(h)}(R) = \sum_1^p \frac{\chi_Q(\gamma_i)\chi_Q(\alpha_i)}{2} \pmod 1. \tag{25}$$

THEOREM 9. *Upper and lower bounds on* $|\mathfrak{L}(n)|$ *are given by,*

$$2^n \leqslant |\mathfrak{L}(n)| = \frac{1}{p} 2^{(n)(n+1)/2} \prod_{i=1}^n (2^i + 1),$$

where p is an integer, $p \geqslant 1$.

PROOF. According to equation (23), the group $\mathcal{H}(n)$ acts on $\mathfrak{L}(n)$ by conjugation. Suppose that $g \in \operatorname{kernel}(\varepsilon\eta)$. By equation (22) we have

$$g^{-1}\mathcal{K}_{h_2,h_1} g = g^{-1}h_2^{-1}\mathcal{K}_{\mathrm{id},\, h_2h_1} h_2 g = \mathcal{K}_{h_2 g, g^{-1}h_1}. \tag{26}$$

By Theorem 8, part (iii), the groups $\mathcal{K}_{h_2 g, g^{-1}h_1}$ and $\mathcal{K}_{h_2,h_1}$ coincide; hence the action of g is trivial. Thus $\hat{\mathcal{H}}(n)$ acts on $\mathfrak{L}(n)$, so that $\hat{\mathcal{H}}(n)$ has a representation as a group of permutations of the set $\mathfrak{L}(n)$. By Theorem 8, part (iii), this action is transitive; hence $|\mathfrak{L}(n)|$ divides $|\hat{\mathcal{H}}(n)| = 2^{n^2}\prod_{i=1}^n(2^i + 1)(2^i - 1)$ (see [Nw, p. 125]).

Let $\hat{\mathcal{H}}_0 \subset \hat{\mathcal{H}}$ be the stabilizer of $\mathcal{K}_{1,s} \in \mathfrak{L}(n)$. Then $|\mathfrak{L}(n)| = |\hat{\mathcal{H}}|/|\hat{\mathcal{H}}_0|$. Let $\mathfrak{F}_0 = \mathfrak{F} \cap s\mathfrak{F}s^{-1}$, and let $\hat{\mathfrak{F}}_0$ be the image of $\mathfrak{F}_0$ under $\varepsilon\eta$. If $f \in \mathfrak{F}_0$, then we may find $f' \in \mathfrak{F}_0$ such that $f = sf'^{-1}s^{-1}$, i.e. $fsf' = s$. Since $\mathcal{K}_{\mathrm{id},g} = \mathcal{K}_{\mathrm{id},gf}$ for each $f \in \mathfrak{F}$, equation (17) now implies

$$f\mathcal{K}_{\mathrm{id},\, s} f^{-1} = \mathcal{K}_{\mathrm{id}, fsf^{-1}} = \mathcal{K}_{\mathrm{id}, fsf'} = \mathcal{K}_{\mathrm{id},\, s}. \tag{27}$$

Thus $\hat{\mathfrak{F}}_0 \subset \hat{\mathcal{H}}_0$, and $|\mathfrak{L}(n)|$ divides $|\hat{\mathcal{H}}|/|\hat{\mathfrak{F}}_0|$. Since

$$|\hat{\mathfrak{F}}_0| = |\mathrm{GL}(n, \mathbf{Z}/2\mathbf{Z})| = 2^{(n)(n-1)/2} \prod_{i=1}^n (2^i - 1),$$

it follows that $|\mathfrak{L}(n)|$ divides $2^{(n)(n+1)/2}\prod_{i=1}^n(2^i + 1)$. This establishes the upper bound in Theorem 9.

