BENDER GROUPS AS STANDARD SUBGROUPS

BY

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Abstract. A subgroup X of a finite group G is called *-standard if
X = X/O(X) is quasisimple, Y = C_G(X) is tightly embedded in G and
N_G(X) = N_G(Y). This generalizes the notion of standard subgroups.

Theorem. Let G be a finite group with O(G) = 1. Suppose X is *-standard
in G and X/Z(X) = L_2(2^n), U_3(2^n) or Sz(2^n). Assume X ⊆ G. Then O(X)
= 1 and one of the following holds:

(i) E(G) ≅ X × X.
(ii) X ≅ L_2(2^n) and E(G) ≅ L_2(2^{2n}), U_3(2^n) or L_3(2^n).
(iii) X ≅ U_3(2^n) and E(G) ≅ L_3(2^{2n}).
(iv) X ≅ Sz(2^n) and E(G) ≅ Sp(4, 2^n).
(v) X ≅ L_2(4) and E(G) = M_{12}, A_9, J_1, J_2, A_7, L_2(25), L_3(5) or U_3(5).
(vi) X ≅ Sz(8) and E(G) = Ru (the Rudvalis group).
(vii) X ≅ L_3(4) and E(G) = G_2(3).
(viii) X ≅ SL(2, 5) and G has sectional 2-rank at most 4.

In particular, if G is simple, G = M_{12}, A_9, J_1, J_2, Ru, U_3(5), L_3(5), G_2(5),
or 3D_4(5).

1. Introduction. This paper is concerned with those finite groups G
containing a standard subgroup of Bender type. Actually we deal with a more
general situation as we allow for cores.

A subgroup X of a finite group G is called *-standard if \( \tilde{X} = X/O(X) \)
is quasisimple, \( Y = C_G(\tilde{X}) \) is tightly embedded in G and \( N_G(X) = N_G(Y) \).
A standard subgroup (in the sense of Aschbacher [1]) is clearly *-standard.

We classify finite groups with a *-standard subgroup of Bender type.

Theorem. Let G be a finite group with O(G) = 1. Suppose X is *-standard
in G and \( \tilde{X}/Z(\tilde{X}) = L_2(2^n), U_3(2^n), \) or Sz(2^n). Assume that X ⊆ G. Then
O(X) = 1 and one of the following holds:

(i) E(G) ≅ X × X.
(ii) X ≅ L_2(2^n) and E(G) ≅ L_2(2^{2n}), U_3(2^n), or L_3(2^n).
(iii) X ≅ U_3(2^n) and E(G) ≅ L_3(2^{2n}).
(iv) X ≅ Sz(2^n) and E(G) ≅ Sp(4, 2^n).

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(v) \( X \cong L_2(4) \) and \( E(G) \cong M_{12}, A_9, J_1, J_2, A_7, L_2(25), L_3(5), \) or \( U_3(5). \)
(vi) \( X \cong Sz(8) \) and \( E(G) \cong Ru \) (the Rudvalis group).
(vii) \( X \cong L_2(8) \) and \( E(G) = G_2(3). \)
(viii) \( A^* \cong SL(2, 5), \) \( G \) has sectional 2-rank at most 4, so by [12], \( E(G) \cong U_3(5), L_3(5), G_2(5), \) or \( 3D_4(5). \)

In particular, if \( G \) is simple then \( G = M_{12}, A_9, J_1, J_2, Ru, U_3(5), L_3(5), G_2(5), \) or \( 3D_4(5). \)

Let \( G \) and \( X \) be as in the main theorem with \( X \not\subset G \) and let \( T_0 \) be a Sylow 2-subgroup of \( Y. \) Then, except in cases (v) and (vi), \( |T_0| = 2 \) and \( T_0 \) induces an outer automorphism on \( E(G). \) This shows that if \( X \) is a standard subgroup and \( m(C_G(Y)) > 1, \) then the conclusion of the main theorem in [3] holds.

The proof of the main theorem involves a “pushing up” procedure. Starting from a Sylow 2-subgroup of \( M = N_G(X), \) we attempt to find a Sylow 2-subgroup of \( G. \) At each stage of the procedure there occurs a certain 2-transitive group and this permutation group either has a regular normal 2-subgroup or a normal subgroup isomorphic to \( L_3(2). \) In all cases except (vi) and (vii) we show that the latter does not occur. When \( E(G) = G_2(3) \) an \( L_3(2) \) does occur at the first step in the process, while for \( E(G) = Ru, \) a factor of \( L_3(2) \) occurs in the second step of the process.

The method of proof eventually reduces us to a situation where we may quote a previous characterization theorem. In particular, we will use the work of Goldschmidt [11] and Gilman and Gorenstein [10] in the identification of \( E(G). \) In the exceptional cases (v), (vi) and (vii) we also use Aschbacher [2], Dempwolff [6], Assa [4], O'Nan [19], and Harada [14].

The paper is organized so that §2 contains preliminary lemmas and §3 basic reductions together with the first step of the “pushing up” process. Then §§4, 5, 6 deal with the cases \( X \cong L_2(2^n), Sz(2^n), U_3(2^n), \) respectively.

2. Preliminaries. The first lemma deals with tightly embedded subgroups in the automorphism group of a Bender group.

(2.1) Let \( X \) be a simple Bender group and \( X < Y < \text{Aut}(X). \) If \( X < F \) and \( F \) is a tightly embedded subgroup of \( Y, \) then one of the following holds:

(i) \( F \cap X \) lies in the normalizer of a Sylow 2-subgroup of \( X, \) has even order, and contains every involution of \( F. \)
(ii) \( F \cap X = 1, |F| = 2, \) and \( F \) induces a field automorphism on \( X. \)
(iii) \( F = (F \cap X) \langle t \rangle, \) where \( |F \cap X| \) is odd, and \( t \) induces a field automorphism of order 2 on \( X \cong L_2(4) \) or \( U_3(2^n). \) If \( X \cong L_2(4), \) then \( F \cap X \cong Z_3, \) and if \( X \cong U_3(2^n), \) \( F \cap X \neq 1 \) is cyclic of order dividing \( 2^n + 1 \) and \( F \cap X \) centralizes \( E(C_X(t)) \cong L_2(2^n). \)

PROOF. Suppose \( t \in F \cap X \) is an involution. Then \( t \) is central in a Sylow
2-subgroup $U$ of $X$, so that $U$ normalizes $F$ and $U(F \cap X)$ is a group. It follows that $U(F \cap X) \leq N_X(U)$ (see (1.6) of [19]) and, consequently, $F \cap X$ fixes a unique point in the usual 2-transitive permutation representation of $X$. From here we have $F \leq N(U)$ as $U$ is the unique Sylow 2-subgroup of the stabilizer of that point. If $F - (F \cap X)$ contained an involution $j$, then $C_X(j) \leq N(F)$, whereas $j$ must induce a field automorphism of $X$ and $C_X(j)$ does not contain a normal Sylow 2-subgroup. We have now verified (i).

Assume now that $|F \cap X|$ is odd and $t$ is an involution in $F$. So $t$ induces a field automorphism on $X$ and, by [22], $X \cong L_2(2^n)$ or $U_3(2^n)$. So $C_X(t) \cong L_2(2^{n/2})$ or $L_2(2^n)$, respectively, and this group normalizes $F$. Let $V$ be a Sylow 2-subgroup of $C_X(t)$. We may assume $C_X(t) \cong L_2(q_0)$ with $q_0 > 4$, as otherwise the result is trivial. So we may write $F \cap X = \langle C_{F \cap X}(v) : v \in V^* \rangle$. If $V \leq U \leq \text{Syl}_2(X)$, then $C_{F \cap X}(v) \leq N_X(U)$ for each $v \in V^*$. Say $F \cap X \neq 1$. Then from the structure of $N_X(U)$ we conclude that $X \cong U_3(2^n)$, $n > 2$, $F \cap X$ is cyclic of order dividing $2^n + 1$, and $[F \cap X, C_X(t)] = 1$.

In any case $[C_X(t), F] < F \cap X$, and the above implies $[C_X(t), F] = 1$ for $q_0 > 4$. This implies that $F = (F \cap X) \langle t \rangle$, and we have either (ii) or (iii).

The next several lemmas deal with 2-groups and their automorphism groups.

(2.2) Let $U$ be a 2-group of order $q^2$ and $Y$ a cyclic group of order $q - 1$ acting fixed-point-free on $U$. Let $V < U$ be $Y$-invariant and such that $U/V$ and $V$ are elementary and equivalent as $F_2(Y)$-modules. Then $U$ is abelian.

**Proof.** Higman [17].

(2.3) Let $UY$ be as in (2.2) and suppose that $T$ is a 2-group of order $q^2$, normalized by $UY$, $[T, U] \leq T \cap U = V$, and $Y$ is fixed-point-free on $T$. Then one of the following holds:

(i) $[T, U] = 1$.

(ii) For any $t \in T - V$, $u \in U - V$, $[t, u] \neq 1$.

**Proof.** This is proved using Lie ring methods. See Dempwolff [6, Lemma 1.1] .

(2.4) Let $U$ be a 2-group and $\langle t \rangle \times Y$ acting on $U$ with $t$ an involution and $Y$ cyclic of order $2^n - 1$. Suppose that $Y$ is regular on $C_U(t)^*$. Then one of the following holds:

(i) $U$ is isomorphic to a Sylow 2-subgroup of $L_2(2^n)$.

(ii) $U$ is isomorphic to a Sylow 2-subgroup of $U_3(2^n)$.

(iii) $U$ is homocyclic of rank $n$ and inverted by $t$.

(iv) $U$ is homocyclic of rank $n$ and each involution in $U\langle t \rangle - U$ is $U$-conjugate to $t$. 

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(v) $U$ is elementary abelian of order $2^{2n}$ and each involution in $U \langle t \rangle - U$ is $U$-conjugate to $t$.

**Proof.** This is essentially contained in Finkelstein [8, Lemmas 2.1 and 2.2]. However, instead of (iv) and (v) he simply states that $U$ is abelian and each involution in $U \langle t \rangle - U$ is $U$-conjugate to $t$. If $U$ is not homocyclic of rank $n$, then using the action of $Y$ we have $|\Omega_1(U)| \geq 2^{2n}$. As $|C_U(t)| = 2^n$, this must be an equality. Here the only involutions in $\Omega_1(U) \langle t \rangle$ are in $\Omega_1(U)$ or in $tC_U(t)$ and

$$t^U \cap \Omega_1(U) \langle t \rangle = t^{\Omega_1(U)} = tC_U(t).$$

Consequently, $U = \Omega_1(U)$ and (v) holds.

(2.5) Let $A = A_1 \times A_0$ be an elementary abelian 2-group, $|A_0| = 2$, $A \triangleleft N$, $R = O_2(N)$. Suppose also that $N$ contains a cyclic subgroup $K$ which operates regularly on $(R/A)^\#$ and on $A_1^\#$. If $C_R(A_0) = A$, then $C_N(A_0)$ covers $N/R$.

**Proof.** Assume $C_R(A_0) = A$. Then the action of $K$ on $A$ forces $A_1 = Z(R)$. Consequently, if $A_0 = \langle t \rangle$, then $t^N \subseteq A_1t$. On the other hand, the hypotheses force $|R/A| = |A_1|$ and $t^R = A_1t$. The result follows.

The following is a useful result of Goldschmidt.

(2.6) Let $T \in \text{Syl}_2(G)$, $W$ a weakly closed subgroup of $T$ (with respect to $G$), and $A$ an abelian subgroup of $C_T(W)$, normal in $T$. Let $\mathcal{S} = \{B \leq T: B \triangleleft A, B \text{ is conjugate to a subgroup of } A\}$ and set $r = \max\{m(B/C_B(W)): B \in \mathcal{S}\}$. Then either

(i) $\Omega_1(A)$ is strongly closed in $T$ (with respect to $G$); or

(ii) there exists $B \in \mathcal{S}$ such that $m(B) + r > m(A)$; also if $t \in T$ is conjugate to an element of $A$, then $m([A, t]) < 2r$, with $m[A, t] < r$ provided $B/C_B(W)$ is elementary for each $B \in \mathcal{S}$.

**Proof.** Theorem 4 of [11].

The following results are the key to the determination of the Sylow 2-subgroup in a group $G$ satisfying the hypotheses of the main theorem.

We consider groups $G$ satisfying the following.

**Hypothesis (\(*\).** (1) $R \triangleleft G$ is elementary and a Sylow 2-subgroup of a tightly embedded subgroup $K$ of $G$.

(2) There is a subgroup $X \triangleleft N_G(R)$ such that $X \leq C_G(R)$, and if $U \in \text{Syl}_2(X)$, then $U$ is elementary of order $q = 2^n > 4$, and $N_X(U)/C_X(U)$ is cyclic of order $q-1$ and is regular on $U^\#$.

(3) For $S \in \text{Syl}_2(N_G(R))$ with $U \times R = V \trianglelefteq S$, $S/V$ is faithful on $U$.

(2.7) Assume that $G$ satisfies Hypothesis (\(*\)). Then one of the following holds:

(a) $S \in \text{Syl}_2(G)$ and $V$ is strongly closed in $S$. 

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(b) $S_1 \in \text{Syl}_2(G)$ with $|S_1 : S| = 2$, $S_1$ acts on $S$ interchanging $U$ and $R$, and $V$ is strongly closed in $S$.

(c) Each of the following holds:

(i) $S = \bigcup \{(R^g) : g \in G, R^g \subset V\}$.

(ii) $N(V)$ is 2-transitive of degree $q$ on $\Delta = R^G \cap V$.

