THE (φ, 1) RECTIFIABLE SUBSETS OF EUCLIDEAN SPACE

BY

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ABSTRACT. In this paper the structure of a subset $E \subset \mathbb{R}^n$ with $H^1(E) < \infty$ has been studied by examining its intersection with various translated positions of a smooth hypersurface $B$. The following result has been established:

Let $B$ be a proper $(n - 1)$ dimensional smooth submanifold of $\mathbb{R}^n$ with nonzero Gaussian curvature at every point. If $E \subset \mathbb{R}^n$ with $H^1(E) < \infty$, then there exists a countably 1-rectifiable Borel subset $R$ of $\mathbb{R}^n$ such that $(E \sim R)$ is purely $(H^1, 1)$ unrectifiable and $(E \sim R) \cap (g + B) = \emptyset$ for almost all $g \in \mathbb{R}^n$.

Furthermore, if in addition $E$ is $H^1$ measurable and $E \cap (g + B) = \emptyset$ for $H^1$ almost all $g \in \mathbb{R}^n$ then $H^1(E \cap R) = 0$. Consequently, $E$ is purely $(H^1, 1)$ unrectifiable.

Introduction. The study of the geometric structure of subsets of $\mathbb{R}^n$ relative to properties of their projections on $k$-dimensional linear subspaces has always played a central role in the progress of geometric measure theory. For example, the proof in [FF] of the existence of solutions for Plateau's problem and the minimal surface problem is dependent on this structure theory. The first results in this direction were obtained by Besicovitch in [BE], where he characterized 1-dimensional rectifiable subsets of $\mathbb{R}^2$ in terms of their projection properties. His results were extended by Federer [F2] to subsets of $\mathbb{R}^n$ and by Brothers [B] to subsets of homogeneous spaces.

Federer showed that if $E \subset \mathbb{R}^n$ with $H^k(E) < \infty$ then there exists a countably $k$-rectifiable Borel subset $R$ of $\mathbb{R}^n$ such that $E \sim R$ is purely $(H^k, k)$ unrectifiable and $L^k[p(E \sim R)] = 0$ for almost all orthogonal projections $p: \mathbb{R}^n \to \mathbb{R}^k$ where $L^k$ is the Lebesgue measure in $\mathbb{R}^k$.

Brothers generalized Federer's results to subsets of a smooth $n$-dimensional Riemannian manifold $X$ with a transitive group of isometries $G$. In order to make the transition from $\mathbb{R}^n$ to $X$ it was necessary to restate the projection properties without referring to projections. This he achieved by replacing

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1This research contains the author's main result in his Ph.D. dissertation at Indiana University (1975).
orthogonal projections of $A \subset \mathbb{R}^n$ into $\mathbb{R}^k$ with intersections $A \cap g(P)$ where $g$ is an isometry of $\mathbb{R}^n$ and $P$ a fixed $(n-k)$-dimensional linear subspace. For example the statement “$p(A)$ has Lebesgue measure zero for almost all orthogonal projections $p : \mathbb{R}^n \to \mathbb{R}^m$ is equivalent to “$A \cap g(P)$ is empty for almost all isometries $g$.” Thus his main result has the following form:

Let $G$ be a Lie group of isometries of $X$ with $\dim G = n(n+1)/2$ and suppose $G$ acts transitively on $X$. Let $B$ be a fixed $(n-k)$-dimensional smooth submanifold of $X$. If $E \subset X$ with $\mathcal{H}^k(E) < \infty$ then there exists a countably $k$-rectifiable Borel subset $R$ of $X$ such that $(E \sim R)$ is purely $(\mathcal{H}^k, k)$ unrectifiable and

$$(E \sim R) \cap g(B) = \emptyset$$

for almost all $g \in G$.

One of the central features of the proof of the above theorem is the use of the fact that the isotropy group at a point $0 \in X$ acts on the tangent space at $0$ as either the orthogonal group or the special orthogonal group. We also note that $\dim G = n(n+1)/2$ implies in the Euclidean case that $G$ is either the full group of isometries or the component of this group which contains the identity. Further, if $X$ is connected and $\dim G = n(n+1)/2$, then $X$ must be of constant curvature. Thus it is natural to ask if it is possible to obtain similar results with less restrictive assumptions on $G$; that is, can Brothers’ results hold if the dimension of $G$ is less than $n(n+1)/2$?

Notice that if we take $k = 1$ and $B = S^{n-1} \subset \mathbb{R}^n$ in Brothers’ theorem then it follows that for almost all translations $g$ of $\mathbb{R}^n$

$$(E \sim R) \cap g(B) = \emptyset.$$ 

On the other hand, standard examples (see for example [F1, 3.3.19]) show that this may not be true if $B$ is a hyperplane. Based upon these examples together with the structure of the proof of his theorem, Brothers conjectured that if $G$ is the group of translations of $X = \mathbb{R}^n$ then his result will hold at least for $(\mathcal{H}^1, 1)$ rectifiability provided it is assumed that the Gaussian curvature of $B$ does not vanish. In this paper we prove this conjecture. Our main result is the following:

**Theorem 1.** Let $B$ be a proper $(n-1)$-dimensional smooth submanifold of $\mathbb{R}^n$ with nonzero Gaussian curvature at every point. If $E \subset \mathbb{R}^n$ with $\mathcal{H}^1(E) < \infty$, then there exists a countably $1$-rectifiable Borel subset $R$ of $\mathbb{R}^n$ such that $(E \sim R)$ is purely $(\mathcal{H}^1, 1)$ unrectifiable and

$$(E \sim R) \cap (g + B) = \emptyset$$

for almost all $g \in \mathbb{R}^n$.

Furthermore, if in addition $E$ is $\mathcal{H}^1$ measurable and $E \cap (g + B) = \emptyset$ for
Theorem 2 is an extension of this result involving measures more general than $H^1$.

The problem of extending these results to the general case where $X$ is a Lie group with an invariant metric is difficult because of noncommutativity. On the other hand, our results clearly hold when $X$ is a torus, hence for the case where $X$ is an Abelian Lie group. In a subsequent paper we will investigate the possibility of extending our results to the case where $k > 1$.

I am indebted to Professor John Brothers for his continuing help, advice and encouragement during the preparation of this paper. He has always been a source of inspiration.

Preliminaries. The purpose of this section is to fix basic notation and terminology; more details may be found in [F1].

If $M$ is an $l$-dimensional manifold of class 1 and $u \in M$, then $T_u(M)$ is the $l$-dimensional real vector space of tangent vectors of $M$ at $u$.

For each finite dimensional vector space $V$ and $l = 0, 1, 2, \ldots, \dim V$, $A_l(V)$ is the associated vector space of $l$ vectors. Furthermore,

$$A_* = \bigoplus_{l=0}^{\dim V} A_l(V)$$

is the corresponding exterior algebra, with exterior multiplication $\wedge$.

Suppose $M$ and $N$ are manifolds of class 1 and $f: M \to N$. If $u \in M$, $w = f(u)$ and $f$ is differentiable at $u$, the differential of $f$ at $u$ is a linear transformation

$$f_\#(u): T_u(M) \to T_w(N).$$

$f_\#(u)$ can be extended to a unique algebra homomorphism

$$f_\#: A_*[T_u(M)] \to A_*[T_w(N)].$$

If $M$ and $N$ are Riemannian manifolds and

$$r = \inf\{\dim M, \dim N\}$$

then the Jacobian of $f$ at $u$ is

$$Jf(u) = \sup\{|f_\#(u)(v)|: v \in A_r[T_u(M)], |v| = 1\}$$

where the indicated norm is induced by the metric on $M$ and $N$.