We will establish the lower bound by showing that the number of distinct groups of the form $\mathcal{K}_{\mathrm{id},h}$ is at least 2^n. In the verification of (iii) in Theorem 8 we noted that each group $\mathcal{K}_{\mathrm{id},h}$ has the form $\mathcal{K}_{\mathrm{id},fs}$ for some $f \in \mathfrak{F}(n)$. Consider the collection of matrices $\{W_j = \operatorname{diag}(\varepsilon_1(j), \ldots, \varepsilon_n(j)) / \varepsilon_i(j) = 0 \text{ or } 1\}$. There are 2^n matrices in this collection. Let these be indexed so that for $j \leqslant n$ we have $\varepsilon_i(j) = \delta_{ij}$. Thus $W_0 = 0$. Set $f_0 = \mathrm{id}$. For each $1 \leqslant j \leqslant n$, the symplectic matrix $F(W_j)$ is equal to $L(f_j)$ for a twist map f_j that performs a twist about a meridian corresponding to ω_{n+j}. For $j > n$, we find that W_j is a sum of the elementary matrices W_i, $i \leqslant n$. Thus we may lift $F(W_j)$ to $\mathfrak{F}(n)$ so

that $F(W_j) = L(f_j)$ where f_j is the composition of the elementary twists f_i, $i \leqslant n$, corresponding to the decomposition of W_j.

Notice that $F(W_i)F(-W_j) \equiv F(W_p)$ (mod 2) for some p such that $W_i - W_j \equiv W_p$ (mod 2). The equality

$$\rho_{\mathrm{id},f_p s} = \rho_{\mathrm{id},f_j^{-1}f_i s}$$

follows now from (ii) in Theorem 8. We claim that $\rho_{\mathrm{id},f_i s} = \rho_{\mathrm{id},f_j s}$ if and only if $\rho_{\mathrm{id},f_p s} = \rho_{\mathrm{id},s}$. Suppose that $\rho_{\mathrm{id},f_j s} \neq \rho_{\mathrm{id},f_i s}$. Then there is an element $k \in \mathcal{K}(n)$ such that,

$$\mu\big(M(kf_j s)\big) - \mu\big(M(f_j s)\big) \not\equiv \mu\big(M(kf_i s)\big) - \mu\big(M(f_i s)\big) \qquad (\mathrm{mod}\ 1). \tag{28}$$

Replacing $M(h)$ by $M(fh)$ for $f \in \mathcal{F}(n)$ does not change the μ-invariant since by (2) the splittings $S(h)$ and $S(fh)$ are equivalent. Because of this (28) implies,

$$\begin{aligned} &\mu\Big(M\big((f_j^{-1}kf_j)f_j^{-1}f_i s\big)\Big) - \mu\Big(M\big(f_j^{-1}f_i s\big)\Big) \\ &\qquad \not\equiv \mu\Big(M\big((f_j^{-1}kf_j)s\big)\Big) - \mu(M(s)) \qquad (\mathrm{mod}\ 1). \end{aligned} \tag{29}$$

Setting $k' = f_j^{-1}kf_j$, we find that k' belongs to one of the subgroups $\mathcal{K}_{\mathrm{id},f_j^{-1}f_i s}$, $\mathcal{K}_{\mathrm{id},s}$ but not the other. The entire argument may now be turned around to show the reverse implication.

From the preceding paragraph we see that to show that the 2^n subgroups $\mathcal{K}_{f_j s}$ are distinct, it is sufficient to show that $\mathcal{K}_{\mathrm{id},f_j s} \neq \mathcal{K}_{\mathrm{id},s}$, $j \neq 0$. To show this we will make use of the Arf invariant in the way previously described. We will describe a simple closed curve R_j in Bd $U(n)$ that separates Bd $U(n)$ so that $a_{M(s)}(R_j) = 0$ but $a_{M(f_j s)}(R_j) = \frac{1}{2}$. If k_j is an associated twist about R_j, then it will follows from (24) that $\rho_{\mathrm{id},s}(k_j) = 0$ but $\rho_{\mathrm{id},f_j s}(k_j) = \frac{1}{2}$ showing that the two subgroups above are not equal.

Consider $W_j = \mathrm{diag}(\varepsilon_1(j), \ldots, \varepsilon_n(j))$, $j \geqslant 1$. By renumbering the basic elements we may assume that $\varepsilon_1(j) = 1$. We will define R_j to be the boundary of a surface Q_j in Bd $U(n)$. Rather than specifying Q_j directly, we will first specify a symplectic basis $\{\gamma, \alpha\}$ for the first homology of Q_j and then use this basis to recover the surface. We consider two cases:

($\varepsilon_2(j) = 0$). In this case set $\gamma = \omega_2 + \omega_{n+2}$, and $\alpha = \omega_1 + \omega_{n+2}$. The linking form χ for Bd $U(n)$ can be determined visually for γ and α. We have, $\chi(\gamma) = 1$ and $\chi(\alpha) = 0$ in $M(s)$, $\chi(\gamma) = 1$ and $\chi(\alpha) = 1$ in $M(f_j s)$. Let R_{cj} and R_{aj} be a pair of simple closed curves in Bd $U(n)$ which intersect geometrically in a single piercing point and which are geometric representatives for the classes γ and α. Let Q_j be a regular neighborhood of $R_{cj} \cup R_{aj}$ in Bd $U(n)$, and take R_j to be Bd Q_j. Formula (25) now shows that $a_{M(s)}(R_j) = 0$ and $a_{M(f_j s)}(R_j) = \frac{1}{2}$.

($\varepsilon_2(j) = 1$). This case proceeds exactly as before except that here γ is defined to be ω_2 rather than $\omega_2 + \omega_{n+2}$. □

REMARK. We were able to prove a little bit more; however we were unable to determine the precise order of $\mathfrak{L}(n)$ for arbitrary n. We showed that the lower bound is strict. Also, we showed that the orbit of $\mathcal{K}_{1,s}$ under $\hat{\mathfrak{F}}$ contains precisely 2^n elements. This followed from a somewhat lengthy argument, too complicated to include in this manuscript. As a consequence, it follows that $|\mathfrak{L}(n)|$ divides $2^n \prod_{i=1}^{n}(2^i + 1)$.

3.3 *Some further remarks.* The results in Theorem 8 suggest a possible means to find homotopy 3-spheres with μ-invariant $\frac{1}{2}$. The method might be described as fixing the surgery while changing the manifold that it takes place in. Begin with a fixed element $k \in \mathcal{K}(n)$ such that $\rho_{\mathrm{id},s}(k) = \frac{1}{2}$. Then look for a sequence of maps $h_2, \ldots, h_1, \ldots$ in $\mathcal{K}(n)$ such that the sum of the ranks of $\pi_1(M(h_i s))$ and $\pi_1(M(h_i ks))$ is a decreasing function of i. If such a sequence could be found then eventually both $M(h_i s)$ and $M(h_i ks)$ would be homotopy 3-spheres. By (ii) in Theorem 8 each $\rho_{h_i,s}(k) = \frac{1}{2}$ so one of the two eventual homotopy spheres $M(h_i s)$ or $M(h_i ks)$ would have μ-invariant $\frac{1}{2}$.

4. Normal subgroups of $\mathcal{K}(n)$ and topological invariants of 3-manifolds. The properties exhibited by our family of groups $\mathcal{K}_{h_2,h_1}$ have an interesting relationship to a more general phenomenon which will be described in this section.

We would like to place an equivalence relation on maps in $\mathcal{K}(n)$ such that $h \sim t$ if and only if $M(h) \cong M(t)$ (recall $\cong$ means oriented equivalence). This may be accomplished by translating the classical Reidemeister-Singer theorem (see **[Rd, ZT]**, **[Sg]**, also **[Cr, NP]**) into an algebraic statement about the groups $\mathcal{K}(n)$.

LEMMA 10. *Let h and t be elements of $\mathcal{K}(n)$. Then the manifolds $M(h)$ and $M(t)$ are equivalent if and only if there exists an integer $p \geqslant 0$ and elements f_1 and f_2 in $\mathcal{K}(n+p)$ such that if $h' = b \# ps^{(1)} \in \mathcal{K}(n+p)$ and $t' = t \# ps^{(1)} \in \mathcal{K}(n+p)$, then $t' = f_2 h' f_1$.*

PROOF. Replacing h and t by h' and t' is the algebraic analogue of the stabilization process used by Reidemeister and Singer (see §2.3). The condition $t' = f_2 h' f_1$ asserts that $S(h')$ and $S(t')$ are equivalent Heegaard splittings (see §2.3). □

For $h \in \mathcal{K}(n)$, we will refer to the collection of maps

$$\{(f_2)(h \# ps^{(1)})(f_1)/p = 0, 1, 2, \ldots, f_1, f_2 \in \mathfrak{F}(n+p)\}$$

as the *stable double coset* (mod $\mathfrak{F}$) defined by h. If $h \in \mathcal{K}(n)$ and $t \in \mathcal{K}(m)$, we define h to be *equivalent* to t, written $h \sim t$, if the stable double cosets of h and t intersect nontrivially. This places an equivalence relation on elements in

the collection of groups $\{\mathcal{H}(n)/n = 0, 1, 2, \dots\}$ such that if $[h]$ denotes the equivalence class of h, then $M(h) \cong M(t)$ if and only if $[h] = [t]$.