(iii) Either $N(V)^A$ contains the Frobenius group of order $q(q - 1)$ as a normal subgroup, or $q = 8$ and $(N(V)^A)' \cong L_3(2)$.

**Proof.** Suppose that $G$ satisfies Hypothesis (*), and that (a) and (b) are false. First note that we may regard $U$ as $F_q$ with $N_X(U)/C_X(U)$ acting as scalar multiplications and $S/V$ acting as field automorphisms.

We first claim that $S \in \text{Syl}_2(G)$. Otherwise we set $S = T$, $V = W = A$ in (2.6). As $q > 4$, $V$ is weakly closed in $S$. So the lemma applies and $r < 1$. But for $q > 4$ this is impossible. Consequently, $S \in \text{Syl}_2(G)$.

As $V$ is weakly closed in $S$, $N_G(S) < N_G(V)$ so $V$ contains more than one conjugate of $R$. Applying (3.6) of [3] (which is independent of any results in this paper) we have (i) and (ii) provided we can show that $G \cap K = \{R, U\}$. So suppose this latter case occurs. Let $y \in N(S) - S$ with $y^2 \in S$. Then $U = R^y$. Set $S_1 = S(y)$. It is easily checked that $V$ is weakly closed in $S_1$, and, since $R^G \cap V = \{R, R^y\} = R^{S_1}$, we have $S_1 \in \text{Syl}_2(G)$. Again we appeal to (2.6) to get a contradiction. At this point we have established (i) and (ii).

Now consider the 2-transitive group $N(V)^A$. The stabilizer of $R$ in $N(V)$ will normalize $X$ and, hence, will normalize $N_X(U) = N_X(V \cap X)$. This implies (using (2) of Hypothesis (*)) that $N(V)^A$ satisfies the conditions of Theorem 1.1 of Hering, Kantor and Seitz [16]. We conclude that either $N(V)^A$ has a regular normal subgroup, so that (iii) holds, or $N(V)^A$ contains $\text{PSL}(2, p)$ acting in its usual 2-transitive representation of degree $p + 1$. Suppose the latter case holds. Then $p + 1 = q = 2^a$ and $p$ is a Mersenne prime. If we consider $N(V)^A \cap N(R)$, then this group acts on $U$ inducing a Frobenius group of order $(q-2)(q-1)$. This forces $\frac{1}{2}(q-2)$ to divide $n$, and hence $n = 3$, completing the proof of (iii).

(2.8) Suppose that $G$ satisfies Hypothesis (*), $V$ is not strongly closed in a Sylow 2-subgroup of $G$, and that conditions (i)-(iii) of (2.7) hold with $(N(V)^A)$ containing a regular normal subgroup. Let $D$ be a 2-complement of $N_X(U)$. Then there is a Sylow 2-subgroup $V_1$ of $O_{2^n}(N(V))$ and a $2^r$-group $D_1$, with the following properties:

(a) $SD_1 < N(V)$, $V_1 = \text{Syl}_2(N(V))$, and $D_1$ induces $D$ on $V$.

(b) $V_1 = U_1R$ with $U_1 \cap R = 1$, where $U_1 = [D_1, V_1]$.

(c) $U < U_1$, and $U_1/U$ and $U$ are equivalent $F_2(D_1)$-modules.

**Proof.** The existence of $V_1$ and $D_1$ satisfying (a) is easy. By (2.7)(i) $U$ is
characteristic in $V$ and we consider $V_1/U$. Suppose that $\Omega_1(V_1/U) = V/U$. Then $V = \Omega_1(V_1)$ is weakly closed in $V_1S$ and $V_1S \in \text{Syl}_2(G)$. However, we can now apply (2.6) to conclude that $V$ is strongly closed in $V_1S$, contradicting our hypothesis.

So $\Omega_1(V_1/U) > V/U$ and, since $D_1$ is transitive on $(V_1/V)\#$, each coset of $V/U$ in $V_1/U$ contains an involution. Since $D$ centralizes $R$ we must have $D_1V_1$ centralizing $V/U$. It follows that $V_1/U$ is elementary. From here (b) follows as well as the first claim in (c). Finally we get the last statement in (c) by letting $r \in R\#$ and noting that the map $u_1U \to [u_1, t]$ is a $D_1$-homomorphism from $U_1/U$ to $U$. The proof is complete.

We next make the observation that the above may be repeated. Namely, suppose that Hypothesis (*) holds for $G$, $V$ is not strongly closed in a Sylow 2-subgroup of $G$, and $N(V)\alpha$ contains a regular normal subgroup. Choose $D_1$ and $V_1$ as in (2.8) and consider $G_1 = G = NG(U)/U$. Then for $g \in G_1$, $Vg \leqslant U\alpha$ implies $Vg = V$ or $Vg \cap V = 1$. With this we can argue as in (2.7) and (2.8).

Suppose now that the process is repeated until at some stage either the induced 2-transitive group does in fact contain $L_3(2)$ as a normal subgroup or the analogue of $V$ in $G_m$ is strongly closed in a Sylow 2-subgroup of $G_m$ and (a) or (b) of (2.7) holds. Assume that the process terminates in the latter way. Then there is a subgroup $D_m$ and a 2-group $V_m$ of $G$ such that $[D_m, V_m] = U_m > U$, $R$ normalizes $U_m$, $R \cap U_m = 1$, $V_m = U_mR$, each $D_m$-composition factor of $U_m$ is isomorphic to $U$ and $D_m$ induces $D$ on $U$. Also $S < N(U_m)$ and $SU_m \in \text{Syl}_2(N_G(V_m\alpha))$.

For $r \in R\#$, $C_{U_m}(r) = N_{U_m}(r) = U$, so $U_m$ satisfies the hypotheses of (2.4). With this notation we can conclude:

(2.9) Let $G$ satisfy Hypothesis (*) and suppose that the above process does not yield the $L_3(2)$ case at any stage. Let $U_m$ be as above. Then one of the following holds:

1. $U_m$ is isomorphic to a Sylow 2-subgroup of $U_3(q)$ or $L_3(q)$ and $U_mS \in \text{Syl}_2(G)$.
2. $U < U_m$ which is homocyclic of rank $n$ and $U_mS \in \text{Syl}_2(G)$.
3. $V$ is strongly closed in a Sylow 2-subgroup of $G$.
4. $U_m = U$ is elementary of order $q^2$.

Proof. We may assume $U < U_m$, as otherwise (3) follows as in (2.7). Also we assume that $U_m$ does not satisfy (v) of (2.4), as otherwise (4) holds. Suppose that $U_mS$ is normalized by an element $y \in G - U_mS$ and $y^2 \in U_mS$. We first show that $y$ normalizes $U$. If $S = RU$, then $U = Z(U_mS)$ and this is clear. Suppose $S > RU$. If $S' < U$, then $S'$ contains an element fused to an element in $R\#$. In this case $U = C_{U_mS}((U_mS))$ so $y$ normalizes $U$ as claimed. So we suppose that $S' < U$ and, hence, $(U_mS)' < U_m$. If $U_m$
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satisfies (i) or (ii) of (2.4) then \((U_mS)' > [U_m, R]\) which is homocyclic of order \(2^{2n}\) and of rank \(n\). So here \(U = \Omega_i((U_mS)')\) if \((U_mS)' = [U_m, R]\) and \(U = Z((U_mS)')\) if \((U_mS)' > [U_m, R]\). Either way \(y \in N(U)\). If \(U_m\) satisfies (iii) or (iv) of (2.4), then \((U_mS)' > U_{m-1}\), so \(U = \Omega_i((U_mS)')\) is normalized by \(y\). So in all cases the claim holds.

In particular, \(y\) normalizes \(C_{U_mS}(U) = U_mR\). But then \(y\) normalizes \((U_mR)' = U_{m-1}\). Hence \(U_mS \leq \text{Syl}_2(N_G(U_{m-1}))\) so that we are in case (b) of (2.7). But then (4) holds. Thus we may assume \(U_mS \in \text{Syl}_2(G)\). We complete the proof by using (2.4) to get the structure of \(U_m\).

3. Initial reductions. Let \(G\) be a finite group having a \(*\)-standard subgroup \(M_1\) such that \(\tilde{M}_1\) is a Bender group and the conclusions of the main theorem are violated. Choose \(|G|\) minimal and \(M_1\) minimal in the group \(G\). Let \(M = N_G(M_1)\) and \(M_0 = C_M(M_1/O(M_1))\).

Choose \(T \in \text{Syl}_2(M)\) and set \(T_i = T \cap M_i\), \(i = 0, 1\). Then \(T = T_1T_0T_3\) where \(T_3\) is cyclic. We set \(q = |\Omega_i(T_1)|\), so that \(q = 2^m\) and \(\tilde{M}_1 \cong L_2(2^m), Sz(2^m), \text{ or } U_3(2^m)\), unless \(M_1\) is a perfect central extension of \(Sz(8)\), when we set \(q = 8, m = 3, \text{ or } M_1 \cong SL(2, 5)\), when we set \(q = 4, m = 2\). Let \(K_1\) be a 2-complement in \(N_{M_1}(T_0T_1)\) and \(K = K_1^{-1}.\) Finally set \(A_i = \Omega_i(T_1)\) and \(A = A_1A_0\).

The above notation will be maintained throughout the rest of the paper.

(3.1) \(M_1 = \langle C_{M_1}(t): t \in \text{Inv}(T_0)\rangle\).

**Proof.** \(C_{M_1}(T_0)\) covers \(M_1/O(M_1)\). So if \(m(T_0) > 1\) the result is clear. If \(m(T_0) = 1\), then it is easy to check that \(C_{M_1}(\Omega_i(T_0))\) is a \(*\)-standard subgroup, so by minimality of \(M_1\) we again have the result.

(3.2) \(F(G) = 1\).

**Proof.** By hypothesis we have \(O(G) = 1\). Suppose \(O_2(G) \neq 1\). For each involution \(t \in T_0\), the tight embedding property implies that \(C_{O_2(G)}(t)\) centralizes \(M_1\). Now (3.1) and the \(P \times Q\) lemma imply that \(\tilde{M}_1 \leq C_G(O_2(G))\). But then \(O_2(G) \leq T_0\), so \(G \leq N(O_2(G)) \leq N(M_1)\) and \(m_1 \leq G\), a contradiction.

(3.3) There does not exist a normal subgroup \(1 < N \trianglelefteq G\) such that \(N\) has Sylow 2-subgroups of class at most 2.

**Proof.** If such an \(N\) exists, then using (3.2) and the result of Gilman and Gorenstein [10], the structure of \(N\) is known. Consideration of the action of \(T_0\) on \(E(N)\) gives a contradiction.

Similarly, we have

(3.4) \(G\) does not contain a normal subgroup \(1 < N \trianglelefteq G\) such that a Sylow
2-subgroup $S$ of $N$ contains an abelian subgroup $A$ with $A$ strongly closed in $S$ with respect to $N$.

**Proof.** Use (3.2) and Goldschmidt's theorem [11].

(3.5) (a) $G = \langle T_0^G \rangle$.
(b) $|G : O^2(G)| < 2$. If the index is 2, then $G = O^2(G)T_0$ and $T_0 \cap O^2(G) = 1$. In particular, $|T_0| = 2$ in this case.

**Proof.** Set $G_0 = \langle T_0^G \rangle$ and suppose $G_0 < G$. If $M_1 \cap G_0 \nleq Z^*(M_1)$, then $M_1 \cap G_0$ is a *-standard subgroup in $N$ and, by minimality of $G$, the structure of $E(G_0)$ is known, from which we have a contradiction.

Suppose that $M_1 \cap G_0 \nleq Z^*(M_1)$. We claim that $T_0 \in \text{Syl}_2(G_0)$. Otherwise, let $X > T_0$ be a 2-subgroup of $G_0$ normalizing $T_0$. Then $X < N(M_1)$, so $[M_1, X] \nleq M_1 \cap G_0 \nleq Z^*(M_1)$. But this forces $X < M_0$ impossible. Consequently, $T_0 \in \text{Syl}_2(G_0)$ and $G = G_0N_G(T_0) = G_0M$. It follows that $M \cap G_0$ is strongly embedded in $G_0$, so using Bender's theorem [5] we have a contradiction. This proves (a).

For (b) use the minimality of $G$.

(3.6) There exists $g \in G - M$ such that $1 \neq R = T_0^g \cap M < T$.
(i) $R \cap T_0 = 1$.
(ii) If $m(T_0) > 1$, then $g$ can be chosen such that $R = T_0^g$.
(iii) If $|R| > 2$, then $\Omega_1(R) \leq \Omega_1(T_1)T_0$.
(iv) If $m(T_0) > 1$, then $T_0$ is elementary abelian.
(v) If $m(T_0) > 3$, then $R = T_0^g$ for all such $g$.

**Proof.** If $m(T_0) = 1$, then we apply (3.2) and the $Z^*$-theorem of Glau-berman. Also, in any case, (i) follows from the tight embedding property. We now assume that $m(T_0) > 1$.

At this point we apply the work of Aschbacher [1]. Theorems 1 and 3 of [1] apply directly, while the proof of Theorem 2 carries over with just one change. Namely at a certain point Aschbacher uses $[M_0, M_0^g] \neq 1$ for any $g \in G$ and his Hypotheses II to conclude that (iii) holds. However, in our case, (iii) follows as in the proof of (2.1). So we may apply the theorems in [1] to obtain (3.6) in the case $m(T_0) > 1$.

(3.7) Suppose that $m(T_0) > 1$. Then:
(i) There is no subgroup $G_0 < G$ such that $T < G_0$, $M_1 = O(M_1)(M_1 \cap G_0)$, and $M_1 \cap G_0$ is a *-standard subgroup of $G_0$, but $M_1 \cap G_0 \nleq G_0$.
(ii) $O(M) = 1$.