If $u = (u_1, \ldots, u_n)$ and $w = (w_1, \ldots, w_n) \in \mathbb{R}^n$, the inner product of $u$ and $w$ is $u \cdot w = \sum_{i=1}^{n} u_i \cdot w_i$.

$e_1, \ldots, e_n$ are the standard orthonormal basis vectors of $\mathbb{R}^n$.

Let $\phi$ be a nonnegative measure on a Riemannian manifold $M$ such that
closed sets are \( \phi \) measurable. In particular, \( H^l \) is the \( l \)-dimensional Hausdorff measure on \( M \).

The family of Suslin subsets of \( M \) contains the Borel subsets of \( M \) and has the following properties:

Each Suslin set is \( \phi \) measurable.

If \( \{ F_i \} \) is a countable family of Suslin sets, then \( \bigcup_{i=1}^{\infty} F_i \) and \( \bigcap_{i=1}^{\infty} F_i \) are Suslin sets.

If \( N \) is a smooth manifold and \( f: M \to N \) is continuous, then \( f(S) \) and \( f^{-1}(T) \) are Suslin sets whenever \( S \) and \( T \) are Suslin subsets of \( M \) and \( N \) respectively.

If \( \mu \) measures \( Y \) and \( A \subset Y \), then \( \mu \upharpoonright A \) is the measure on \( Y \) defined by the formula

\[
\mu \upharpoonright A(S) = \mu(A \cap S) \quad \text{for } S \subset Y.
\]

If \( f: Y \to Z \), then \( f_\#(\mu) \) is the measure on \( Z \) defined by

\[
f_\#(\mu)(S) = \mu[f^{-1}(S)] \quad \text{for } S \subset Z.
\]

\( R \subset M \) is \( k \)-rectifiable if there exists a Lipschitzian function mapping some bounded subset of \( \mathbb{R}^k \) onto \( R \).

\( R \subset M \) is countably \( k \)-rectifiable if \( R \) is the union of a countable family of \( k \)-rectifiable sets.

\( E \subset M \) is countably \((\phi, k)\) rectifiable if there exists a countably \( k \)-rectifiable set \( R \) with \( \phi(E \sim R) = 0 \).

\( E \subset M \) is \((\phi, k)\) rectifiable if \( \phi(E) < \infty \) and \( E \) is countably \((\phi, k)\) rectifiable.

\( E \subset M \) is purely \((\phi, k)\) unrectifiable if \( E \) contains no \( k \)-rectifiable set \( R \) with \( \phi(R) > 0 \).

\[
U_k(u, r) = \mathbb{R}^k \cap \{ w : |w - u| < r \}
\]

for \( r > 0 \), \( u \in \mathbb{R}^k \).

If \( r > 0 \), \( s > 0 \), \( u \in \mathbb{R}^n \) and \( Y \subset \mathbb{R}^n \), then

\[
X(u, r, Y, s) = \mathbb{R}^n \cap \{ w : \text{dist}(w, Y) < s \text{ dist}(w, u) \} \cap U_n(u, r).
\]

Throughout this paper \( B \) will denote a proper \((n - 1)\)-dimensional submanifold of class \( \infty \) of \( \mathbb{R}^n \) with nonzero Gaussian curvature at every point. If \( s > 0 \) and \( g \in -B \) denote

\[
K_{g,s} = \{ h : |h - g| < s \} \cap (-B).
\]

Also,

\[
K_{g,s}(B) = \{ h + b : h \in K_{g,s}, b \in B \}.
\]

If \( F: U \to \mathbb{R} \) is a \( C^2 \) function, where \( U \) is an open subset of \( \mathbb{R}^n \), we denote

\[
D_iF(x) = \partial F(x)/\partial x_i, \quad i = 1, 2, \ldots, n.
\]
\( D_\gamma F(x) = \partial^2 F(x)/\partial x_i \partial x_j, \quad i, j = 1, 2, \ldots, n. \)

We will denote by \( f_0 : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) the map defined by
\[
f_0(a, b) = a - b.
\]

If \( A \subset \mathbb{R}^n \), then \( S_{A,1} \) is the set of \((a, b) \in \mathbb{R}^n \times B\) such that for some \( \epsilon > 0 \)
\[
\lim_{s \rightarrow 0^+} \sup_{0 < r < \epsilon} \phi[A \cap X(a, r, a - b + B, s)](rs)^{-1} = 0.
\]
\( S_{A,2} \) is the set of \((a, b) \in \mathbb{R}^n \times B\) such that for all \( \epsilon > 0 \)
\[
\lim_{s \rightarrow 0^+} \sup_{0 < r < \epsilon} \phi[A \cap X(a, r, a - b + B, s)](rs)^{-1} = \infty.
\]
\( S_{A,3} = \mathbb{R}^n \times B \cap \{(a, b) : a \in \text{Clos}A \cap (a - b + B) \sim \{a\})\} \).

**Lemma 1.** For any \( g \in -B \) there exist positive constants \( r_1, s_1, \alpha, \beta \) such that if \( 0 < s < s_1 \), then
\[
(i) \ X(0, r_1, g + B, s) \subset K_{g, \alpha}(B),
(ii) \ X(0, \infty, g + B, s) \supset K_{g, \beta}(B) \sim \{0\}.
\]

**Proof.** Without loss of generality we may assume \( g = 0 \in B \). By a proper choice of coordinate axes we may assume that
\[
B \cap U_0(0, 1) = \{(x, f(x)) : x \in U\},
\]
where \( 0 \in U \), \( U \) is an open subset of \( \mathbb{R}^{n-1} \) which contains \( \{x \in \mathbb{R}^{n-1} : |x| < 1/2\} \) and \( f : U \rightarrow \mathbb{R} \) is of class \( C^\infty \) and such that
\[
f(0) = 0, \quad D_if(0) = 0 \quad \text{for } i = 1, 2, \ldots, n - 1.
\]

We may also assume that \( f \) is Lipschitzian and if \( |x| < 1/2 \), then
\[
|f(x)| < |x|,
\]
\[
|D_j f(x)|, |D_j f(x)|, |D_j f(x)| < Cn^{-3}
\]
for \( j, l, m = 1, 2, \ldots, n - 1 \), where \( C > 1 \).

Our assumption that the Gaussian curvature does not vanish at any point of \( B \) leads to the fact [KN, Volume 2, p. 17] that
\[
det(D_{ij} f(0)) \neq 0, \quad l, j = 1, 2, \ldots, n - 1.
\]

Let \( L \) be the linear transformation of \( \mathbb{R}^{n-1} \) with the matrix \( (D_{ij} f(0)) \). Then there exists \( 0 < V < C/2 \) such that whenever \( y \in S^{n-2} \subset \mathbb{R}^{n-1}, \)
\[
L(y) \cdot e_j > V \quad \text{for some } j \in \{1, 2, \ldots, n - 1\}.
\]

**Part 1.** Fix \( x, x_0 \in \mathbb{R}^{n-1} \) and \( s \in \mathbb{R} \) such that \( 0 < |x_0| < (V/8C), \quad |x - x_0| < (sV|x_0|/16C), \quad \) and \( 0 < s < (V/8Cn) \). Then the set of numbers \( f(x + t) - f(t) - f(x_0) \) corresponding to \( t = t_1 e_j \) and \( l \in \{1, 2, \ldots, n - 1\} \) with \( |t_i| < s \) contains the interval \( \{r : |r| < sV|x_0|/16\}. \)

**Proof.** Let us consider the sets

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\[ S_1 = \{ x: 0 < |x| < (V/4C) \} \cap \mathbb{R}^{n-1}, \]
\[ S_2 = \{ t: 0 < |t| < (V/2C) \} \cap \mathbb{R}^{n-1}. \]

Let \( x \in S_1, \ t \in S_2 \). Expanding \( f(x + t) \) about \( x \) and \( f(t) \) about \( 0 \), we have

\[
f(x + t) - f(t) = f(x) + Df(x)(t) + \sum_{j=1}^{n-1} t_j R_{ji},
\]

where

\[
R_{ji} = \int_0^1 (1 - \theta) \{ D_{ij} f(x + \theta t) - D_{ij} f(\theta t) \} \, d\theta.
\]

By assumption (3)

\[
|R_{ji}| \leq |x|C/2.
\]

Also,

\[
Df(x)(t) = \sum_{j=1}^{n-1} t_j \left[ \sum_{i=1}^{n} x_i D_{ij} f(0) + \frac{1}{2} \sum_{m,l=1}^{n-1} x_m x_l D_{ml} f(\theta x) \right]
\]

where \( 0 < \theta_j < 1 \).