Recall that in §2.3 we described a procedure for "extending" a map $b \in \mathcal{H}(n)$ to a (nonunique) map $b' = b \# \mathrm{id} \in \mathcal{H}(n+1)$. Suppose now, that for each $n = 0, 1, 2, \dots$, we are given a subgroup $\mathfrak{B}(n)$ of $\mathcal{H}(n)$. The class of groups $\{\mathfrak{B}(n)/n = 0, 1, 2, \dots\}$ will be said to have the *nested extension property* if for any $b \in \mathfrak{B}(n)$, and any extension $b = b \# \mathrm{id}$ to any element of $\mathcal{H}(n+1)$, we have $b' \in \mathfrak{B}(n+1)$. Examples of classes of subgroups with this property are the collections $\{\mathcal{F}(n)\}$ and $\{\mathcal{H}(n)\}$ defined in §2.1. Another example is the collection of subgroups of the groups $\mathcal{H}(n)$ generated by all twists about separating curves.

Let $\{\mathfrak{B}(n)/n = 0,1, 2, \dots\}$ be any class of groups which has the nested extension property and for which $\mathfrak{B}(n) \lhd \mathcal{H}(n)$ for each n. Consider the quotient groups $\mathcal{H}(n)/\mathfrak{B}(n)$ and the natural homomorphism

$$\varphi_n\colon \mathcal{H}(n) \to \mathcal{H}(n)/\mathfrak{B}(n).$$

The stable double coset of an element $h \in \mathcal{H}(n)$ is mapped into a well-defined stable double coset in $\mathcal{H}(n)/\mathfrak{B}(n)$ because the nested extension property insures that if $b \in \ker \varphi_n$, then any extension $b' = b \# \mathrm{id}$ of b is in $\ker \varphi_{n+1}$. It therefore makes sense to speak of the equivalence relation $\sim \varphi$ which is induced on elements of $\mathcal{H}(n)/\mathfrak{B}(n)$ by the relation $\sim$ in $\mathcal{H}(n)$, $n = 0, 1, 2, \dots$, and of equivalence classes $[h]_\varphi$ in $\mathcal{H}(n)/\mathfrak{B}(n)$ under the relation $\sim \varphi$. Moreover, it is immediate that invariants of a class $[h]_\varphi$ are topological invariants of $M(h)$.

An example, is perhaps, in order. Consider the collection of groups $\{\mathcal{H}(n)/n = 0, 1, 2, \dots\}$. Then $\tilde{\mathcal{H}}(n) = \mathcal{H}(n)/\mathcal{K}(n)$, and corresponding to φ_n is the natural map

$$\eta\colon \mathcal{H}(n) \to \tilde{\mathcal{H}}(n)$$

in the notation of §2.1. Let $h \in \mathcal{H}(n)$, and suppose that $\eta(h)$ is represented by the symplectic matrix

$$L(h) = \begin{Vmatrix} R & S \\ P & Q \end{Vmatrix}.$$

One may show without difficulty that the elementary invariants of the submatrix P are invariants of $[h]_\eta$; these are of course topological invariants of $M(h)$ because P is a relation matrix for $H_1(M(h); \mathbf{Z})$. In the case when $H_1(M(h); \mathbf{Z})$ is torsion-free, the equivalence class of h can be seen to be completely determined by the elementary invariants (the proof is a little complicated); however if torsion is present, more subtle invariants may be found for $[h]_\eta$ (see **[Rd, HI]**, **[Sf, VI]**, and **[Br, EH, §4]**).

We now study the analogous situation as it arises in connection with our groups $\mathcal{K}_{h_2,h_1}$.