**Proof.** Suppose that $m(T_0) > 1$. First we show that (i) implies (ii). So assume (i) to hold, but (ii) false. Let $p$ be a prime divisor of $|O(M)|$ and $P_0$ a $T$-invariant Sylow $p$-subgroup of $O(M)$. Extend $P_0$ to a $T$-invariant Sylow
$p$-subgroup, $P$, of $M_0 \cap C(T_0T_1O(M)/O(M))$. As $[M_1,P] < [M_1,M_0] < O(M)$, $N_{M_1}(P)$ covers $\tilde{M}_1$.

Let $g \in G - M$ be as in (3.6)(ii). Since $P = \langle C_T(t): t \in (T_0^g)^* \rangle$, $P < M^g$.

It is easily checked that if $\tilde{M}_1 \neq L_2(4)$, then $T_1T_0/T_0$ is the unique group of its isomorphism type in $T/T_0$. Applying this to $T^g/T_0^g$ we have $T_1T_0 \simeq (T_1^gT_0^g)^{\tilde{M}_1} \times \tilde{M}_1^g$ and the structure of $M_1^g$ forces $P < M_1^g$. If $\tilde{M}_1 \simeq L_2(4)$ or $\text{SL}(2,5)$, this also holds, so in all cases $N_{M_1}(P)$ covers $\tilde{M}_1^g$. Setting $G_0 = N_G(P)$ it is easily checked that $M_1 \cap G_0$ is *-standard in $G_0$. So it suffices to prove (i).

We apply induction to $G_0/O(G_0)$. Since $m(T_0) > 1$, we must be in case (v) or (vi) of the main theorem. Let $T \subseteq S \in \text{Syl}_2(G_0)$, $S_0 \in \text{Syl}_2(N_G(S))$. First, assume $S \subseteq S_0$.

If $G_0/O(G_0) \simeq A_9$ or $S_9$, then $T_0 \sim T_1$ in $G_0$ and $Z_2(S)$ is a klein group which we may take to be $\langle t' \rangle \times \langle t^s \rangle$ for $t \in T_0$ and $s \in S$. As $tt^s \in Z(S)$, $S_0 = SC_{S_0}(t)$, a contradiction.

Suppose that $G_0/O(G_0) \simeq J_2$ or $\text{Aut}(J_2)$. Again we check centralizers to see that for each $t \in T_0^g$, $tS_0 \subseteq tG_0$. Using the results in [13] we see that $S$ contains precisely 8 conjugates of $T_0$ and $tG_0 \cap S$ is contained in the union of those conjugates. As $S$ is transitive on $T_0^g \cap S$, the tight embedding property gives $S_0 \subseteq SN(T_0)$, and again we have a contradiction.

Next suppose that $G_0 = G_0/O(G_0) \simeq J_2$. Then $T_0 \cap G_0' = \langle t' \rangle$ for some involution $t$ and $C_{G_0}(t) \simeq S_5 \times \langle t' \rangle$, modulo $O(G_0)$. We have $S > T$ and $T$ contains a Sylow 2-subgroup of $C_{G_0}(t)$, which has the form $T_1\langle a \rangle \times \langle t \rangle$ for some involution $a$. Set $A = \langle a \rangle \times Z(T_1\langle a \rangle) \times \langle t \rangle$. Then by Theorem 2 of Harada [15], $G$ is of known type. In particular, $G_0O(G) = G$ and certainly $S \in \text{Syl}_2(G)$.

Finally we assume that $G_0 = G_0/O(G_0) \simeq Ru$. Here we use information about $S$ available in Dempwolff [6]. In his notation $S = V$ and $V$ contains a normal subgroup $W$ such that $F = W' = A_1$ and $W/A_1 = W/F$ is elementary of order $2^g$ on which $N_{G_0}(\overline{W}/F) \simeq \text{GL}(3,2)$ acts irreducibly. Checking centralizers we see that $G_0$ controls the fusion of its involution so that $S_0$ cannot fuse an involution in $T_0^g$ into another $G_0$-class of involutions.

Using the argument in Lemma 2.2 of [6] we conclude that $S_0 < N_G(W)$. So $S_0$ permutes the involutions in $W - W' = W - F$. However, Lemmas 2.7 and 2.8 of [6] show that $S$ is transitive on $T_0^g \cap W$. Consequently, $S_0 = SN_{S_0}(T_0) = S$, a contradiction.

Now that $S \in \text{Syl}_2(G)$ we can obtain a contradiction by quoting an appropriate characterization theorem giving the structure of $G/O(G)$. For all cases except $G_0/O(G_0) \simeq Ru$ we can use the result of Gorenstein and Harada [12]. In the remaining case we quote the recent result of Assa [4]. At this point (3.7) is proved.
(3.8) $\tilde{M}_1 \cong L_2(4)$ or $\text{SL}(2, 5)$.

**Proof.** If $m(T_0) = 1$ and $\tilde{M}_1 \cong L_2(4)$, we can quote Theorem 2 of Harada [15] to get a contradiction. If $m(T_0) = 1$ and $\tilde{M}_1 \cong \text{SL}(2, 5)$ let $\langle r \rangle = \Omega_1(T_0)$. Then it is easily seen that $r$ is a 2-central involution in $G$. Since $C(t)$ has Sylow 2-subgroups of sectional rank at most 4 we again have a contradiction.

Suppose $m(T_0) > 1$. By (3.7) $O(M) = 1$, so $O(M_1) = 1$. If $M_1$ is a standard subgroup of $G$, then we quote Aschbacher [2], while if $M_1$ is not standard it is because $E(M_0)$ is conjugate to $M_1$ and $E(M) = M_1 \times M_1^g$ for some $g \in G$. In particular, $T_0$ is a klein group and we can quote Smith [21].

(3.9) $T_0 \cap T_1 = 1$.

**Proof.** Suppose false. Then $M_1/O(M_1)$ is a perfect central extension of $S_7(8)$ by $Z_2$ or $Z_2 \times Z_2$. First suppose that $m(T_0) = 1$. Here $T = T_0T_1$ and $\Omega_1(T) = \Omega_1(T_0)$ (as $T_0 = \Omega_1(T_0) < T_1$ and $\Omega_1(T_0/T_0) = \Omega_1(T_0/T_0)$). Also $[T, \Omega_1(T)] = T_0 \cap T_1$. Consequently, $N_G(T) < N_G(T_0 \cap T_1)$ and it follows that $T \in \text{Syl}_2(G)$. But then $T_1$ is a strongly closed subgroup of $T$, contradicting (3.4).

If $m(T_0) > 1$, then $T_0$ is elementary abelian by (3.6)(iv). Here $\Omega_1(T) = \Omega_1(T_1)T_0$ and the above argument again gives a contradiction.

(3.10) $T \not\subseteq \text{Syl}_2(G)$.

**Proof.** If $m(T_0) > 1$, then $T_0$ is elementary by (3.6), so in all cases $\Omega_1(T_0T_1) = \Omega_1(Z(T_0T_1))$. Suppose that $t \in T - T_0T_1$ is a conjugate of an involution in $T_0$. Then $\tilde{M}_1 = U_2(q)$ or $L_2(q)$ and $C_{\tilde{M}}(t) = L_2(q)$ or $L_2(\sqrt{q})$, respectively. Moreover, all involutions in $C_{\tilde{M}}(t)$ are fused to $t$. Clearly, $C_{\tilde{M}}(t') < C_G(t')$ and, by (3.8), $C_{\tilde{M}}(t)$ is simple so $C_{\tilde{M}}(t')$ covers $C_{\tilde{M}}(t)$. Now we conclude that some conjugate $t^g$ of $t$ induces a nontrivial inner automorphism of $M_1$.

Assume that $T \in \text{Syl}_2(G)$. If $\tilde{M}_1 \cong L_2(q)$ we use (3.6)(iv) and then (2.6) to conclude that $\Omega_1(T_0T_1) = T_1\Omega_1(T_0)$ is strongly closed in $T$. This contradicts (3.4). If $\tilde{M}_1 \cong S_7(9)$, then $T = T_1 \times T_0$ and again $\Omega_1(T)$ is strongly closed and abelian.

Suppose that $\tilde{M}_1 \cong U_3(q)$ and let $t^g$ be as in the first paragraph. The group $M^g$ contains $C_{\tilde{M}}(t^g)$ and $C_{\tilde{M}}/O(M_1^g)$ has order $(q + 1)q^3$ or $\frac{1}{2}(q + 1)q^3$. A 2-complement in $N(T_1) \cap C_{\tilde{M}}(t^g)$ acts fixed-point-freely on $T_1/\Phi(T_1)$, and from the structure of $M^g$ we conclude $\Omega_1(T_0) \not\subseteq M^g$.

In particular, (3.9) implies that $t^g \not\subseteq T_1$. We may assume that $\Omega_1(T_0^g) < T$ (this is clear if $m(T_0) = 1$, and if $m(T_0) > 1$ we use (3.6)(ii) and (2.1)). Let $\Delta = \Omega_1(T_0)G \cap V$. Since $N_{\tilde{M}}(V)$ contains a cyclic group acting regularly on $\Omega_1(T_1)$ and since $\Delta \not\subseteq \langle \Omega_1(T_0), \Omega_1(T_1) \rangle$, we argue as in (2.7) to conclude that
$N(V)^A$ is 2-transitive of degree $q$. But $T_0 T_1 \leq C(V)$ and $|T : T_0 T_1| < n < q$. This is a contradiction.

(3.11) Let $T < S \in \text{Syl}_2(G)$. Then $N_S(T) < N_G(T_0 T_0)$.

**Proof.** Suppose $\tilde{M}_1 \cong U_3(q)$. Then from (3.6) and (3.8) it is easy to see that $\Omega_1(T_0 T_1)$ is weakly closed in $T$ with respect to $T$ and $T_0 = C_T(\Omega_1(T_0 T_1))$. If $\tilde{M}_1 \cong U_3(q)$, then we may assume $T > T_1 T_0$. In this case $\Omega_1(Z(\Omega_1(T'))) = J > A_1$ is normalized by $N_S(T)$, and since $T_1 T_0 / J$ is the unique group of its isomorphism type in $T / J$, we have the result.

(3.12) $T_0$ is elementary abelian.

**Proof.** By (3.6)(iv) we may assume that $m(T_0) = 1$. Choose $y \in N_S(T) - T$. By (3.11) $y \in N(T_1 T_0)$. Also $T_0 \cap T_0 = 1$ and $T_0 \cong T_1$. The Krull-Schmidt theorem implies that $T_0 < T_0 Z(T_1)$, and the result follows from the fact that $Z(T_1)$ is elementary.

(3.13) Let $L = N_G(A)$ and $A = \{A_0^g : g \in G, A_0^g \leq A\}$ is a disjoint union of $q$ conjugates of $A_0$.

(i) $A - A_1 = \bigcup \{(A_0^g)^x : x \in G, A_0^g \leq A\}$ is a disjoint union of $q$ conjugates of $A_0$.

(ii) $A_1$ is strongly closed in $A$ with respect to $G$.

(iii) $L$ induces a 2-transitive group on $A$.

**Proof.** This follows exactly as in the proof of (2.7) once we show $\Delta \neq \{A_0, A_1\}$. Suppose that, in fact, $\Delta = \{A_0, A_1\}$ and let $y \in N_S(T) - T$ for $T < S \in \text{Syl}_2(G)$ (here we use (3.10)). By (3.11) $y \in N(T_1 T_0) < N_A(T)$ as $y \notin T$ we must have $A_0^y = A_1$. If $M_1 \cong L_2(q)$, then $A_1 < (T_1 T_0)'$ and $A_0 < (T_1 T_0)'$, impossible. Therefore $\tilde{M}_1 \cong L_2(q)$. But now $G$ satisfies the conditions of Hypothesis $(\ast)$ of §2 ($R = T_0, K = M_0, X = C_{M_0}(T_0), U = T_1$). So (2.7) implies that $A$ is strongly closed in a Sylow 2-subgroup of $G$, contradicting (3.4).

(3.14) Let $L = N_G(A)$ be as in (3.13).

(i) $L^A$ contains $O_2(L^A)$ as a regular normal subgroup of order $q$.

(ii) $O_2(L^A) K_1^A \leq L^A$ is a 2-transitive Frobenius group.

**Proof.** It suffices to show that $L^A$ contains a regular normal subgroup. Here we use the proof of (2.7)(iii). If $L^A$ does not contain a regular normal subgroup then we must have $(L^A)' \cong L_2(2)$ and $q = 8$. So $M \cong L_2(8), S_8(8)$, or $U_3(8)$. Since $L$ has a 7-element acting nontrivially on $A_1$, $L$ induces $L_2(2)$ on $A_1$.

Let $T < S_1 \in \text{Syl}_2(N_G(A))$. Then $S_1$ contains an element $x$ inducing an automorphism of order 4 on $A$ and satisfying $C_A(x) < A_1$. From the Jordan form of $x$ acting on $A$ we conclude that $|A_0| = 2$. 

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First suppose that $\tilde{M}_1 \cong S_2(8)$. The stabilizer $J$ in $L$ of an element $yA \in (T/A)^*$ induces $S_4$ on $A_1$. But also $J$ must stabilize $[T, y]$, a Klein group in $A_1$ and $y^2$, an involution in $A_1$. This is impossible.