Now let \( y = (x/|x|) \in S^{n-2} \). Then from (5), for some \( l \)

\[
0 < V < \left| \sum_k (x_k/|x|) D_{lk} f(0) \right|.
\]

Consequently by setting \( t_j = 0 \) for \( j \neq l \) in (8), we get

\[
t_l D_{lj} f(x) = |x|t_l H,
\]

where

\[
H = \sum_k (x_k/|x|) D_{lk} f(0) + (|x|/2) \sum_{k,m} (x_m/|x|)(x_k/|x|) D_{mk} f(\theta x).
\]

By (9) and (3), we conclude that

\[
|H| > V - V/8 > V/2.
\]

Therefore from (6), (10) we get

\[
f(x + t) - f(t) - f(x) = |x|t_l H + t_l^2 R_{lj} \text{ with}
\]

\[
|H| > V/2 \text{ and } |R_{lj}| < (|x|C/2) \text{ for}
\]

\[
t = t_l e_l \in S_2 \text{ and } x \in S_1.
\]

Now let \( x, x_0, s \) be such that \( 0 < |x_0| \leq (V/8C), |x - x_0| < (sV|x_0|/16C) \)

and \( 0 < s < (V/8Cn) \). Notice that \( x \in S_1 \). Also,

\[
|f(x) - f(x_0)| < (sV|x_0|/16).
\]

Now by (11), for \( t = t_l e_l \in S_2 \),
\( f(x + t) - f(t) - f(x_0) = |x| t H + t^2 R_H + f(x) - f(x_0), \)

where \(|H| > V/2\) and \(|R_H| < |x| C/2\).

Suppose \( H > 0 \). Since \( s < V/2C \), \( t = s e_1 \in S_2 \). By setting \( t_i = s \), we obtain

\[
\begin{align*}
f(x + t) - f(t) - f(x_0) &> (|x|/2)(Vs - Cs^2) - sV|x_0|/16 \\
&> (|x_0|/2)(1/2)(Vs - Cs) - sV|x_0|/16 > sV|x_0|/16.
\end{align*}
\]

Similarly, putting \( t_i = -s \), we obtain

\[
\begin{align*}f(x + t) - f(t) - f(x_0) &< -iF|x_0|/16.
\end{align*}
\]

Since for fixed \( x, x_0, s \), \( f(x + t) - f(t) - f(x_0) \) with \( t = t_i e_i \) is a continuous function of \( t \) on the interval \([-s, s]\), we conclude that Part 1 holds. Finally, in the case where \( H < 0 \) we reach the same conclusion by replacing \( s \) by \(-s\).

**Part 2.** \( K_{0.4s}(B \cap U_n(0, 1)) \supseteq U_n((x_0, f(x_0)), rVs/64C) \) where \( 0 < s < V/8Cn, 0 < |x_0| < V/8C \) and \( r = |(x_0, f(x_0))| \).

**Proof.** Let

\[
S_{x_0} = R^{n-1} \times \{x: |x - x_0| < sV|x_0|/16C\}
\]

\[
\times R \cap \{\xi: |\xi - f(x_0)| < (sV|x_0|/16)(1 - sV/8C)\}.
\]

Since \( |x_0| < 1/2, r < 2|x_0| \) by (2). Also,

\[
(Vs|x_0|/16)(1 - sV/8C) > rVs/64C.
\]

Thus

\[
U_n((x_0, f(x_0)), rVs/64C) \subseteq S_{x_0}.
\]

So, let \((x, \xi) \in S_{x_0}\). We observe that \( x \in S_1 \). Since \( |\xi - f(x_0)| < sV|x_0|/16 \), by Part 1 there exists \( t = t_i e_i \) such that \( |t| < s \), and

\[
f(x + t) - f(t) - f(x_0) = \xi - f(x_0).
\]

Notice that by (2) \((-t, -f(t)) \in K_{0.4s} \subset -B\). Also,

\[
(x + t, f(x + t)) \in B \cap U_n(0, 1).
\]

Thus

\[
(x, \xi) \in K_{0.4s}(B \cap U_n(0, 1)).
\]

This together with (13) establishes our claim.

**Part 3.** \( K_{0.4s}(B) \supseteq X(0, V/16C, B, sV/128C) \) where \( 0 < x < V/8nC \).

**Proof.** If \( w \in X(0, V/16C, B, sV/128C) \), then there exists \( b \in B \) such that \( |b - w| < r_0 sV/128C \), where \( r_0 = |w| \). Now \( r = |b| > |w| - |b - w| > r_0/2 \) and \( r < 2r_0 < V/8C \). Therefore \( b = (x_0, f(x_0)) \) with \( |x_0| < V/8C \). Hence
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\[ w \in U_n((x_0, f(x_0)), rsV/64C) \subset K_{0,4s}(B \cap U_n(0, 1)) \]

by Part 2.

From Part 3 we get (i) of Lemma 1 immediately.

Turning to the proof of (ii) of the lemma, we claim that

(14) \[ K_{0,2s}(B) \sim U_{n}(0, 1/16) \subset X(0, \infty, B, 32s), \]

where \( 0 < s < V/8nC < 1/32 \). Indeed, if \( u \in K_{0,2s}(B) \sim U_{n}(0, 1/16) \), then \( u = g + w; g \in K_{0,2s}, w \in B \). But then

\[
\left[ \text{distance}(u, B) \right]/|u| < 32s.
\]

Next let \( u_1 \in K_{0,2s}(B) \cap U_{n}(0, 1/16) \). This means that \( u_1 = g_1 + w_1 \), with \( g_1 \in K_{0,2s} \) and \( w_1 \in B \), and \( |u_1| < 1/16 \). Therefore

(15) \[ K_{0,2s}(B) \cap U_{n}(0, 1/16) \subset K_{0,2s}(B \cap U_{n}(0, 1/8)). \]

So let us assume \( u \in K_{0,2s}(B \cap U_{n}(0, 1/8)) \). Then \( u = (t, -f(-t)) + (x, f(x)) \) where \( (t, -f(-t)) \in K_{0,2s} \) and \( (x, f(x)) \in B \cap U_{n}(0, 1/8) \). Write \( y = x + t \), so that \( |y| < 1/4 \). Thus \( y \in U \). Also,

\[
f(x) - f(-t) = f(x) - f(x - y) = f(y) + Df(y)(x - y) + \sum_{j,l} (x_j - y_j)(x_l - y_l)R_{jl}
\]

where

\[
R_{jl} = \int_0^1 (1 - \theta) \{ D_{j,l}f(y + \theta(x - y)) - D_{j,l}f(\theta(x - y)) \} \, d\theta
\]

and, as in (7), \( |R_{jl}| < |y|C/2 \). Note that \( |Df(y)| \leq C|y| \). We thus conclude that

\[
|f(x) - f(-t) - f(y)| \leq C|y|^{2s} + n^2(2s)^2|y|C/2.
\]

Writing \( v = (y, f(y)) \) we find that

(16) \[ |f(x) - f(-t) - f(y)| < 4|v|Cn^2s, \]

hence \( \text{distance}(u, B) < |u - v| < 4|v|Cn^2s \). Since \( |v| < 2|u| \) by (2) we conclude that \( u \in X(0, \infty, B, 8Cn^2s) \). Hence

\[ K_{0,2s}(B) \cap U_{n}(0, 1/16) \subset X(0, \infty, B, 8Cn^2s), \]

by (15). This together with (14) establishes (ii).