Let $\mathcal{C}(n) = \bigcap \mathcal{K}_{h_2,h_1}$ where the intersection is taken over the finite set $\mathcal{L}(n)$ defined in §3, or equivalently the intersection is taken over all admissible sequences $\langle h_2, h_1\rangle$. Since $\mathcal{L}(n)$ is a complete conjugacy class of subgroups of $\mathcal{K}(n)$, the group $\mathcal{C}(n)$ is normal in $\mathcal{K}(n)$, also the collection $\{\mathcal{C}(n)/n = 0, 1, 2, \dots\}$ satisfies the nested extension property. Thus we have natural homomorphisms

$$\psi_n\colon \mathcal{K}(n) \to \mathcal{K}(n)/\mathcal{C}(n), \quad n = 0, 1, 2, \dots,$$

and any invariants of $[h]_\psi$ are topological invariants of $M(h)$. We remark that the homomorphism η factors through ψ_n (because $\mathcal{C}(n) \subset \mathcal{K}(n)$); hence invariants of $[h]_\psi$ include all invariants of $[h]_\eta$. Even more, we have the following theorem:

THEOREM 11. *Let $h \in \mathcal{K}(n)$ and $t \in \mathcal{K}(m)$. Suppose that $M(h)$ and $M(t)$ are $\mathbf{Z}/2\mathbf{Z}$-homology spheres and that $[h]_\psi = [t]_\psi$.*

Then $\mu(M(h)) = \mu(M(t))$.

PROOF. Since $[h]_\psi = [t]_\psi$, we may find integers $p, q \geqslant 0$ with $n + p = n + q$ and maps f_1 and f_2 in $\mathcal{F}(n+p)$ such that if $h' = h \# ps^{(1)}$ and $t' = t \# qs^{(1)}$, then

$$\psi_{n+p}(t') = \psi_{n+p}(f_2 h' f_1).$$

Thus $f_2 h' f_1 = kt'$ for some $k \in \ker \psi_{n+p} = \mathcal{C}(n+p)$. Since $k \in \mathcal{C}(n+p) = \bigcap \mathcal{K}_{h_2,h_1}(n+p)$ we have that $k \in \mathcal{K}_{\mathrm{id},t'}(n+p)$; hence $\mu(M(kt')) = \mu(M(t'))$. Also

$$\mu(M(kt')) = \mu(M(f_2 h' f_1)) = \mu(M(h')).$$

Therefore $\mu(M(t')) = \mu(M(h'))$. Since $M(h) \cong M(h')$ and $M(t) \cong M(t')$ our proof is complete. $\square$

A partial converse to Theorem 11 holds.

PROPOSITION 12. *Let $\{\mathfrak{D}(n)\}$ be any class of nested normal subgroups of the groups $\mathcal{K}(n)$ with the two properties*

P1. $\mathfrak{D}(n) \subseteq \mathcal{K}(n)$, $n = 0, 1, 2, \dots$.

P2. *If $[h]_\zeta = [t]_\zeta$, where $\zeta_n\colon \mathcal{K}(n) \to \mathcal{K}(n)/\mathfrak{D}(n)$ and where $M(h)$ and $M(t)$ are $\mathbf{Z}/2\mathbf{Z}$-homology spheres, then $\mu(M(h)) = \mu(M(t))$.*

Then $\mathfrak{D}(n) \subseteq \mathcal{C}(n)$ for $n = 0, 1, 2, \dots$.

PROOF. Let $k \in \mathfrak{D}(n)$ and let $\langle h_2, h_1\rangle$ be an admissible sequence of elements of $\mathcal{K}(n)$. By hypothesis $\mu(M(h_2 k h_1)) = \mu(M(h_2 h_1))$; hence $k \in \mathcal{K}_{h_2,h_1}(n)$. This holds for any admissible sequence $\langle h_2, h_1\rangle$; therefore $k \in \mathcal{C}(n)$ and we have $\mathfrak{D}(n) \subseteq \mathcal{C}(n)$. $\square$

REMARK. Invariants of $[h]_\zeta$ are, of course, well defined for all manifolds $M(h)$, not just for the class of $\mathbf{Z}/2\mathbf{Z}$-homology spheres. We conjecture that, if $M(h)$ and $M(t)$ are *not* $\mathbf{Z}/2\mathbf{Z}$-homology spheres, then $[h]_\zeta = [t]_\zeta$ if and only if $[h]_\eta = [t]_\eta$, where η is the canonical homomorphism from $\mathcal{H}(n)$ to $\mathcal{H}(n)/\mathcal{H}(n)$ and ζ_n is the canonical homomorphism from $\mathcal{H}(n)$ to $\mathcal{H}(n)/\mathfrak{D}(n)$.