Next suppose that $\tilde{M}_1 \cong U_3(8)$. We argue as follows, referring the reader to p. 17 of [9] for the structure of $T_1$. Let $z \in C_{A_1}(x^2) \cap [A_1, x^2]$. The square roots in $T_0T_1$ of $z$ form 9 cosets $x_iA_i$, $i = 1, \ldots, 9$, permuted by $x$. Hence one coset at least, say $x_1A_1$, is fixed by $x$. Then, since $A < Z(T_0T_1)$, $x$ acts on the 4-element set $\{[x_i, x_j]: i = 2, \ldots, 9\}$, which an easy computation shows is not the case.

Now assume that $\tilde{M}_1 \cong U_3(8)$. Here $T = T_0T_1$ is elementary of order $2^4$. We claim that $S_1 \in Syl_2(G)$. For suppose $g \in N(S_1) - S_1$ with $g^2 \in S_1$. Then $A^g < S_1$, but $A \neq A^g$. As $A^g$ centralizes $A \cap A^g$, $|A \cap A^g| = 4$ and $A \cap A^g < A_1$. So there is a conjugate $A_0^g = \langle t^z \rangle \subseteq S_1 - T$. We may assume $t^z A < Z(S_1/A_1)$. Then $t^z$ has two nontrivial Jordan blocks on $A$ and, hence, $C_{S_1}(t^z)$ covers $S_1/A_1$. This forces $C_{S_1}(t^z)$ to involve $D_8$, a contradiction. This proves the claim.

Finally we observe that $S_1$ has sectional 2-rank 4 so that the theorem of Gorenstein and Harada [12] gives a contradiction.

We remark that the only groups $G$ in the main theorem satisfying $(L^4)' \cong L_3(2)$ are those with $G' \cong G_2(3)$.

Notation (3.15). As in (2.8) we now have the existence of certain subgroups of $L$. Let $L_0$ be the subgroup of $L$ stabilizing each element of $\Delta$. Then either $T_1T_0 \in Syl_2(L_0)$ or $\tilde{M}_1 = U_3(q)$, $|T \cap L_0: T_0T_1| = 2$, and $T \cap L_0 \in Syl_2(L_0)$. Choose $R > T \cap L_0$, a 2-subgroup of $L$ so that $R^A$ is the regular normal subgroup in $L^A$. We may assume that $T < N(R)$. Except in the case $T \cap L_0 > T_0T_1$, we may choose a subgroup $D_1 < N(R)$ of odd order with $D_1$ inducing $K_1$ on $T_0T_1$ and $T_3 < N(D_1)$. In those cases set $R_1 = R$. If $T \cap L_0 > T_0T_1$, then $K < L_0$ and $K$ induces a cyclic group of order $q + 1$ or $\frac{1}{3}(q + 1)$ on $T_0T_1$ normalized by $\langle R, K_1 \rangle$. From here it is easy to see that $R/T_0T_1$ is elementary and that $R$ contains a subgroup $R_1$ of index 2 such that $R_1 > T_0T_1$, $R_1$ covers $R/R \cap L_0$, and $K_1$ normalizes $R_1$ module $O(L_0)$. So here we choose $D_1 < N(R_1)$ of odd order with $D_1$ inducing $K_1$ on $T_0T_1$ and $T_3 < N(D_1)$.

Set $R_0 = [R_1, D_1]$.

(3.16) (i) $T_1 \trianglelefteq R$ and $[T_1, R_1] \trianglelefteq A_1$.
(ii) $R_0 \cap A_0 = 1$ and $R_0A_0T_3 \subseteq Syl_2(L)$.
(iii) $R_0 = T_1R_2$ with $T_1 \cap R_2 = A_1$, $R_2$ abelian, and $R_2/A_1$ and $A_1$ are isomorphic $F_2(D_1)$-modules.

PROOF. We have $A \triangleleft R$, $[A, R] \trianglelefteq A_1$ and $T_1T_0 \triangleleft R$ (as $T_1T_0 = R_1 \cap$
C(A) and R ⊊ R_1 T_3). First we show that T_A ⊊ R. If T_A = A_1, this follows from (3.13)(ii). Suppose that \( \tilde{M}_1 \cong U_3(q) \). As \( q > 2 \), \( K \cong C(A) \) and \( [K, T_1] = T_1 \). It follows that if \( g \in R, A_T^g \cong A, T_1 \cong M_f \). So \( T_1 = T_0 T_1 \cap M_f \) and \( g \in N(T_0) \). In particular, \( R \cong N(T_0) \). Now suppose that \( \tilde{M}_1 \cong Sz(q) \). If \( R/A \) is not elementary abelian, then since \( D_1 \) is transitive on \( (R/T)^# \) and on \( (T/A)^# \), we have \( \Omega_4(R/A) = T/A \). But then \( \Omega_4(R) = A, R \in \text{Syl}_2(G) \), A is strongly closed in R, and we contradict (3.4). So \( R/A \) is elementary abelian.

Let \( X/A \) be a \( D_1 \)-invariant complement to \( T_0 T_1 / A \) in \( R/A \). We use the action of \( D_1 \) to see that \( X/A_1 \) is elementary abelian. Indeed, if \( X/A_1 \) is not abelian, choose \( A_2 / A_1 \) a hyperplane in \( A/A_1 \) with \( X' < A_2 \). Then since \( D_1 \) is irreducible on \( X/A, X/A_2 \) is extraspecial, contradicting the fact that \( n \) is odd.

So \( X/A \) is a \( J_1 \)-invariant complement to \( T_0 T_1 / A \) in \( R/A \). We use the action of \( D_1 \) to see that \( X/A_1 \) is elementary abelian. Indeed, if \( X/A_1 \) is not abelian, choose \( a \in A \) as \( R/A \) is abelian. Since \( t \) centralizes \( x^2 \in A_1 \), we must have \( x^2 = (xa)^2 = x^2a^2[x, a] = x^2[x, a] \). Consequently, \( [x, a] = 1 \) and, as \( x \in X \), \( A \), this forces \( a \in A_1 \). We conclude that \( [T_1, X] < A_1 \) and \( T_1 < R \) as claimed.

Now we complete the proof of (i); that is, we show \( [T_1, R_1] < A_1 \). If \( T_1 = A_1 \) this is obvious. In the other cases we have the result since \( D_1 \) acts irreducibly on \( T_0 T_1 / A_1 \), and \( T_0 T_1 / A_1 \cap Z(R/A_1) \neq 1 \).

A previous argument shows that \( R_1 / A \) is elementary if \( \tilde{M}_1 \cong Sz(q) \). We claim that \( R_1 / A \) is elementary in all cases. If not, then as before \( \Omega_4(R_1 / A) = T_0 T_1 / A \) and \( \Omega_4(R_1) = A \). If \( \tilde{M}_1 \cong L_2(q) \) it is then easy to see that \( A \) is weakly closed in \( RT_2, RT_3 \in \text{Syl}_2(G) \), and by (2.6) (using \( q > 4 \) \( A \) is strongly closed in \( RT_3 \). This contradicts (3.4). Now assume that \( \tilde{M}_1 \cong U_3(q) \) and let \( D = D_1 \). Then \( D < L_0 \) and, as \( q > 2 \), \( [D, T_1] = T_1 \). But also \( [D, R_1] < T_0 T_1 \). Consequently, \( [R_1, D, R_1] < [T_0 T_1, R_1] < A \) and \( [D, R_1, R_1] < A \). By the 3-subgroup lemma \( [R_1, R_1, D] < A \) and so \( R_1 < A \). That is, \( R_1 / A \) is abelian and, since \( D_1 \) acts irreducibly on \( T_0 T_1 / A_1 \), we conclude that \( \Phi(R_1 / A) = 1 \) and \( R_1 / A \) is elementary.

Choose a \( D_1 \)-invariant complement \( X/A \) to \( T_0 T_1 / A \) in \( R_1 / A \). We next claim that \( X/A_1 \) is elementary abelian. If not then there is an element \( x \in X \) with \( x^2 \in A - A_1 \). Then \( x^2 \) is \( R_1 \)-conjugate to an involution in \( A_0 \). Therefore, \( x \) is \( R_1 \)-conjugate to a member of \( T \), a contradiction.

We now set \( R_2 = [D_1, X] \). Then \( A_1 \leq R_2 \) and \( R_2 \cap A_0 = 1 \). As \( R_1 / A_1 \) is the direct sum of \( T_1 / A, R_2 / A_1 \), and \( A / A_1 \), we have \( R_0 = T_1 R_2 \). This proves (ii) and the first two parts of (iii). If \( t \in A_0^# \) then the map \( r_2 A_1 \to [r_2, t] \) is a \( D_1 \)-isomorphism from \( R_2 / A_1 \) to \( A_1 \). Apply (2.2) to complete the proof of (3.16).

At this stage we have begun the process of building a Sylow 2-subgroup of \( G \). We will complete the proof of the main theorem by taking the cases \( \tilde{M}_1 \cong L_2(q), Sz(q), U_3(q) \) separately.
4. $\mathcal{M}_1 \cong L_2(2)$. In this section we assume that $\mathcal{M}_1 \cong L_2(q)$. Recall that we are after a contradiction and that, by (3.8), $q > 4$.

For this case the group $G$ satisfies the conditions of $(\ast)$ in §2 (setting $R = A_0$, $K = M_0$, $U = A_1$). We could immediately apply (2.9) provided we knew that at each stage of the process described in §2 the 2-transitive group did not involve $L_3(2)$. So we first prove this.

Suppose that at some stage $L_3(2)$ does occur. Then $\mathcal{M}_1 \cong L_2(8)$ and $T = T_0 \times T_1$. By (3.14) $L^A$ does contain a regular normal subgroup, so that the difficulty occurs at stage $m + 1$ of the inductive process, where $m > 1$. Consequently, there is a subgroup $U_m > A_1$ and a subgroup of odd order $D_m$, such that $D_m A_0 < N(U_m)$, $D_m$ acts on $A, A_0$ as does $D_1$, each $D_m$-composition factor of $U_m$ is isomorphic to $A_0$, and if $U_{m-1} = [U_m, A_0]$, then $N = N_G(U_{m-1}) \cap N_G(U_m A_0)$ induces $L_3(2)$ on $U_m A_0 / U_{m-1}$, 2-transitive on $\Omega = (A_0 U_{m-1} / U_{m-1})^N$. Also $U_m$ is normal in $G$ (see (2.7)).

We claim that $U_m$ is homocyclic of rank $n$, $|A_0| = 2$, and $A_0$ inverts $U_m$. To see this, note that for $t \in A_0^\#$, $t^N$ contains elements in $U_m t$. So $t$ inverts elements of $U_m$, and, using the action of $D_m$, $t$ inverts an element of each coset of $U_{m-1}$ in $U_m$. But now (2.4) implies that $U_m$ is abelian, so $t$ inverts $U_m$ and $U_m$ is homocyclic of rank $n$. As $t \in A_0^\#$ was arbitrary, $A_0 = \langle t \rangle$ and we have the claim.

Next note that $t U_m = t^N$ and the Thompson transfer lemma implies that $t \notin O^2(N)$. In particular, $O^2(N)$ has index 2 in $N$, is complemented by $\langle t \rangle$, and a Sylow 2-subgroup of $N$ has the form $S = S_0 \langle t \rangle$, where $S_0 \cap \langle t \rangle = 1$ and $U_m < S_0 \in Syl_2(O^2(N))$. Then $S_0 / U_m \cong D_8$. As $U_m > A_1$ has exponent at least 4, $U_m$ is weakly closed in $S$, and since $U_m A_0 = C_S(\langle t \rangle)$, $S \in Syl_2(G)$. In addition it is clear that $t$ does not fuse into $S_0$ so by transfer $G$ contains a normal subgroup $G_0$ of index 2. Clearly, $S_0 \in Syl_2(G_0)$. At this point we have the structure of $G_0$ by appealing to [15] or to [19]. In either case we have a contradiction.

We may now apply (2.9) to get the subgroup $U_m > A_1$. Here $S = A_1 A_0 T_3 \in Syl_2(M)$. By (3.13), (2.9)(3) does not hold.

(4.1) $U_m$ is not isomorphic to a Sylow 2-subgroup of $U_3(q)$ or $L_5(q)$.

**Proof.** Deny. Then $U_{m-1}$ is homocyclic of exponent 4 and, since for each $t \in A_0^\#$, $t^{U_m} = U_{m-1} t$, we have $t$ inverting $U_{m-1}$. In particular, $A_0 = \langle t \rangle$. By (2.9) $U_m S = U_m A_0 T_3 \in Syl_2(G)$. Now $A_0 T_3$ is abelian, and if $A_0 T_3$ is cyclic, then we transfer out $A_0 T_3$ and contradict (3.3). So we may assume that $T_3 A_0 = T_3 \times A_0$ and $T_3 \neq 1$. Each involution in $T_3 U_m - U_m$ centralizes a homocyclic subgroup of order $q = 4^{n/2}$ and rank $n/2$ in $U_{m-1}$. Each involution in $U_m$ has centralizer of order at least $q^2$. So $t^G \cap U_m T_3 = \emptyset$ and $G$ contains a normal subgroup $G_0$ of index 2.
By (3.3) and transfer we may assume that $x^G_0 \cap U_m \neq \emptyset$, where $\langle x \rangle = \Omega_i(T_3)$. Say $y = x^g \in U_m$. Then either $y \in A_1$ and $U_m \leq C(y)$ or $y \in U_m - A_1$, $U_m$ is isomorphic to a Sylow 2-subgroup of $L_3(q)$, and $C_{U_m}(y)$ contains an elementary abelian subgroup of order $q^2$. However, $t \not\sim tx$ (for the same reason that $t \not\sim x$), and it follows that $B = C_{M}(x)/\langle x \rangle$ is a $\ast$-standard subgroup in $C_G(x)/\langle x \rangle$ with $C(B/O(B)) \cap C_G(x)$ having $\langle t, x \rangle/\langle x \rangle$ as Sylow 2-subgroup. From the minimality of $G$ we have a contradiction.