**Lemma 2.** Let \( g \in -B \). Then there exist positive numbers \( r_2, s_2, H_1, H_2 \) and \( \delta > 1 \) such that if \( 0 < s < s_2 \) and \( 0 \neq w \in K_{g,s}(B) \cap U_n(0, r_2) \), then

(i) \( H^{n-2}[(w - B) \cap K_{g,s}] \geq H_1s^{n-2}, \)

(ii) \( H^{n-2}[(w - B) \cap K_{g,s}] \leq H_2s^{n-2}. \)

**Proof.** Assuming \( g = 0 \in B \) and choosing \( f \) and \( U \) as in the proof of
Lemma 1 we denote $U_0 = -U$ and define $h$: $U_0 \to \mathbb{R}$ by $h(x) = -f(-x)$. Thus

$$-B \cap U_0(0,1) = \{(x,h(x)): x \in U_0\}.$$  

Since $\det(D_y h(0)) \neq 0$, we can (with the use of an orthogonal change of coordinates) assume $h$ has a Taylor expansion of the form

$$h(x) = \sum_{i=1}^{n-1} k_i x_i^2 + \sum_{i,j,k=1}^{n-1} \alpha_{ijk}(x)x_i x_j x_k,$$

with $k_i \neq 0$, $|\alpha_{ijk}|$ and $|D_y \alpha_{ijk}|$ bounded and $C^\infty$ on $U_0$. We will write

$$K_0 = \min(|k_1|, \ldots, |k_{n-1}|) > 0 \quad \text{and} \quad K = \max(1, |k_1|, \ldots, |k_{n-1}|).$$

For $i = 1, 2, \ldots, (n - 1)$ we will write $\pi_i x = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ whenever $x = (x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) \in \mathbb{R}^{n-1}$.

**PART 1.** There exist $C > 2$, $0 < \gamma < \min(1/16, K_0/\sqrt{4C(2n)^{1/2}})$, and a $C^\infty$ function $\Phi$ defined on

$$\Omega = \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}$$

$$\cap \{(x,\rho,\eta,\xi): |x| < 1/4, |\rho| < 1/4, |\eta|^2 + |\xi|^2 = 1\}$$

such that if $0 < s < \gamma$ and $0 \neq w = \rho(\rho, \eta) \in K_{a,s}(B) \cap U_0(0, \gamma)$ with $|\rho|^2 + |\eta|^2 = 1$, then the following are true:

(i) $(w-B) \cap K_{a,s} = \{(x, h(x)) \in K_{a,s}: \Phi(x, \rho, \eta, \xi) = 0\}$.

(ii) There exists $i \in \{1, 2, \ldots, n - 1\}$ (which depends only on $\gamma$) such that

$$|\rho_i| > (2n)^{-1/2},$$

and for $|x| < \gamma$,

$$|D_i \Phi(x, \rho, \eta)| > 1/C$$

and

$$|D_j \Phi(x, \rho, \eta)| / |D_i \Phi(x, \rho, \eta)| < C \quad \text{for} \quad j = 1, 2, \ldots, n - 1, j \neq i.$$

(iii) If for $i = 1, 2, \ldots, n - 1$ we write $\Phi_i(x, \rho, \eta) = \Phi(x, \rho, \eta)$ where $x = (0, \ldots, 0, x_i, 0, \ldots, 0)$, $|x_i| < 1/4$, then there exist $C^\infty$ functions $\phi_i, g_i$, each having domain $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} \cap \{(x_i, \rho, \eta, \xi): |x_i|, |\rho| < 1/4, |\eta|^2 + |\xi|^2 = 1\}$ such that $|g_i|, |\phi_i|, |D_i \phi_i| < C$ and

$$\Phi_i(x, \rho, \eta, \xi) = 2k_i x_i \rho_i - \rho \sum_{i=1}^{n-1} k_i \rho_i^2 - \eta$$

$$+ g_i(x, \rho, \eta, \xi) x_i^2 + \rho \phi_i(x, \rho, \eta, \xi).$$

**PROOF.** By choosing $\rho_0$ sufficiently small ($0 < \rho_0 < 1/8$) we may assume that

$$B \cap U_0(0, 2\rho_0) \sim \{0\} \subset \mathbb{X}(0, \infty, T_0(B), 1/8).$$
Now if \( u \in X(0, \infty, B \cap U^0_n(0, 2\rho_0), s_0) \) where \( 0 < s_0 < 1/8 \) then there exists \( 0 \neq b \in B \cap U^0_n(0, 2\rho_0) \) such that \( |u - b| < |u|s_0 \). But by the above assumption there exists \( w \in T^0_n(B) \) such that \( |b - w| < |b|/8 < 9|u|/64 \). Hence \( u \in X(0, \infty, T^0_n(B), 1/2) \). Assuming \( s_0 < s_1 \), we apply Lemma 1(ii) with \( \beta s_1 = \rho_1 = \min(\beta s_0, \rho_0) \) to obtain

\[
[\left. K_{0,s_1}(B \cap U^0_n(0, 2\rho_1)) \right] \approx \{0\} \subset X(0, \infty, T^0_n(B), 1/2).
\]

Consider now \( x \in \mathbb{R}^{n-1} \) and \( (t, \theta) \in \mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n \) with \( |(t, \theta)| < 1/4 \). Applying (1) and writing \( (t, \theta) = \rho(y, \eta) \) where \( |(y, \eta)| = 1 \), we obtain

\[
h(x) - h(x - t) - t = \rho \Phi(x, \rho, y, \eta)
\]

where

\[
\Phi(x, \rho, y, \eta) = \sum_{i=1}^{n-1} 2k_i x_i y_i - \rho \sum_{i=1}^{n-1} k_i y_i^2 - \eta + R(x, \rho, y, \eta)
\]

with

\[
R(x, \rho, y, \eta) = \sum_{i,j,k=1}^{n-1} \alpha_{ijk}(x) \left\{ -y_j x_j x_k - y_k x_i x_j - y_j x_i x_k + \rho x_j y_j y_k + \rho x_k y_j y_j + \rho x_j y_j y_j - \rho y_j y_j y_j \right\}
\]

\[
+ \sum_{i,j,k,l=1}^{n-1} \left\{ y_j \int_0^1 D_j \alpha_{i j k} (\tau x + (1 - \tau)(x - \rho y)) \, d\tau \right\} \times (x_i - \rho y_j)(x_j - \rho y_j)(x_k - \rho y_k).
\]

Obviously \( \Phi \) is \( C^\infty \) with domain

\[
\Omega = \{(x, \rho, y, \eta) : |x| < 1/4, |\rho| < 1/4, |y|^2 + |\eta|^2 = 1\}.
\]