The groups $\mathcal{C}(n)$ are not given explicitly by our construction. The proposition below exhibits some nontrivial elements in these groups and gives upper bounds for the orders of the groups $\mathcal{C}(n)$ in the groups $\mathcal{H}(n)$.

PROPOSITION 13. *For each $n \geqslant 2$, let $\mathcal{H}^2(n)$ denote the subgroup of $\mathcal{H}(n)$ consisting of elements that can be expressed in the form*

$$k = \left(k_p^2\right)^{\delta_1}\left(k_{p-1}^2\right)^{\delta_2}\cdots\left(k_1^2\right)^{\delta_p} \bmod\left[\mathcal{H}(n), \mathcal{H}(n)\right]$$

where $k_1, \ldots, k_p$ are elements of $\mathcal{H}(n)$.

Then $\mathcal{H}^2(n)$ is a nontrivial subgroup of $\mathcal{H}(n)$, even when homeomorphisms isotopic to the identity are factored out.

Also, the order of $\mathcal{C}(n)$ in $\mathcal{H}(n)$ is bounded above by 2^{m^2} where m is the order of the group $\mathrm{Sp}(2n, \mathbf{Z}/2\mathbf{Z})$.

PROOF. It is clear from the nature of $\mathbf{Z}/2\mathbf{Z}$ that $\mathcal{H}^2(n)$ belongs to each $\mathcal{H}_{h_2,h_1} = \ker \rho_{h_2,h_1}$. That $\mathcal{H}^2(n)$ contains nontrivial elements, even when isotopy classes are factored out, follows from the fact that $\mathcal{H}(n)/\mathrm{Isotopy}$ contains no elements of finite order (Equation 6.2, page 49 of [**Nl, ST**]).

From Theorem 8, the group $\mathcal{C}(n)$ is the intersection of fewer than m^2 subgroups of $\mathcal{H}(n)$, each having index 2 in $\mathcal{H}(n)$. Therefore by a theorem due to Poincaré (see [**Kr**, p. 62]), $\mathcal{C}(n)$ itself has index not greater than 2^{m^2}. $\square$

One would expect that more detailed knowledge about $\mathcal{C}(n)$ would lead to a better understanding of the μ-invariant. One would also expect that other nested sequences of normal subgroups of $\mathcal{H}(n)$ would yield new topological invariants of 3-manifolds.

There is an intriguing question suggested by the fact that quotient group $\mathcal{H}(n)/\mathrm{Isotopy}$ is a residually finite group (see [**Gr**]). Consider the set of all classes $\{\{\mathfrak{B}_i(n)\}: i \in I\}$ of nested normal subgroups of the groups $\mathcal{H}(n)$ subject to the restriction that the index of each $\mathfrak{B}_i(n)$ in $\mathcal{H}(n)$ always be *finite*. Let $\varphi_{i,n}$: $\mathcal{H}(n) \to \mathcal{H}(n)/\mathfrak{B}_i(n)$ be the natural homomorphism for each subgroup.

Question. If $M(h) \not\cong M(t)$, does there exist some $i \in I$ such that $[h]_{\varphi_{i,n}} \neq [t]_{\varphi_{i,n}}$?

Another possible line of study is suggested by the "additive" structure in the collection of groups $\{\mathcal{H}(n)\}$. Suppose that for some nested normal class

of subgroups $\{\mathfrak{B}(n)\}$, it is possible to identify, in some natural way, the factor groups $\mathfrak{H}(n)/\mathfrak{B}(n)$ with subgroups G_n of an abelian group G. Then the addition defined in §2.3 allows us to look for another characteristic in the nesting, namely that the equivalence classes satisfy $[h \# h']_\varphi = [h]_\varphi + [h']_\varphi$. This condition would provide a topological invariant for 3-manifolds which expressed the invariant for a connected sum of manifolds as the sum of the invariants for the components. This might be useful, for example, in attempting to get at finer invariants of homology cobordism classes of **Z**-homology spheres than the μ-invariant.

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