(4.2) $U_m$ is not homocyclic.

Proof. Suppose $U_m$ is homocyclic. Then (2.9)(2) implies that $A_1 < U_m$ and $S = U_mA_0T_3 \in \text{Syl}_2(G)$. So $q > 8$ by (3.8). It is now easy to show that $U_m$ is weakly closed in $S$.

We apply (2.6) to the weakly closed subgroup $U_m$ of $S$ and its subgroup $A_1$ (so $T = S$, $W = U_m$, $A = A_1$). Let $r$ be the integer given in (2.6).

As $U_mA_0 < C(A_1)$, $r < 1$. But from (2.6)(ii) and the fact that $q > 4$ we see that, in fact, $r = 0$. By (3.4) $A_1$ is not strongly closed in $S$, so there is a conjugate $x \in U_m(A_0^\#)$ of an involution of $A_1$. Say $t \in A_0^\#$ and $x \in U_m t$.

Then $x \not\in U_{m-1}t = tU_m$ and so $t$ must invert $U_m$. As $C_{A_0}(U_m) = 1$, $U_m t$ is the unique coset of $U_m$ in $U_m A_0^\#$ that contains involutions not in $A$. Also we note that each element of $U_m t - U_{m-1}t$ is conjugate to $x$.

Suppose $U_m A_0 \in \text{Syl}_2(G)$. If $U_m$ has exponent 4, then $U_m A_0$ has class 2, against (3.3). If $U_m$ has exponent greater than 4, then $U_{m-1}A_0^\#$ consists of involutions so each element of $A_0^\#$ inverts $U_{m-1}$, forcing $|A_0^\#| = 2$. But now we transfer out $A_0$ from $G_0$ and again contradict (3.3).

Thus we may choose $x \in T_3 - U_m A_0$ with $x^2 \in A_0$. $x$ clearly has no conjugates in $U_m A_0$, and if $x^2 \not= 1$ then $x^2$ is an involution in $A_0$ and so has no conjugate in $S - U_m A_0$ (check centralizers). Hence $x \not\in O^2(G)$ by transfer. By (3.5)(b), $|A_0| = 2$, $x$ is an involution, and $xt \in O^2(G)$, where $A_0 = \langle t \rangle$. But we can transfer out $xt$ also, a contradiction.

(4.3) $U_m$ is not elementary abelian of order $q^2$.

Proof. Suppose that $U_m$ is elementary abelian of order $q^2$. Then $U_m = R_2$ and, for $a \in A_0^\#$, $aR_2 - aA_1$ contains no involutions. So $A = \langle A_0^\# \cap R \rangle$ and $N_G(R) < N_G(A)$. Let $S \in \text{Syl}_2(G)$ with $RT_3 \leq S$. Then $N_S(R) = RT_3$.

Suppose that there are no involutions in $RT_3 - R$. Then $N_S(RT_3) < N_S(A) = RT_3$ so $RT_3 \in \text{Syl}_2(G)$. Also $R_2 \not\leq RT_3$ must be strongly closed in $S$, contradicting (3.4). So we may assume that there is an involution $x \in T_3 R - R$, and since $R/A$ is a free $F_2(<x>)$-module, we may take $x \in N(A_0)$. Let $t \in A_0^\# \cap C(x)$.

As $q > 4$, $R_2$ is weakly closed in $RT_3 = R_2 A_0T_3 = S_0$. Let $S_1 = N_S(R_2)$. If $S_1 < S_0$, then using (2.6) and (3.4) we obtain a contradiction. So assume $S_1 > S_0$. If $a \in N_{S_1}(S_0) - S_0$ then $A_0^\# \cap R_2 A_0^\# = 1$. As $T_3 R / R$ is cyclic this
forces $|A_0| = 2$ and we may assume that $T_3A_0 = T_3 \times A_0$ with $T_3$ cyclic.

Now, let bars denote images modulo $R_2$. Since $C_\Sigma(\ell) = \langle \ell \rangle \times \overline{T_3}$, we may apply Lemma 2.20 in [18] to conclude that either (i) $\overline{\ell} \in Z(\overline{S_1})$, or (ii) $\overline{S_1}$ has a subgroup $\overline{S_2}$ of index 2 with $\overline{S_2} = \overline{D\overline{T_3}}$, $D = \langle \overline{\ell}, \overline{t} \rangle$ dihedral (with $(\overline{\ell}t)^2 = 1$) and $\overline{T_3}$ acting on $\overline{D}$ centralizing $\overline{\ell}$ and normalizing $\langle \overline{\ell} \rangle$; also $|\overline{T_3} \cap \overline{D}| = 2$ and the involutions $\overline{\ell}t$ are fused in $\overline{S_2}$. Set $\langle s \rangle = \Omega_1(\overline{T_3})$. Then $\langle \overline{s} \rangle = Z(\overline{D})$.

Let $z \in S_1 - R_2$ be an involution, and suppose $m([R_2, z]) < 2$. Then $\overline{z} \not\in C(\overline{\ell})$, so we are in case (ii) above. If $\overline{z} \in \overline{D\overline{T_3}}$, then $\overline{z} = \overline{z_1z_2}$ with $z_1 \in D$, $z_2 \in T_3$, and $|z_2| = 4$. But $\overline{\ell} \sim z_2$, so $m([R_2, z]) < 4$. Hence $q = 16$, $m([R_2, z]) = 2$. If $\overline{z} \not\in \overline{D\overline{T_3}}$, then $\langle \overline{z}, \overline{t} \rangle > \overline{D}$ is dihedral of order $2|\overline{D}|$ and, hence, we can write $\overline{s}$ as a $\frac{1}{4}|\overline{D}|$th power of a product of $\overline{z}$ and a conjugate. In particular, $m([R_2, s]) < 4$, so $q = 16$ and $m([R_2, z]) = 2$. Also, $|\overline{D}| = 4$ and so in this case $\overline{D\overline{T_3}} = C_\Sigma(\overline{\ell})$ and no involution of $\overline{D\overline{T_3}}$ satisfies $m([R_2, z]) < 2$.

At this point one can argue that $R_2$ is weakly closed in $S_1$. So $S_1 \in \text{Syl}_2(G)$ and using (2.6) we have $R_2$ strongly closed in $S_1$. This is a contradiction.

At this stage we have considered all cases of (2.4) and we conclude that there are no counterexamples to the main theorem with $\tilde{M}_1 \cong L_2(q)$, $q = 2^n > 4$.

5. $\tilde{M}_1 \cong \text{Sz}(q)$. Recall the notation of §3 and assume $\tilde{M}_1 \cong \text{Sz}(q)$. $R \in \text{Syl}_2(N(A))$, $R = T_1R_2A_0$, $R_2$ is abelian, and $[D_1, R] = T_1R_2$. Let $Y = N_G(R) < N_G(A_1)$ (as $A_1 = Z(R)$) and consider the induced group $Y^*$ on $\Delta = \{(A/A_1)^Y\}$.

We will obtain a contradiction to the standing assumption that $G$ is a counterexample to the main theorem.

(5.1) Suppose $T_1$ is isomorphic to the Sylow 2-subgroup of $\text{Sz}(8)$. Then $\text{Aut}(T_1)$ does not involve $L_3(2)$.

PROOF. Suppose $X = \text{Aut}(T_1)$ does induce $L_3(2)$. Then $\text{Aut}(T_1)$ induces $L_3(2)$ on $T_1/A_1$ and on $A_1$, and looking at the action of an element of order 7 in $X$ we see that the representations of $X$ on $T_1/A_1$ and on $A_1$ are contragredient. Choose a basis $x_1A_1$, $x_2A_1$, $x_3A_1$ of $T_1/A_1$ and a klein group $X_0 < X$ centralizing $\langle x_1A_1, x_2A_1\rangle$. Then $X_0$ centralizes $\langle x_1^2, x_2^2\rangle$, whereas $X_0$ centralizes no klein group in $A_1$. This is a contradiction.

(5.2) If $q = 8$ and $T_0 \cong Z_2 \times Z_2$, then $G \cong \text{Ru}$, the Rudvalis group.

PROOF. Dempwolff [6] (see the appendix).

(5.3) $|\Delta| < 5q$.

PROOF. By Lemma 1.8 of [6], $R - A_1$ contains at most $q(2q|A_0| - |A_0| + q - 2)$ involutions. Each conjugate of $A$ contains $q|A_0| - q$ involutions outside
and by the tight embedding property, \( A^g \neq A \) for \( g \in Y \) implies \( A^g \cap A = A \). The result follows.

(5.4) \(|\Delta| = q \) and \( Y^* \) is 2-transitive on \( \Delta \). Either \( Y^* \cong L_3(2) \) or \( Y^* \) contains a regular normal subgroup.

**Proof.** Let \( N^* \) be a minimal normal subgroup of \( Y^* \). We note that 
\[ |\Delta| = 1 + k(q - 1) \]
where \( k \geq 1 \) is an integer. This follows since \( D_1^* \) is semiregular on \( \Delta - \{ A/A_1 \} \). Also \( |Y^*| = |\Delta|v \) where \( q - 1|v \) and \( v \) is odd.

We claim that \( N^* \cong L_3(2) \) or \( N^* \) is a \( p \)-group for some prime \( p \). First note that by (3.3) \( R \not\subseteq \text{Syl}_2(Y) \). So \( |Y^*| \) is even. By (5.3) \( k \leq 5 \). Consequently, \( k = 1, 3, \) or \( 5 \). Suppose that \( k = 3 \) or \( 5 \). Then \( 8 \nmid |Y^*| \). By Feit and Thompson [7] and Gorenstein and Walter [14], if the claim is false then \( N^* \cong \text{PSL}(2, q_1) \) for some prime power \( q_1 \). Suppose this occurs. Let \( P^* \) be a Sylow \( p \)-subgroup of \( Y^* \) for a primitive divisor \( p \) of \( q - 1 \). If \( P^* \cap N^* \neq 1 \), then \( N_{N^*}(P^* \cap N^*) \) has order twice an odd number. This implies that some involution in \( Y^* \) normalizes a conjugate of \( P^* \). But \( P^* \) fixes just one point of \( \Delta \), and the stabilizer of this point in \( Y^* \) has odd order. So \( P^* \cap N^* = 1 \) and by the Frattini argument \( P^* \) normalizes a Sylow 2-subgroup of \( N^* \). But then \( P^* \) centralizes this subgroup (as \( p \neq 3 \)) and we again have a contradiction.

Finally consider the case \( k = 1 \). Here \( Y^* \) is 2-transitive. By Hering, Kantor and Seitz [16] either \( N^* \) is a \( p \)-group or \( N^* \cong \text{PSL}(2, q_1) \) for some \( q_1 \). As in (2.7) we must have \( q_1 = 7 \) and \( q = 8 \) in the latter case.

(5.5) \( Y \) does not induce \( L_3(2) \) on \( R/A_1 \).

**Proof.** Suppose \( Y \) induces \( L_3(2) \) on \( R/A_1 \). Then \( q = 8 \) and, by (5.2), \(|T_0| = 2 \) or \( 8 \). The nontrivial irreducible constituents of \( D_1 \) on \( R/A_1 \) are \( T_1/A_1 \) and \( R_2/A_1 \). These are inequivalent. Also the irreducible \( F_2 \)-modules of \( L_3(2) \) have degrees \( 1, 3, 3, 8 \). Suppose that the representation of degree \( 8 \) is a \( Y \)-composition factor on \( R/A_1 \). Since any \( F_2 \)-module affording this representation is injective and projective, \( R/A_1 \) is completely reducible as an \( F_2(Y) \)-module. But then there is an involution \( t \in A_0 \) such that \( A_1 \langle t \rangle \leq Y \). As \( t^G \cap A_1 \langle t \rangle = t^{R_1} \cap A_1 \langle t \rangle \), \( Y \not\leq R_2C(t) \), a contradiction.

Let \( V/A_1 \) be a minimal normal subgroup of \( Y/A_1 \) contained in \( R/A_1 \). Then \( |V/A_1| = 2 \) or \( 8 \) and, as above, the case \( |V/A_1| = 2 \) gives a contradiction. So \( |V/A_1| = 8 \) and from the action of \( D_1 \) on \( R/A_1 \) we have \( V = T_1 \) or \( R_2 \). By (5.1) \( V \neq T_1 \), so \( R_2 \not\leq Y \).

As \( Y^2 \cong L_3(2) \) and \( |T_0| = 2 \) or \( 8 \), \( D_1 \) must contain an element \( g \) with \( g \) inducing an element of order 3 on \( R \) and \( C_{A_0}(g) \neq 1 \). Say \( 1 \neq t \in C_{A_0}(g) \). Then \( t \) normalizes \( C_{T_1R_1}(g) \). Now \( C_{T_1R_1}(g) \) has order \( 8 \) as \( g \) induces the regular module for \( Z_3 \) on each of \( A_1, T_1/A_1 \), and \( R_2/A_1 \). So \( C_{T_1R_1}(g) = \langle t_1, r_2 \rangle \) for some \( t_1 \in T_1 - A_1, r_2 \in R_2 - A_1 \). Therefore \([r_2, t] = t_2^2 \).
First suppose that $R_2$ is elementary abelian. Let $x \in Y$ be such that $x^A$ inverts $g^A$ and $x \in N\langle g\rangle$. Then $x \in N(C_R(g)) = \langle t_1, r_2 \rangle C_{A_0}(g)$. As neither $T_1 t$ nor $R_2 t$ contain involutions not in $A$, we have $t^* \in t_1 t_2 t A_1$. In particular, $t_1 t_2$ must be an involution. This forces $I = t_1 t_2 / t_1, r_2$. So $[t_1, r_2] = 1$ and, by (2.3), $[T_1, R_2] = 1$. At this point $Y$ normalizes $C_R(R_2) = T_1 R_2$ and, arguing as in (5.1) (choosing bases in $T_1 R_2 / R_2$ rather than in $T_1 / A_1$), we obtain a contradiction.