For each \( i = 1, 2, \ldots, (n - 1) \) we write

\[
D_i \Phi = \partial \Phi / \partial x_i = 2k_i y_i + T_{i,x} + T_{i,\rho}
\]

where \( T_{i,x} \) stands for the sum of all terms containing at least one \( x_j \) but not \( \rho \) as a factor and \( T_{i,\rho} \) stands for the sum of all terms with \( \rho \) as a factor. Since \( |\alpha_{ijk}|, |D_i \alpha_{ijk}| \) are all bounded, by assuming \( |x|, |\rho| < \gamma' < 1/4 \) we can make

\[
|T_{i,x}|, |T_{i,\rho}| < K_0/(2(2n)^{1/2}).
\]

Also, for each \( i \in \{1, 2, \ldots, n - 1\} \), by setting \( x_j = 0 \) for \( j \neq i \) in \( R(x, \rho, y, \eta) \) we will get functions of \( x_i, \rho, y, \eta \) in this form:

\[
g_i(x_i, \rho, y, \eta) x_i^2 + \rho \phi_i(x_i, \rho, y, \eta).
\]

We may assume
where $C > 3K(2n)^{1/2}/K_0 > 2$. Let

$$\gamma = \min\{\gamma', 1/16, K_0/[4C(2n)^{1/2}], \rho_1\}.$$ 

Fix $0 < s < \gamma$ and $0 \neq w \in K_{0,s}(B) \cap U_n(0, \gamma)$. Then

$$K_{0,s}(B) \cap U_n(0, \gamma) = K_{0,s}(B \cap U_n(0, 2\gamma)) \cap U_n(0, \gamma),$$

$$\{w - B\} \cap K_{0,s} = \{w + K_{0,4\gamma}\} \cap K_{0,s}.$$ 

Therefore, writing $w = (t, \theta) = \rho(y, \eta)$ with $|(y, \eta)| = 1$ and using (3) we obtain

$$K_0 \cap \{w - B\} \cap K_{0,s} = \{(x, h(x)) \cap K_0; \Phi(x, \rho, y, \eta) = 0\}$$

giving us (i) of Part 1.

Furthermore, if $0 \neq w \in K_{0,s}(B) \cap U_n(0, \gamma)$ where $0 < s < \gamma$ and $|x| < \gamma$, then from (4) and (2) we have $0 \neq w \in \mathbb{R}_0 \cap U_n(0, 1/2)$. Since $T_0(B) = \mathbb{R}^{n-1} \times \{0\}$ and $w = \rho(y, \eta)$, $|y|^2 + |\eta|^2 = 1$, there exists $i \in \{1, 2, \ldots, n - 1\}$ for which $|y_i| > (2(n - 1))^{-1/2} > (2n)^{-1/2}$. But then

$$|D_1\Phi(x, \rho, y, \eta)| = 2|k_iy_i + T_{i,x} + T_{i,y}| > K_0/(2n)^{1/2} > 1/C.$$ 

Also, for $j = 1, 2, \ldots, n - 1, j \neq i$, we have

$$|D_1\Phi(x, \rho, y, \eta)/D_1\Phi(x, \rho, y, \eta)| < 3(2n)^{1/2}K_0^{-1/2}.$$ 

This proves (ii) of Part 1.

**Part 2.** Let $0 < s < \gamma/4nC$ and $\rho, \eta \in \mathbb{R}, y \in \mathbb{R}^{n-1}$ be given with $|y|^2 + |\eta|^2 = 1, |\rho| < \gamma$ and $|y_i| > (2n)^{-1/2}$. Let $|a_i| < Csn$ be such that $\Phi_i(a_i, \rho, y, \eta) = 0$. If $|x_i| < Csn$ and $\Phi_i(x_i, \rho, y, \eta) = 0$, then $x_i = a_i$.

**Proof.** Suppose $|c_i| < Csn$. Then

$$|\Phi_i(c_i, \rho, y, \eta)| = |2k_i(a_i - c_i)y_i + \left[g_i(a_i, \rho, y, \eta) - g_i(c_i, \rho, y, \eta)\right]c_i^2|$$

Applying the mean value theorem to $g_i$ and $\Phi_i$ and using the bounds on $g_i, D_1g_i, D_1\Phi_i$, one can show that since $Csn < \gamma$ this expression is not less than

$$|a_i - c_i|K_0/(2n)^{1/2}.$$

**Part 3.** If $0 < s < \gamma/4nC$ and

$$0 \neq w = \rho(y, \eta) \in K_{0,s/C}(B \cap U_n(0, 2\gamma)) \cap U_n(0, \gamma)$$

with $|(y, \eta)| = 1$, then there exists $i \in \{1, 2, \ldots, n - 1\}$ (depending only on $y$) such that $|y_i| > (2n)^{-1/2}$. Corresponding to such an $i$ there is a $C^\infty$ function

$$\psi_{w,i}: U_{n-2}(0, s/C) \to \mathbb{R}$$

such that
(i) If \(|z| < s/C\), then
\[ \Phi(\sigma_i(z), \rho, \gamma, \eta) = 0 \]
where we define
\[ \sigma_i(z) = (z_1, \ldots, z_{i-1}, \psi_{w,i}(z), z_i, \ldots, z_{n-2}) \quad (\psi_{w,i}(z) \text{ in } i\text{th place}). \]

(ii) \(|\psi_{w,i}(0)| < 2ns\).

(iii) If \(x \in \mathbb{R}^{n-1}\), \(|x| < s/C\) and \(\Phi(x, \rho, \gamma, \eta) = 0\), then
\[ x = \sigma_i(\pi_i x). \]

It follows that
\[ H_{n-2}^2((w - B) \cap K_{0.6ns}) \supset H_1s^{n-2} \]
where \(H_1 = C^{2-n}H_{n-2}[U_{n-2}(0, 1)]\).

Proof. We have \(w = g_0 + b_0\) where \(g_0 \in K_{0.5/C}\) and \(b_0 \in B \cap U_n(0, 2\gamma)\). Set \(g_0 = (x_0, h(x_0))\) where \(x_0 \in \mathbb{R}^{n-1}\). By Part 1(i), \(\Phi(x_0, \rho, \gamma, \eta) = 0\). In view of (ii) of Part 1, we may assume that \(\gamma < 2/n\) and \(\delta > 0\) and \(\Phi(\sigma(x_0), \rho, \gamma, \eta) = 1/C\) for \(|x| < \gamma\). We will identify \(\mathbb{R}^{n-2} \times \mathbb{R} = \mathbb{R}^{n-1}\); denote \(x_0 = (z_0, \xi_0)\). By the implicit function theorem there exist a \(\delta > 0\) and a \(C^\infty\) function \(\psi_0: U_{n-2}(z_0, \delta) \to \mathbb{R}\) such that \(\psi_0(z_0) = \xi_0\) and \(\Phi(z, \psi_0(z)) = 0\), \(|z - z_0| < \gamma/2\) for \(|x| < \gamma/2\), and the relations
\[ (x', \rho, \gamma, \eta) = 0, \quad |x' - z_0| < \delta, \quad |\pi_{n-1}x' - z_0| < \delta \]
hold only in case \(x'_{n-1} = \psi_0(\pi_{n-1}x')\). Since \(\rho, \gamma, \eta\) are all fixed we may write \(\Phi(z, \psi_0(z))\) in place of \(\Phi((z, \psi_0(z)), \rho, \gamma, \eta)\). We claim that \(\delta\) may be assumed to be greater than \(2s/C\).

Let \(S\) be the set of all \(\delta > 0\) corresponding to which there exists \(\psi_0\) as above and let \(\delta_0 = \operatorname{lub} S\). We may assume \(\delta_0 < 2s/C\). Since the \(\psi_0\) are unique we conclude that \(\delta_0 \in S\) with \(\cup \{\psi_0: \delta \in S\} = \psi_0^c\). Set \(\psi = \psi_0^c\) by Part 1(ii), \(|D_0\psi| < C, j \in \{1, 2, \ldots, n - 2\}\). Thus \(\psi\) is Lipschitz, and hence uniformly continuous in \(U_{n-2}(z_0, \delta_0)\). Therefore \(\psi\) has a continuous extension to the Closure of \(U_{n-2}(z_0, \delta_0)\). Let \(z\) be such that \(|z - z_0| = \delta_0\) and denote \(\xi = \psi(z)\). Since \(\Phi\) is continuous, \(\Phi(z, \xi) = 0\). By the mean value theorem and continuity of \(\psi\), \(|\xi| < 2ns\). Also \(|x| < 3s\), so \(|(z, \xi)| < 4ns < \gamma\). Thus \(D_{n-1}\Phi(z, \xi) \neq 0\). Hence \(\psi\) has a \(C^\infty\) extension to a neighborhood of the compact set \(\operatorname{Clos} U_{n-2}(z_0, \delta_0)\) [H, p. 23] which contradicts the maximality of \(\delta_0\). Writing \(\psi_{w,i} = \psi\) we conclude that (i) and (ii) hold. Note that it also follows from what we have shown that
\[ ((z, \xi), h(z, \xi)) \in K_{0.6ns} \quad \text{for } |z| < s/C \text{ and } \psi(z) = \xi. \]
Thus by Part 1(i) we conclude that
\( W^{-2}(w - B) \cap K_{0.6n} \) 
\( \geq W^{-2}(P_{n-1}(w - B) \cap K_{0.6n}) \)
where \( P_{n-1}(x_1, \ldots, x_{n-1}, x_n) = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} \)
\( \geq W^{-2}(U_{n-2}(0, 1)](s/C)^{n-2}. \)

Finally, fix \( x \in \mathbb{R}^{n-1} \) such that \( |x| < s/C \) and \( \Phi(x, \rho, y, \eta) = 0. \) Choosing \( \psi_{w,n-1} \) as above with \( x_0 \) replaces by \( x \) we infer from Part 2 that \( \psi_{w,n-1}(0) = \psi_{w,n-1}(0). \) Thus by (\( \ast \))
\( \psi_{w,n-1}[U_{n-2}(0, s/C)] = \psi_{w,n-1}[U_{n-2}(0, s/C)] \)
which proves (iii).

**Part 4.** There exist \( 0 < \alpha_0 < 1, 0 < s_0 < \gamma/4nC, 0 < \delta_0 < \gamma \) and for each \( i \in \{1, 2, \ldots, n-1\} \) a positive integer \( m_i \) such that the following is true:

Let 
\[ W = \mathbb{R}^{n-1} \times \mathbb{R} \cap \{(y, \eta): |(y, \eta)| = 1, |\eta| < \alpha_0 s_0\}. \]

For each \( i \in \{1, 2, \ldots, n-1\} \) and \( j \in \{1, 2, \ldots, m_i\} \) there exist \( \delta_j > 0, (y_{ij}, \eta_{ij}) \in W_i = W \cap \{(y, \eta): |(y, \eta)| = 1, |y| > [2(n-1)]^{-1/2} \) and \( \theta_j: \mathbb{R}^{n-2} \cap \{z: |z| < \delta_j \} \times \mathbb{R} \cap \{\rho: |\rho| < \delta_j \} \times (\mathbb{R}^{n-1} \times \mathbb{R}) \cap \{(y, \eta): |(y, \eta)| = 1, |(y, \eta) - (y_{ij}, \eta_{ij})| < \delta_j, |y| > (2n)^{-1/2} \to \mathbb{R} \) all having the same Lipschitz constant \( H_0, \) such that for each \( (y_0, \eta_0) \in W \) there exist \( i \) and \( j \) for which:

(i) \( \mathbb{R}^{n-1} \times \mathbb{R} \cap \{(y, \eta): |(y, \eta)| = 1, |(y, \eta) - (y_0, \eta_0)| < \delta_0 \} \subset \mathbb{R}^{n-1} \times \mathbb{R} \cap \{(y, \eta): |(y, \eta)| = 1, |(y, \eta) - (y_{ij}, \eta_{ij})| < \delta_j, |y| > (2n)^{-1/2} \).

(ii) If \( |\rho| < \delta_o, (y', \eta') \in \mathbb{R}^{n-1} \times \mathbb{R} \cap \{(y', \eta'): |(y', \eta')| = 1, |(y', \eta') - (y_0, \eta_0)| < \delta_0 \) and \( |z| \in \mathbb{R}^{n-2} \) with \( |z| < \delta_o \) then
\[ |\theta_j(z, \rho, y, \eta)| < C_{\delta_0} \eta, \]

and
\[ \Phi(\sigma_j(z, \rho, y, \eta), \rho, y, \eta) = 0 \]

where we define
\[ \sigma_j(z, \rho, y, \eta) = (z_1, \ldots, \theta_j(z, \rho, y, \eta), \ldots, z_{n-2}) \quad (\theta_j \text{ in } i\text{th place}). \]

**Proof.** Using differentiability of \( h \) together with Lemma 1 and the fact that \( 0 \in B, \) we infer the existence of \( 0 < s_0 < \gamma/4nC, \) and \( 0 < \alpha_0 < 1 \) such that
\[ \text{Clos } X(0, \gamma/2, T_0(B), \alpha_0 s_0) \subset K_{0.6/C}(B) \cap U_n(0, \gamma) \]

\[ \subset X(0, \infty, T_0(B), 1/2) \cup \{0\}. \]

We may assume that \( i = n-1; \) fix \( (y, \eta) \in W_{n-1} \) and let \( 0 < |\rho| < \gamma/2. \)

Then
$0 \neq \rho(y, \eta) \in \text{Clos } X(0, \gamma/2, T_0(B), \alpha_0\delta_0) \subset K_{0,s/C}(B) \cap U_n(0, \gamma)$.

Let $\psi_{\rho(y, \eta)n-1}$ be the $C^\infty$ function found in Part 3. Observing that $i$ is independent of $\rho$, we conclude using Part 3(i), (ii) and continuity of $\Phi$ at $\rho = 0$ that there exists $a_{n-1} \in \mathbb{R}$ such that $\Phi((0, a_{n-1}), 0, \gamma, \eta) = 0$ and $|a_{n-1}| < 2ns < C \eta$. Also we infer from Part 1(ii) that $|D_{a_{n-1}}\Phi((0, a_{n-1}), 0, \gamma, \eta)| > 1/C$. Using the implicit function theorem at each of the points $((0, a_{n-1}), 0, \gamma, \eta) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}$ where $(\gamma, \eta) \in W_{n-1}$ together with the Lebesgue covering lemma, we easily infer (i) and (ii) by noting that if $(\gamma_0, \eta_0) \in W$ then

$\begin{align*}
(\gamma_0, \eta_0) \in \text{Clos } X(0, 1, T_0(B), \alpha_0\delta_0) \subset X(0, \infty, T_0(B), 1/2) \cup \{0\};
\end{align*}$

consequently there exists an $i \in \{1, 2, \ldots, n - 1\}$ such that

$|\gamma_0| > [2(n - 1)]^{-1/2}$

and so $(\gamma_0, \eta_0) \in W_i$.