Thus $R_2$ is homocyclic and with $t_1, r_2,$ and $t$ as before, $R_2 = C_{R_2}(\langle g \rangle)$ and $t$ inverts $r_2$. As $t$ commutes with the action of $D_1$ on $R_2$, $t$ inverts $R_2$. If $T_1 < C(R_2)$, then $T_1 R_2 \langle t \rangle$ is the extended centralizer in $R$ of $R_2$ and $T_1 R_2$, $T_1 R_2 \langle t \rangle \leq Y$. For $y \in Y$, $t^* \in T_1 R_2 t$. Also $[T_1, R_2 \langle t \rangle] = 1$ and $t$ inverts $R_2$, so $t^* \in T_1 R_2$. Thus $R_2 t = t^Y$ and $T_1 = C_{T_1, R_2}(\langle t^Y \rangle) < Y$. This contradicts (5.1). Therefore $T_1 < C(R_2)$ and by (2.3) no element in $T_1 - A_1$ commutes with an element in $R_2 - A_1$. The extended centralizer of $R_2$ in $R$ is $R_2 \langle t \rangle$, so $R_2 \langle t \rangle \leq Y$. Let $J / R_2 \langle t \rangle < R / R_2 \langle t \rangle$ be a minimal normal subgroup of $Y / R_2 \langle t \rangle$. If $J < R_2 A_0$, then $A_0$ would contain a klein group with each involution inverting $R_2$. This is ridiculous. So $Y$ induces $L_3(2)$ on $J / R_2 \langle t \rangle$ and $J = R_2 \langle t \rangle T_1$. As $D_1$ has inequivalent representations on $T_1 / A_1$ and on $R_2 / A_1$, the representation of $Y$ on $J / R_2 \langle t \rangle$ is the contragredient of the representation of $Y$ on $R_2 / A_1$.

We now complete the proof of (5.5) using an argument in Dempwolff [6] (see the end of the proof of Lemma 3.4 in [6]). Let $r_2 \in R_2 - A_1$ with $a = r_2^2$ and let $K = C_Y(\langle a \rangle)$. As the representation of $Y$ on $J / R_2 \langle t \rangle$ is contragredient to the representation of $Y$ on $A_1$, $K$ fixes no involution in $J / R_2 \langle t \rangle = T_1 R_2 \langle t \rangle / R_2 \langle t \rangle$. Choose $t_1 \in T_1$ with $[t_1, r_2] = a$ and $x \in K$ with $(t_1 R_2 \langle t \rangle)^x \neq t_1 R_2 \langle t \rangle$. Say $t_1^x = t_2 t_0^* r$, with $t_2 \in T_1 - A_1$, $a = 0$ or 1, and $r \in R_2$. Then we have $a = a^x = [t_1^x, r_2^x]$. But $r_2^x \in r_2 A_1$ as $R_2$ is homocyclic and the squaring map is a $Y$-isomorphism from $R_2 / A_1$ to $A_1$. So

$$a = [t_2 t_0^* r, r_2^x] = [t_2, r_2^x][t_0^* r, r_2^x] = [t_2, r_2][t_0^* r, r_2].$$

We know $[t, r_2] = a$ and, since $[t_2, r_2] \neq 1$, this forces $a = 0$ and $[t_2, r_2] = a$. But then $t_2 t_1^{-1} \in C_{T_1}(r_2) < A_1$, whereas $t_1 R_2 \langle t \rangle \neq t_2 R_2 \langle t \rangle$. This is the final contradiction.

At this stage we know that $Y^A$ contains a regular normal subgroup.

(5.6) $T_1 R_2 < Y$ and $R_2 < Y$.

Proof. Let $V / A_1$ be minimal normal in $Y / A_1$ with $V < R$. If $D_1 < C(V / A_1)$, then $V < A$, which is impossible. Also $D_1$ must act irreducibly on $V / A_1$, so $V = T_1$ or $R_2$. Suppose that $V = T_1$. If $[R_2, T_1] = 1$, then $R_2 A_0 = C_R(T_1) < Y$. In this case $R_2 = R_2 A_0 - \bigcup_{g \in A_2} A_0)^* < Y$. Suppose that $[R_2, T_1] \neq 1$. Then by (2.3) $x \in T_1 - A_1, y \in R_2 - A_1$ implies that $[x, y] \neq 1$. Consequently neither $x y$ nor $x y t$ centralizes $x$, where $t \in A_0$. So $A =
Let \( V/R_2 \) be normal in \( Y/R_2 \), with \( V \leq R_2A_0 \). Choose \( V \) maximal such that \( T_1V \triangleleft Y \). Say \( V > R_2 \). Then \( V = R_2(A_0 \cap V) \) and \( A_0 \cap V \) is tightly embedded in \( Y \). So \( V \) contains \( q^2 \) conjugates of \( A_0 \cap V \) and each element of \( V - R_2 \) is an involution. It follows that \( |A_0 \cap V| = 2 \) and \( A_0 \cap V = \langle t \rangle \) inverts \( R_2 \). Let \( U \leq Y \) be a \( D_1 \)-invariant Sylow 2-subgroup of \( Y \), containing \( R \). Then \( A_1 = Z(U) \) and we may assume \( T_1 \triangleleft C(R_2) \), for otherwise \( T_1R_2 = C_R(R_2) \triangleleft Y \). Say \( u \in U \), \( x \in T_1 - A_1 \), and \( x^u = ytx \) with \( y \in R_2 \). For any \( r \in R_2 \), we have

\[
\left[ x, r \right] = \left[ x, r \right]^u = \left[ xyt, r \right] = \left[ x, r \right] \left[ t, r \right],
\]

whence \( [t, r] = 1 \). This is certainly false, so \( V = R_2 \) as required.

We conclude that \( T_1R_2 \triangleleft Y \), proving the result.

Let \( U > R \) be a Sylow 2-subgroup of \( Y \), invariant under \( D_1 \). Setting \( U_1 = [U, D_1] \) we have \( U = U_1A_0 \) and \( U_1 \cap A_0 = 1 \).

(5.7) \( R_2 = Z(T_1R_2) \).

**Proof.** Suppose \( [T_1, R_2] \neq 1 \). Consider the map \( u_1 \to [t, u_1] \), where \( u_1 \in U_1 \) and \( t \in \mathbb{A}_0^n \) is fixed. Then considering this map from \( U_1/T_1R_2 \) to \( T_1R_2/A_1 \) and noting the map commutes with the action of \( D_1 \), we see that \( [t, U_1]A_1 = T_1 \) or \( R_2 \) (use the fact that \( T_1/A_1 \) and \( R_2/A_1 \) are inequivalent irreducible \( F_2(D_1) \)-modules). If \( [t, U_1]A_1 = T_1 \), then \( U_1 \) normalizes \( T_1 \). As in (5.6) this forces \( C_R(T_1) = A \) to be normal in \( U \), which is a contradiction.

Consequently, \( [U_1, A_0] = R_2 \) and \( R_2A_0 - R_2 = \cup_{g \in Y} A_0^g \). Then \( A_0 = \langle t \rangle \) inverts \( R_2 \). We note \( U_1/T_1R_2 \cong R_2/A_1 \cong A_1 \) as \( F_2(D_1) \)-modules. Also \( U_1/R_2 \) is elementary abelian, as otherwise \( T_1R_2/R_2 = \Omega_1(U_1/R_2) \) and, for \( x \in T_1 - R_2 \), \( xR_2 = u^2R_2 \) for some \( u \in U_1 \). But then \( [x, R_2] = [u^2, R_2] = 1 \), contradicting (2.3). As \( T_1 \triangleleft U_1 \), (2.3) implies that \( U_1/A_1 \) has derived group \( R_2/A_1 \) and for each \( t_1 \in T_1 - A_1 \) and \( r_2 \in R_2 - A_1 \), there is some element \( u \in U_1 \) with \( t_1^u = t_1r_2a \) with \( a \in A_1 \). But \( u \) centralizes \( t_1^u \), so \( t_1^u = t_1r_2^u(t_1, r_2) \) and \( r_2^u = [t_1, r_2] \). So \( t_1 \) inverts \( r_2 \). But \( t_1 \) was arbitrary and, by (2.3), \( C_R(r_2) = A_1 \). This is a contradiction.

(5.8) **There are members of \( A \) not in \( R_2A_0/A_1 \). Consequently, \( \langle x \in A \rangle = R/A_1 \).**

**Proof.** Suppose that \( A \subseteq R_2A_0/A_1 \). If \( R_2 \) is elementary, then there are no involutions in \( R_2A_0 - A_1A_0 - R_2 \), a contradiction. Consequently, \( R_2 \) has exponent 4 and \( A_0 = \langle t \rangle \) inverts \( R_2 \).

Also \( Y \) normalizes \( C_R(\langle A \rangle) = T_1 \). Consider \( N = N_G(T_1) \) and apply (2.9) to obtain the structure of a Sylow 2-subgroup \( U_2\langle t \rangle \) of \( N \). By (5.1) \( N \cap N(R) \) does not involve \( L_2(2) \), so \( U_2/T_1 \) is homocyclic of rank \( n \), elementary of order \( q^2 \), the Sylow 2-subgroup of \( L_3(q) \), or the Sylow 2-subgroup of \( U_3(q) \).
We may assume that $D_1$ normalizes $T_1/A_1$ and $A_1$. Consequently, $\Phi(U_2) \leq C_{U_2}(T_1) = U_3$. So $U_2/U_3$ is elementary abelian. Also $T_1 \leq U_3$, so we may write $U_2/U_3 = T_1U_2/U_3 \times U_4/U_3$, with $U_4 D_1$-invariant. Each nontrivial $D_1$-composition factor of $U_4$ is equivalent to $A_1$, whereas $T_1U_3/U_3$ is $D_1$-equivalent to $T_1/A_1$, which is not equivalent to $A_1$. Consequently, $t$ normalizes $U_4$.

By (2.9) $U_4$ is elementary of order $q^2$, homocyclic of rank $n$, The Sylow 2-subgroup of $U_3(q)$, or the Sylow 2-subgroup of $L_3(q)$. We conclude that $U_2/T_1$ is elementary or homocyclic of rank $n$. Also $U_3 > R_2$. We must have $U_4/R_3$ of order 1 or $q$.

If $U_3(t) \in Syl_2(G)$, then we get a contradiction as follows. First we claim that $tG \cap U_2 = \emptyset$. For suppose $t \in U_2$. As each of $U_2/T_1$ and $U_2/U_3$ are abelian we have $U_2 < T_1 \cap U_3 = A_1$. Consequently, $|C_{U_2}(t)| > |U_2|/q$ and it follows that $|U_2| = q^3$. Therefore $U_2 = T_1U_2 = T_1R_2$. But then $R_2 < C(t^2)$, which is impossible. This proves the claim.

Now apply the Thompson transfer lemma to conclude $t \not\in G'$. Clearly $U_2 < G'$ by the action of $D_1$. Notice that $U_2/A_1$ is abelian, so that $U_2$ has class 2. This is against (3.3).

We now have that $U_3(t) \not\in Syl_2(G)$. Assume also that $U_4$ is not abelian. Then for $x \in U_4 - R_2$ and $t_1 \in T_1$,

$$(t_1 xt)^2 = t_1(xt)t_1(xt) = t_1^2(xt)[xt, t_1](xt) = t_1^2(xt)^2[xt, t_1, xt] \in (xt)^2A_1$$

because $t_1^2 \in A_1$ and $[T_1, U_3(t)] < A_1$. As $(xt)^2 \in R_2 - A_1$, so such element is an involution. In particular,

$$tG \cap U_2(t) \subseteq T_1U_3(t) = T_1R_2(t) = R.$$ 

It is easy to see that there are no involutions in $T_1^#R_2t$, and the claim above shows that $tG \cap U_2 = \emptyset$. Consequently, $tG \cap U_3(t) = R_2t = tU_2(t)$, and it follows that $U_2(t) \in Syl_2(G)$, a contradiction. Therefore $U_4$ is a homocyclic of rank $n$.

Suppose $U_4 = U_3 < C(T_1)$. Let $g \in N(U_3(t)) = P$ with $t^g = t_1u_4t$, $t_1 \in T_1 - A_1$, and $u_4 \in U_4$. One checks directly that it is not possible for $C(t^g) \cap T_1U_4$ to cover $T_1U_4/U_4$ and so this is impossible. Therefore for $g \in P$, $t^g \in U_4(t)$ and, hence,

$$C(t^B) \cap T_1U_4(t) = T_1 \triangleleft B,$$

a contradiction. Therefore $U_4 > U_3 > R_2$.

Now $Z_2(U_3(t)) = T_1R_2$ so $P$ normalizes each of $T_1R_2$, $R_2 = Z(T_1R_2)$, $T_1U_4 = U_3(t) \cap C(R_2)$, and $T_1U_3 = (T_1R_2)Z(T_1U_4)$. Consider $P = N(U_3(t))$. We want to apply (2.9) to $P/T_1U_3$. First we must show that $P$ does not involve $L_3(2)$. 

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Suppose that \( P \) involves \( L_3(2) \). The representations of \( P \) on \( T_1R_2/R_2 \) and on \( U_2/T_1U_3 \) are contragredient. Say \( P < N(U_4) \). Then we argue as in (5.5) to get a contradiction. Namely, choose \( u \in U_4 - U_3 \) and set \( K = C_p(uU_3) \). Choose \( t_1 \in T_1 - A_1 \) with \([t_1, u] = a\), where \( \langle a \rangle = \Omega_4(\langle u \rangle) \). As \( K \) fixes no involution in \( T_1R_2/R_2 \), we choose \( k \in K \) with \( t_1^r \neq t_1R_2 \). Say \( t_1^r = t_2r \), \( r \in R_2 \), and \( u^k = uu_3 \), with \( u_3 \in U_3 \). Then
\[
a = a^k = [t_1^r, u^k] = [t_2r, uu_3] = [t_2, u].
\]
But then \( t_1t_2^{-1} \in C(u) \), contradicting (2.3).

Therefore \( P \not< N(U_4) \) and we note that this implies \( U_4 \cap U_3^g = U_3 \) for \( g \in P - N(U_4) \). Otherwise \( C(U_4 \cap U_3^g) > \langle U_4, U_3^g \rangle > U_4 \), against (2.3). Choose a Sylow 3-subgroup \( J \) of \( P \) normalized by \( t \). Then a Sylow 2-subgroup of \( N_p(J) \) has the form \( Q = \langle t_1, u_4 \rangle \langle x, t \rangle \), where \( t_1 \in T_1 - A_1 \), \( u_4 \in U_4 - U_3 \), and \( Q/\langle t_1, u_4 \rangle \) is a klein group. Modulo \( U_3 \) we must have \( u_4^x = t_1u_4 \) and \( t^x = t_4u_4 \) or \( t_4u_4 \). Say \( t^x = t_4u_4 \) modulo \( U_3 \). As \( t^x \) inverts \( U_4 \), we compute and get a contradiction. Therefore \( P \) does not involve \( L_3(2) \) and (2.9) applies.

By (2.9) and a previous argument we obtain a \( D_1 \)-invariant subgroup \( U_5 > U_4 \) such that \( D_1 \) is transitive on \( (U_5/U_4)^* \) and \( T_1U_5/\langle t \rangle \in \text{Syl}_2(P) \). We have \( T_1R_2 = \Omega_4(T_1U_4) \), so \( [T_1, U_5] < R_2 \). If \( u \in U_5 \) and \( t_1 \in T_1 - A_1 \), then \( t_1^u = t_1r_2 \) for \( r_2 \in R_2 \). But \( u \) fixes \( t_1^u \), so that \( r_2^2 = 1 \) and \( r_2 \in A_1 \). Consequently, \( u \in N(T_1) \), contradicting \( U_2(\langle t \rangle) \in \text{Syl}_2(N(T_1)) \). This completes the proof of (5.8).

(5.9) \( R_2 \) is elementary abelian.

PROOF. By (5.8) there are elements \( t_1 \in T_1 - A_1 \), \( r_2 \in R_2 \), \( t \in A_0^* \) with \( t_1r_2t \) conjugate to an involution in \( A_0 \). Say \( t^u = t_1r_2^v \) for \( u \in U_1 \). Then
\[
T_1 \cap T_1^u = C_{U_1}(t) \cap C_{U_1}(t^u).
\]
As \( T_1 \) centralizes \( r_2t \), we conclude from the structure of \( T_1 \) that \( T_1 \cap T_1^u < A_1 \langle t_1 \rangle \).

Choose \( t_2 \in T_1 - A_1 \). Then \( t_2^d = t_2d \) for some \( d \in R_2 \). If \( t_2 \not\in A_1 \langle t_1 \rangle \), then, by the above, \( d \not\in A_1 \). On the other hand \( d^2 = 1 \). So \( R_2 - A_1 \) contains involutions and, consequently, \( R_2 \) is elementary.

(5.10) \( U_1/R_2 \) is elementary abelian.

PROOF. \( U_1/A_1 \) is of class at most 2 and \( U_1/R_2 \) is abelian by (2.2). For \( x, y \)
in $U_1$, $[x, y^2] = [x, y]^2$ modulo $A_1$, so that $[x, y^2] \in A_1$. Therefore

$$[x^2, y^2] = [x, y^2]^2 = [x, y^2] = [x, y^2]^2 = 1.$$ 

If $U_1/R_2$ were homocyclic of exponent 4, then $\Omega_1(U_1/R_2) = T_1R_2/R_2$ and, by the above, $T_1$ is abelian. This is absurd. $U_1/R_2$ must be elementary.

(5.11) $R_2 = Z(U_1)$.

**Proof.** It suffices to prove that $R_2 \leq Z(U_1)$. Suppose otherwise. Consider the semidirect product $R_2(U_1/T_1R_2)D_1$, of order $q^3D_1$. By (5.10) and (2.2), we may apply (2.3). As we are assuming that $R_2$ is not central in $U_1$, we conclude that if $r_2 \in R_2 \neq A_1$ and $u \in U_1 - T_1R_2$, then $[u, r_2] \neq 1$.

Recall the proof of (5.9) and the notation. For $t_2 \in T_1 - A_1 \langle t_1 \rangle$, $t_2^2 = t_2d$ for $d \in R_2 - A_1$. But $u^2 \in R_2$, so $t_2 = t_2u = t_2dd^u$ and $d = d^u$. This contradicts the above paragraph.

We can now obtain a contradiction in the case $\tilde{M}_1 \cong Sz(q)$. For each involution $t \in A_0^\#$, both $R_2$ and $U_1/R_2$ are free $F_2(t)$-modules.

Suppose $A_0 = \langle t \rangle$. Then the above implies $t^G \cap U = U_1t = tU$, and so $U = U_1 \langle t \rangle \in Syl_2(G)$. By the Thompson transfer lemma $t \not\in G'$, although $U_1 \leq G'$ (use the action of $D_1$). As $U_1$ has class 2 we have a contradiction to (3.3).

Now assume that $|A_0| > 2$. Fix $u \in U_1 - T_1R_2$ and consider the involutions $t''$ for $t \in A_0^\#$. We have $t'' = t_1r_2$ for $t_1 \in T_1 - A_1$, $r_2 \in R_2$. As $u^2 \in R_2$, $t''^2 \in T_1$. But $t''^2 = t_1r_2t_1r_2$ and $r_2 = r_2^u$. So $t_1t_1'' \in A_1$ and $T_1 \cap T_1'' > A_1\langle t_1 \rangle$. On the other hand, $T_1 \cap T_1'' \leq C(A_0) \cap C(A_0^\#)$. Choosing $t' \neq t$ in $A_0^\#$ we have $(t'')^u = t't_1r_2$. We claim that $t_1A_1 \neq t_1A_1$. Otherwise $tt'' \in A_0$ and $(tt'')^u \in tt'R_2$. But as $R_2$ is a free $F_2(t''t')$-module, this implies that $u \in R_2C(t'')$, which is false. Therefore $t_1A_1 \neq t_1A_1$. Consequently, $t_1$ does not centralize $(t'')^u$. This is a contradiction, as $t_1 \in T_1''$.

6. $\tilde{M}_1 \cong U_2(q)$. In this section we assume $\tilde{M}_1 \cong U_2(q)$, $q = 2^n > 4$, and obtain a contradiction. Let $D_1$ be as in §3 and $D = D_1^{-1}$. Also $R_0 = T_1R_2$.

(6.1) $|A_0| = 2$.

**Proof.** Assume $|A_0| > 2$. By (3.7) $O(M) = 1$. Let $K_1$ and $K = K_1^{-1}$ be as in §3. Then $T_1K \leq C(A)$ and for each $A_0^\# \leq A$, $(C_{MT}(K))' \cong L_2(q)$. Set $G_0 = N_G(K)$.

By induction $E(G_0/O(G_0)) \cong A_0$, $I_2$, or $M_{12}$. The first case is out by (3.13)(ii). Suppose $E(G_0/O(G_0)) \cong J_2$. Then $|A_0| = q = 4$, $R_2$ is elementary abelian, and $[A_0R_2, T_1] = 1$ (for this use the 3-subgroups lemma together with $[R_2, K] = 1$, $[T_1, K] = T_1$, and $[R_2, T_1] < A_1$). Also $G_0$ contains an involution $g$ interchanging $A$ and $R_2$. (Sylow 2-subgroups of $G_0$ are of type
$L_3(4)\langle \sigma \rangle$ where $\sigma$ is a graph-field automorphism. See [13].) Hence $g$ normalizes $G_2(C_2(\langle A, R_2 \rangle)) = T_1$. Since $[g, K] = 1$ and $K$ is irreducible on $T_1/A_1$, $g$ centralizes $T_1/A_1$ and, hence, also $A_1$ which is not the case.

Finally we suppose $E(G_0/O(G_0)) \simeq M_{12}$. Then $G_0/O(G_0) \cup \text{Aut}(M_{12})$ and $T_0 \not< G_0$. It follows that there is an involution in $N(T_0) \cap C(K)$ not centralizing $T_0$ (e.g. see the table in [3], which gives centralizers of involutions for $\text{Aut}(M_{12})$). This is impossible.

(6.2) $[T_1, R_2] = 1$.

**Proof.** We have $[D, R_2] = 1$ and $[D, T_1] = T_1$. So $[D, R_2, T_1] = 1$ and $[R_2, T_1, D] < [A_1, D] = 1$. The 3-subgroups lemma implies the result.

(6.3) (i) $T_1R_2$ is characteristic in $T_1R_2A_0$.
(ii) $R_2$ is characteristic in $T_1R_2$ and in $T_1R_2A_0T_3$.

**Proof.** The abelian subgroups of maximal order in $R_1A_0 = T_1R_2A_0$ are the groups $BR_2$ where $B < T_1$ and $B \simeq Z_4^n$. For if $B_1 < T_1R_2A_0$ is abelian and $B_1 \not< R_0$, then $[B_1, x] \neq 1$ for any $x \in R_2 - A_1$, $|B_1 \cap R_0| < 2^{2^n}$ and $|B_1| < |A_0|^{2^{2^n}} < 2^{3n} = |BR_2|$. So $R_0 = J(R_0A_0)$ and $R_0$ is characteristic in $R_0A_0$. Also $R_2 = Z(R_0)$. Similarly, $R_0 = J(T_1R_2A_0T_3)$ and we get the result.

**Notation.** Let $A_0 = \langle t \rangle$ and $Y = N(T_1R_2A_0)$. So $Y < N(R_2)$ by (6.3). Set $\Omega = \{AR_2/R_2\}$ and let $Y^*$ be the induced group of $Y$ on $\Omega$. Set $S_1 = T_1R_2A_0T_3$ and $S_1 < S = \text{Syl}_2(G)$ with $S \cap Y \in \text{Syl}_2(Y)$.

(6.4) $\Omega \neq \{AR_2/R_2\}$.

**Proof.** Suppose that $\Omega = \{AR_2/R_2\}$ so that $A_0R_2 < Y$. Then

\[
Y < N(T_1R_2A_0 \cap C(AR_2)) = N(T_1).
\]

If $R_2$ is elementary, then $A_1\langle t \rangle = \langle t^\sigma \rangle \cap R_2\langle t \rangle$ and so $S_1 = S$. We can obtain the structure of $S \cap Y$ in case $R_2$ is homocyclic of rank $n$ as follows. 

First note that $Y$ cannot involve $L_3(2)$ in its action on $R_2T_1/T_1$, since using the squaring map $Y$ would then involve $L_3(2)$ in its action on $A_1$. But then $Y$ involves $L_3(2)$ in its action on $T_1$ and the argument in (3.14) give a contradiction. Now apply (2.9) to the group $Y/T_1$ (setting $U = R_2T_1/T_1$ and $R = A_0T_1/T_1$). We may assume $S \cap Y = U_1T_3A_0$ where $T_1R_2 < U_1$ and $U_1$ is invariant under $T_3A_0$ and there is a subgroup $E_1 < N(U_1)$ such that $E_1$ acts as $D_1$ on $T_1R_2$. So $E = E_1^{-1}$ centralizes $U_1/T_1$. Also $E_1$ is irreducible on $T_1/A_1$ and $E$ does not centralize $T_1/A_1$. It follows that $U_1/A_1 = T_1/A_1 \times U_2/A_1$ for $U_2/A_1 = C_{U_1/A_1}(E)$. Also $U_2 = C_{U_1}(E)$ and $U_2/A_1 \simeq U_1/T_1$. By (2.4) and the fact that $R_2$ is homocyclic of rank $n$, $U_2$ is homocyclic of rank $n$, or $U_2$ is isomorphic to a Sylow 2-subgroup of $U_3(q)$ or $L_3(q)$. As in (6.2) $[U_2, T_1] = 1$.