**Part 5.** There exists $0 < \beta_0 < 1$ such that if $0 < s < \min(\beta_0\delta_0, \delta_0)$, $0 < r < \delta_0$ and $0 \neq w = \rho(0, \eta, \eta) \in K_{0,s/C}(B) \cap U_n(0, r)$ with $|\gamma, \eta| = 1$, then there exist $i \in \{1, 2, \ldots, n - 1\}$ and $j \in \{1, 2, \ldots, m_i\}$ such that whenever $0 \neq w = \rho(y, \eta, \eta) \in K_{0,s/C}(B) \cap U_n(0, r)$ with $|\gamma, \eta| = 1$ and $|\gamma, \eta| - (\gamma_0, \eta_0) < \delta_0$, we have

\[
K_{0,s/C} \cap (w - B) \subset \{(\sigma_i(z, \rho, \gamma, \eta), h \circ \sigma_i(z, \rho, \gamma, \eta)) : |z| < s/C\}.
\]

It follows that

$H^{n-2}[K_{0,s/C} \cap (w - B)] < H_2(s/C)^{n-2}$

where $H_2$ is a real number which does not depend on $s$, $w$ and $w_0$.

**Proof.** We note that there exists $0 < \beta_0 < 1$ such that

$K_{0,\beta_0 s/C}(B) \cap U_n(0, \gamma/2) \subset \text{Clos } X(0, \gamma/2, T_0(B), \alpha_0\delta_0).$

Therefore $(\gamma_0, \eta_0) \in W$; choosing $i \in \{1, 2, \ldots, n - 1\}$ and $j \in \{1, 2, \ldots, m_i\}$ by Part 4(i) we see that if $0 \neq w = \rho(y, \eta, \eta) \in K_{0,s/C}(B) \cap U_n(0, r)$ with $|\gamma, \eta| = 1$ and $|\gamma, \eta| - (\gamma_0, \eta_0) < \delta_0$, then $|\gamma| > (2n)^{-1/2}$. Thus if $(x, h(x)) \in K_{0,s/C} \cap (w - B)$, then by Parts 1 and 3 there exists a $C^\infty$ function $\psi_{\omega,i}$ such that

$\begin{align*}
x = \sigma_i(\pi_i x) = (x_1, \ldots, \psi_{\omega,i}(\pi_i x), \ldots, x_{n-1}).
\end{align*}$

We infer from Part 4(ii) and Part 2 that we must have

$\theta_j(\pi_i x, \rho, \gamma, \eta) = \psi_{\omega,i}(\pi_i x),$

hence $x = \sigma_i(\pi_i x, \rho, \gamma, \eta)$. The desired inclusion follows.

Now the map $F$ defined by

$F(x) = (x, h(x)), \quad x \in U_0,$
is Lipschitzian since $h$ is Lipschitzian on $U_0$; let $M$ be a Lipschitz constant for $F$. Furthermore, $1 + H_0$ is a Lipschitz constant for $\sigma_y$. Therefore,

$$H^{n-2}[K_{\phi,C} \cap (w - B)] < M^{n-2}H^{n-2}\left[\mathbb{R}^{n-1} \cap \{ \sigma_y(z, \rho, \gamma, \eta): |\gamma| < s/C \} \right] < H_2(s/C)^{n-2}$$

where $H_2 = [M(1 + H_0)]^{n-2}H^{-2}[U_{n-2}(0, 1)]$. This completes the proof of Part 5.

Combining Part 3 and Part 5 and noting (4) we get the desired lemma.

**Lemma 3.** Let $A$ be a Suslin subset of $\mathbb{R}^n$. Then $H^{n-1}$ almost all $g \in -B$ satisfy one of the following three conditions:

(i) For some $e > 0$,

$$\lim_{s \to 0^+} \sup_{0 < r < e} (rs)^{-1} \Phi[A \cap K_{g,s}(B) \cap U_n(0, r) \sim \{0\}] = 0.$$  

(ii) For all $e > 0$,

$$\lim_{s \to 0^+} \sup_{0 < r < e} (rs)^{-1} \Phi[A \cap K_{g,s}(B) \cap U_n(0, r) \sim \{0\}] = \infty.$$  

(iii) $0 \in \text{Clos}[A \cap (g + B) \sim \{0\}]$.

**Proof.** The proof is similar to that of [B, 3.7].

Let $r_2$ be as in Lemma 2. We may assume $A \subset U_n(0, r_2)$. Consider the map $F: -B \times B \to \mathbb{R}^n \times -B$ with $F(g, b) = (g + b, g)$. Denote $\Phi = F(-B \times B)$. Also, let

$$\pi_1: \mathbb{R}^n \times (-B) \to \mathbb{R}^n \text{ with } \pi_1(u, g) = u$$

and

$$\pi_2: \mathbb{R}^n \times (-B) \to -B \text{ with } \pi_2(u, g) = g$$

denote the projection maps.

For $u \in \mathbb{R}^n$ denote $\Phi_u = \pi_1^{-1}(u) \cap \Phi$. Thus

$$\Phi_u = \{(u, g): g \in (u - B) \cap (-B)\},$$

and it follows that for $g \in -B$ and $s > 0$

$$\Phi_u \cap \pi_2^{-1}(K_{g,s}) = \{(u, h): h \in K_{g,s} \cap (u - B)\}.$$

Let $\phi'$ be the measure on $\Phi$ such that for $S \subset \Phi$,

$$\phi'(S) = \int_A H^{n-2}(\Phi_u \cap S) \, d\phi_u.$$  

Now, for each positive integer $n$ we introduce the measure $\Psi_n$ over $(-B)$ defined by

$$\Psi_n(T) = \sup_{0 < r < 1/n} \phi'[\pi_1^{-1}(U_n(0, r)) \cap \pi_2^{-1}(T) \cap \Phi]r^{-1}$$
for $T \subset -B$. Let
\[ P_r = (-B) \cap \left\{ g: \lim_{s \to 0^+} \left[ \Psi_{s} (K_{g,s}) / s^{\alpha-1} \right] = 0 \right\}, \]
\[ Q_r = (-B) \cap \left\{ g: \lim_{s \to 0^+} \left[ \Psi_{s} (K_{g,s}) / s^{\alpha-1} \right] = \infty \right\}, \]
\[ R_r = \pi_2 \left[ \pi_1^{-1} (A \cap U_n(0, r^{-1})) \cap \Phi \right], \]
\[ P = \bigcup_{r=1}^{\infty} P_r, \quad Q = \bigcap_{r=1}^{\infty} Q_r, \quad R = \bigcap_{r=1}^{\infty} R_r. \]

It follows that $H^{\alpha-1}((-B) \sim (P \cup Q \cup R)) = 0$.

Using Lemma 2 we complete the proof by proceeding as in the proof of [B, 3.7].

**Lemma 4.** Let $A$ be a Suslin subset of $R^n$ with $\phi(A) < \infty$. Then
\[ \phi \times H^{\alpha-1} [ A \times B \sim (S_{A,1} \cup S_{A,2} \cup S_{A,3}) ] = 0. \]

**Proof.** Let us fix $a \in R^n$ and $(a, b) \in \{a\} \times B$. We note that $\tau_{a}(A)$ is a Suslin set. Also $(\tau_{a})_{\#} \phi$ is a nonnegative measure such that closed sets are $(\tau_{a})_{\#} \phi$ measurable. Consequently, replacing $A$ and $\phi$ in Lemma 3 by $\tau_{a}(A)$ and $(\tau_{a})_{\#} \phi$ we see that
\[ H^{\alpha-1} [ \{a\} \times B \sim S_{A,1} \cup S_{A,2} \cup S_{A,3} ] = 0. \]

We infer from [B, 4.1–4.2] that $S_{A,1}$, $S_{A,2}$, $S_{A,3}$ are Suslin sets. Also, $\phi(A) < \infty$, hence we can apply Fubini's theorem to obtain our assertion.

**Lemma 5.** If $A$ is a purely $(\phi, 1)$ unrectifiable Suslin subset of $R^n$ such that $\phi(A) < \infty$ and $\phi(W) = 0$ whenever $W \subset A$ and $H^1(W) = 0$, then
\[ \phi \times H^{\alpha-1} [ A \times B \cap S_{A,1} ] = 0. \]

**Proof.** This is the special case of [B, 4.6] where $G = X = R^n$.

**Lemma 6.** If $A$ is a Suslin subset of $R^n$ and $\phi(A) < \infty$, then
\[ H^\alpha [ f_0 (A \times B \cap S_{A,2}) ] = 0. \]

**Proof.** This is immediate from [B, 4.7] with $G = X = R^n$.

**Lemma 7.** Let $A$ be a Suslin subset of $R^n$ with $H^1(A) < \infty$. Then
\[ H^\alpha [ f_0 (A \times B \cap S_{A,3}) ] = 0. \]

**Proof.** Since $B$ is separable it is sufficient to show that
\[ H^\alpha [ f_0 (A \times B_0 \cap S_{A,3}) ] = 0 \]
where $B_0 = B \cap U_0$, $U_0$ being an open subset of $R^n$ with $H^{\alpha-1}(B_0) < \infty$. 
Now, by [F1, 2.10.45] and [F1, 2.10.25] we conclude that
\[ \int_{R^n} H^n(A \times B_0 \cap f_0^{-1} \{ g \}) \, dH^g \leq C_1 H^n(A \times B_0) < \infty \]
where \( C_1 \) is a positive constant. Thus
\[ H^n \left( R^n \cap \left\{ g : H^0(A \times B_0 \cap f_0^{-1} \{ g \}) = \infty \right\} \right) = 0, \]
and it is not difficult to show that
\[ f_0[A \times B_0 \cap S_{A,3}] \subset \left\{ g : H^0(A \times B_0 \cap f_0^{-1} \{ g \}) = \infty \right\}. \]

**Theorem 1.** Suppose \( E \subset R^n \) with \( H^1(E) < \infty \). Then there exists a countably 1-rectifiable Borel subset \( R \) of \( R^n \) such that \( (E \sim R) \) is purely \((H^1, 1)\) unrectifiable and
\[ (E \sim R) \cap (g + B) = \emptyset \]
for \( H^n \) almost all \( g \in R^n \).

Furthermore, if in addition \( E \) is \( H^1 \) measurable and \( E \cap (g + B) = \emptyset \) for \( H^n \) almost all \( g \in R^n \), then \( H^1(E \cap R) = 0 \), hence \( E \) is purely \((H^1, 1)\) unrectifiable.

**Proof.** Since \( H^1 \) is Borel regular, we may assume \( E \) to be Borel. By maximizing the finite measure \( H^1 |_E \) on the class of countably 1-rectifiable Borel subsets of \( R^n \) and using [F1, 3.2.14] we obtain a countably 1-rectifiable Borel subset \( R \) of \( R^n \) such that \( A = (E \sim R) \) is purely \((H^1, 1)\) unrectifiable. Applying [B, 4.1 and 4.2] with \( G = X = R^n \) we infer that \( S_{A,1}, S_{A,2}, S_{A,3} \) are Suslin sets. Using Lemmas 4–7 together with [F1, 2.10.25] we easily conclude that
\[ H^n \left[ f_0(A \times B) \right] = 0 \]
which is equivalent to \( A \cap (g + B) = \emptyset \) for \( H^n \) almost all \( g \in R^n \).

Now let \( E \) be \( H^1 \) measurable with \( E \cap (g + B) = \emptyset \) for \( H^n \) almost all \( g \in R^n \). Observe that by [F1, 3.2.28] we may assume \( R \) to be a subset of a proper 1-dimensional submanifold \( R_0 \) of class 1 of \( R^n \). We may also assume \( 0 \in R \) and \( Re_n = T_0(R_0) \) where \( \{e_1, \ldots, e_n\} \) is the standard orthonormal basis for \( T_0(R^n) = R^n \). Let
\[ M = S^{n-1} \cap \{ u : u \cdot e_n = 0 \}. \]

Assuming \( B \) is oriented with a unit normal vector field \( v \), let us consider the Gauss map \( \eta : B \to S^{n-1} \) defined by \( \eta(b) = v(b) \in R^n \) for \( b \in B \). Since the Gaussian curvature is nonzero at every point of \( B \), \( \eta \) has nonzero Jacobian at every point. Therefore by the inverse function theorem \( \eta(B) \) is an open subset of \( S^{n-1} \) and thus \( H^{n-1}[\eta(B)] > 0 \). There therefore exists \( b \in B \) at
which \(\eta(b) \not\subset M\). We can assume \(b = 0\); thus \(e_n \not\subset T_0(B)\) and we see that \(R e_n + T_0(B) = \mathbb{R}^n\).

For \(r > 0\) denote \(B_r = B \cap U_n(0, r)\). Since \((E \cap R) \times B_r\) is \(H^n\) measurable application of [F1, 3.2.3] gives

\[
\int_{(E \cap R) \times B_r} J(f_0| R_0 \times B) \, dH^n = \int_{\mathbb{R}^n} H^0\left[\left(\int_0^1 (E \cap R_0) \times B_r \right)^{-1}\{g\}\right] \, dH^n g.
\]

Now the integral on the right is zero by our hypothesis. Moreover, if \(\{u_1, \ldots, u_n-1\}\) is an orthonormal basis of \(T_0(B)\) then

\[
J(f_0| R_0 \times B)(0, 0) = |e_n \wedge u_1 \wedge \cdots \wedge u_{n-1}| > 0.
\]

Consequently, \(H^n((E \cap R) \times B_r) = 0\) for some \(r > 0\), whence we conclude using [F1, 3.2.25] that \(H^1(E \cap R) = 0\).

If \(u \in \mathbb{R}^n\), the \(k\)-dimensional upper density of \(\phi\) at \(u\) is

\[
\theta^{*k}(\phi, u) = \lim \sup_{r \to 0^+} \alpha(k)^{-1} r^{-k} \phi(U_k(u, r))
\]

where \(\alpha(k)\) is the volume of the unit \(k\)-ball \(U_k(0, 1)\).

**Theorem 2.** Suppose \(W \subset \mathbb{R}^n\), \(\phi(W) < \infty\), \(\phi(S) = 0\) whenever \(S \subset W\) and \(H^1(S) = 0\) and \(\theta^{*k}(\phi \nmid W, u) > 0\) for \(\phi\) almost all \(u \in W\). Then there exists a countably \((\phi, 1)\)-rectifiable and \(\phi\) measurable set \(Q\) such that \((W \sim Q)\) is purely \((\phi, 1)\) unrectifiable and

\[
(W \sim Q) \cap (g + B) = \emptyset
\]

for \(H^n\) almost all \(g \in \mathbb{R}^n\).

Furthermore, if in addition \(W\) is a Borel set such that \(W \cap (g + B) = \emptyset\) for \(H^n\) almost all \(g \in \mathbb{R}^n\), then \(W\) is purely \((\phi, 1)\) unrectifiable.

**Proof.** Applying Theorem 1 one proceeds in a manner similar to the proof of [B, 5.3].

**Bibliography**


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