If $R_2$ is elementary, set $U_2 = R_2$, so that we have the group $U_2$ in all cases.
We next determine $S$. If $U_2 = R_2$ is elementary then we have already noticed that $S = S_1 = T_1 R_2 T_3 A_0$. Suppose $U_2$ is nonabelian. Then
\[ N_S (S \cap Y) < N_S (Z(\Omega_1(S \cap Y))) = N(A_1) \]
and $T_1 U_2 / A_1$ is unique of its isomorphic type in $(S \cap Y) / A_1$. Consequently, $T_1 U_2 \leq N_S (S \cap Y)$. Fix $s \in N_S (S \cap Y) - S \cap Y$. Then $C_{T_1 U_2} (t^s) \approx T_1$. If $t^s \in T_1 U_2 t$, then, as $U_2 < C(T_1)$, $t^s \in U_2 t$. But
\[ t^o \cap U_2 t = R_2 t = t U_2, \]
so $s \in U_2 C(t) < S < Y$, a contradiction. So $t^s \in T_1 U_2 (T_3 A_0 - A_0)$. $t^s$ normalizes $T_1$ and $U_2$, and $C_{T_1 U_2 / A_1} (t) \approx C_{T_1 U_2 / A_1} (t^s)$ has order $q^3$. It follows that $t^s$ centralizes $U_2 / A_1$ and, hence, $U_2 (t^s) / A_1$. Then for $x \in U_2 < t^s$,
\[ [x^2, t^s] = [x, t^s]^x [x, t^s] = 1 \]
so $t^s$ centralizes $R_2$.

Therefore $R_2^{t^s} < T_1$ and is inverted by an element of $T_1 U_2 T_3 A_0$. Hence, $A_0 T_3$ is noncyclic and
\[ R_2^{t^s} / A_1 = C_{T_1 / A_1} (\Omega_1 (T_3)). \]

Thus $|N_S (S \cap Y) : S \cap Y | = 2$. We easily see that $A_1$ is characteristic in $N_S (S \cap Y)$, and so $T_1 U_2$ is characteristic in $N_S (S \cap Y)$. From the action of $N_S (S \cap Y)$ on $T_1 U_2 / A_1$ we now see that $N_S (S \cap Y) = S$. Let $\Omega_1 (T_3) = \langle \sigma \rangle$. Without loss $t^s \in t o T_1 U_2$. Since $T_1 / A_1$ is a free $F_2 \langle \sigma \rangle$-module we may assume that $t^s \in t o U_2$. Also $t^s$ centralizes $U_2 / A_1$, so $t o$ centralizes $U_2 / A_1$ and, hence, $U_2 (t^s) / A_1$. Arguing as in the previous paragraph we obtain $t o \in C(R_2)$. Write $t^s = t o u_2$ with $u_2 \in U_2$. Then $u_2 \in C_{U_2} (R_2) = R_2$. We are assuming that $R_2$ is not elementary abelian, so for $t^s$ to be an involution we must have $u_2 = A_1$. Conjugating $t^s$ by an element of $T_1$ we may assume that $t^s = t o$.

Now by (2.6), $T_1 U_2 / A_1$ is strongly closed in $S / A_1$, whence by G"oldschmidt [11], the action of $E_1$, and the knowledge of the appropriate Schur multipliers, we conclude that $\overline{T_1} \overline{U_2} \leq N(A_1)$ (where bars refer to images in $N(A_1) / O(N(A_1))$). Let $x \in S - Y$ and suppose $x \sim t$. $T_1 U_2 / A_1$ is an $F_2 \langle x \rangle$-module, so if $C / A_1 = C(x) \cap T_1 U_2 / A_1$, then $|C| > q^3$. As $x$ centralizes $C / A_1$, $|C(x)| > q^2$. Any subgroup $X$ of $T_1$ of order $q^2$ contains $A_1$. For otherwise, with $H$ a subgroup of $A_1$ of index 2 containing $X \cap A_1$, and bars indicating images modulo $H$, $\overline{T_1}$ is extraspecial of order $2 q^2$ and $\overline{X}$ is a subgroup of order $> q$ meeting $Z(\overline{T_1})$ trivially. This is impossible. Choose $g$ with $x^g = t$ and $C_S (x)^g < S$. By the above we have $t_1 \in T_1$ satisfying
\[ t_1 t_1 t_1 = t x t_1 \in t o A_1, \]
so we may assume that $[t, t^s] = 1$. Considering the action of $x$ on $C_G (t) \cap C_G (t^s)$ we have $|C_A (x)| = q$ or $\sqrt{q}$. Since $|C| > q^3$ and all involutions in
$A_1x$ are in $A_4(x)x$, we conclude that $|C_c(x)| > q^2$ or $q^2\sqrt{q}$, respectively, and in either case $\Phi(C_c(x)) \leq A_4(x)$.

Set $J = \langle x \rangle \times \langle t^x \rangle \langle C_c(x) \rangle$. Then $J^g \leq C_c(t) = T_1T_3A_0$. As $\langle x \rangle \times C_c(x)$ has class at most 2, $\langle (x) \times C_c(x) \rangle^g \leq T_1\langle t, v \rangle$. Suppose $C_A(x) = A_1$. Then for $r_2 \in R_2 - A_1$, $x$ centralizes the involution $r_2t_2$, and so $C_c(x)$ contains an elementary abelian group of order $q^2$. This implies that $\langle (x) \times C_c(x) \rangle^g \leq T_1\langle t, v \rangle$ contains an elementary subgroup of order $2q^2$, a contradiction. Therefore $|C_A(x)| = \sqrt{q}$. Then $y \in C_c(x)$ implies that

$$\left| C_c(x), y \right| < \sqrt{q} \quad \text{and} \quad |C_c(\langle x, y \rangle)| > q^2.$$

Therefore

$$y^g < T_1\langle t \rangle \quad \text{and} \quad (C_c(x)(\langle x \rangle))^g < T_1\langle t \rangle.$$

As $C_c(x)(\langle x \rangle)$ has order at least $2q^2\sqrt{q}$, we must have $A_1 < \Phi(C_c(x))$, whereas $\Phi(C_c(x)) \leq A_4(x)$ has order at most $\sqrt{q}$. This is a contradiction. We have now shown that $tG \cap S \subseteq S \cap Y$.

Now $t \neq v$ as $v$ centralizes an elementary abelian subgroup of $T_1U_2$ of order $q^2$. Hence $tG \cap \langle v \rangle T_1U_2 = \emptyset$. Next suppose that $v \sim x$ with $x$ extremal and $x \in T_1U_2$ or $x \in S - Y$. Choose $g \in G$ with $v^g = x$, $C_s(v)^g < C_s(x)$. Then $t^g \sim_s t$, because we have shown that $t^G \cap S < T_1U_2\langle v \rangle t$ and $S$ controls fusion in $t^G \cap T_1U_2\langle v \rangle t$. So $t^g = t$, without loss. Therefore $g \in N(M_1)$ and $v^g \in C(t)$. Then $v^g \in S \cap Y$, and, $x \notin T_1U_2$. This is a contradiction. Therefore $v$ is extremal.

We have $C_s(v) = FT_3\langle t \rangle \langle s_1 \rangle$, where $F = C_{T_1U_2}(v)$ is elementary of order $q^2$, and $s_1 = 1$ or $s_1 \in S - Y$. By the above $v$ is extremal in $S$ so that $C_s(v) \in Syl_2(C_G(v))$. Set $S_0 = C_s(v)$ and $I = C_{C_c(v)}$. We study the strong closure, $I$, of $F(v)$ in $S_0$ (with respect to $I$). We have already seen that $t^G \cap F(v) = \emptyset$. Since $t^G \cap Ft = A_1t = t^F$, and since no element of $Ft - A_1\langle t \rangle$ is an involution, we conclude that no element of $Ft$ is conjugate to an element of $F^\#$. Similarly, $t \sim tv$ and $t^G \cap Ftv = A_1t^v = (tv)^F$ imply that no element of $Ftv$ is conjugate to an element of $F$. Let $a \in (F(v))^\#$ and suppose $a^t \cap S_0 \not\subseteq F(v)$. Say $b \in (a^t \cap S_0) - F(v)$.

No element of $FT_3 - F(v)$ is an involution, so, by the above, $b \in S_0 - FT_3\langle t \rangle$. If $F(v)$ is not weakly closed in $S_0$ (with respect to $I$), then for some $g \in I, F(v)^g \neq (F(v))^g < S_0$, so $F(v) \cap (F(v))^g$ is maximal in $F(v)$, and we may assume that $b$ centralizes a maximal subgroup of $F(v)$. If $F(v)$ is weakly closed in $S_0$, then this also holds, as is seen from (2.6). We claim that $b$ must centralize $F$. As $b$ normalizes $T_1U_2$, $b$ normalizes $C_{T_1U_2/A_1}(F) = \hat{F}/A_1$, where $\hat{F} = T_1R_2, T_1 < T_1$, and $T_1$ is homocyclic of rank $n$. In particular, $\hat{F}$ is abelian, $\otimes_1(\hat{F}) = F$, and $\otimes_1(\hat{F}) = A_1$. Consider the action of $b$ on $\hat{F}/A_1$. Since $t^b \in tvF, t^b \in tvA_1$, and so $T_1 \cap T_1^b < C_{T_1}(tv) = A_1$, Consequently, $\hat{F}/A_1 = T_{11}/A_1 \times T_{11}/A_1$ and $\hat{F}/A_1$ is a free $F_2(b)$-module. If $b$ centralizes
Ax, then $g \in F$ implies that $gg^b$ is an involution centralized by $b$, and it follows that $b$ must centralize $F$. If $b$ does not centralize $A_1$, then $[F, b] = [A_1, b]$ has order 2 and $F/A_1 = C(b) \cap \hat{F}/A_1$. In this case there is an element $g \in T_1$ with $gg^b$ of order 4 (just choose $g$ so that $(g^2)^b \neq g^2$). But then $gg^b A_1 \not\subset F/A_1$ although $b$ centralizes $gg^b A_1$. This is a contradiction, so the claim is proved.

In view of the above claim we conclude that either $J = F\langle v \rangle$ or $J = F\langle v \rangle \langle b \rangle = C_{S_0}(F\langle v \rangle)$. Clearly $J$ is weakly closed in $S_0$, and using (2.6) we have $J$ strongly closed in $S_0$. We can then apply Goldschmidt [11] and conclude that $E(I/O(I)) \cong L_2(q^2)$ or $L_2(q) \times L_2(q)$. Let $H$ be a 2-complement in $N(F) \cap C(v)$. We consider the group $N = N_0H$, where $N_0 = N_G(F\langle v \rangle) \cap C_G(F)$. Then $N_0 \leq N$ and, since $v^g \in F$, $N$ acts on $F_v$. Also $P = N_0 \cap T_1U_2$ has order $q^4$ (check this directly using the fact that $[T_1, U_2] = 1$). As $C_p(v) = F$, $P$ is transitive on $F_v$. Consequently, $N_0$ and $N$ are each transitive on $F_v$.

In $N_0$ the stabilizer of $v$ must centralize $F\langle v \rangle$. So let $N_1 = C_{N_0}(F\langle v \rangle)$. Then $N = N_1PH$ and $H$ acts on $PN_1/N_1$. Since $C_5(v) \in \text{Syl}_2(C_G(v))$, and since $C_5(v)$ normalizes $N_0$, we conclude that $C_5(v) \cap N_1 \in \text{Syl}_2(N_1)$. Consequently, $F\langle v \rangle$ has index at most 2 in a Sylow 2-subgroup of $N_1$, and $\hat{S} = S \cap N_0 \in \text{Syl}_2(N_0)$. Now $N_0(\hat{S})$ covers $N_0/N_0$ and we observe that a 2-complement of $N_0(\hat{S})$ induces an abelian group on $\hat{S}$. For let $H_1$ be a 2-complement in $N_0(\hat{S})$. Then $H_1/C_{H_1}(F) \cong H/C_H(F)$ is abelian. Also $C_{H_1}(F) < N_1$, so $[P, C_{H_1}(F)] \subseteq F\langle v \rangle \langle g \rangle$, where $g^2 \in F\langle v \rangle$ and $g \in N(F\langle v \rangle)$. Since $C_{H_1}(F)$ centralizes $F$ and normalizes $F\langle v \rangle$, $[P, C_{H_1}(F)] = 1$, proving the claim. Since $E_{\hat{S}}^{g+1} < FH$ and $[\hat{S}, E_{\hat{S}}^{g+1}] = P$, $N_0(\hat{S}) < N_0(P)$. Therefore $N_0(\hat{S}) < N(P') = N(A_i)$. But $H\langle \hat{t} \rangle$ acts irreducibly on $F$ and $N_0(\hat{S})$ induces $H/C_H(F)$ on $F$. This is a contradiction.

We are left with the possibility that $U_2$ is homocyclic of rank $n$. If $U_2 = R_2$, then $S = S_1 = T_1R_2T_3A_0$, so suppose $U_2 > R_2$. Checking centralizers we see that

$$R_2t \subseteq t^{N(S \cap Y)} \subseteq U_2t.$$ 

So

$$N(S \cap Y) \leq N(C_{S \cap Y}(t^{S \cap Y})) \leq N(T_1) \text{ or } N(T_1\Omega_1(T_3)),$$

and as $T_1$ is characteristic in $T_1\Omega_1(T_3)$, we have $N(S \cap Y) < N(T_1)$. Consider the group $I = C_G(T_1)E_1$. We have $(S \cap Y) \cap I = U_2\langle \hat{t} \rangle$. We apply (2.9) to $I$. Even though we may have $q = 4$, the arguments used in the proof of (2.9) all carry through in this case as well. We conclude that there is a homocyclic group $\hat{U}_2 > U_2$ such that $\hat{U}_2$ has rank $n$ and $\hat{U}_2\langle \hat{t} \rangle \in \text{Syl}_2(C(T_1))$. We may assume that $T_3 < N(\hat{U}_2)$. Letting $S_2 = T_1\hat{U}_2T_3A_0$, we may assume $S_2 < S$. Choose $g \in N_5(S_2) - S_2$. Then $g \in N(Z(\hat{U}_1(S_2))) = \ldots$
$N(A_1)$, and since $T_1 \hat{U}_2 A_0 / A_1$ is the unique group of its isomorphism type in $S_2 / A_1$, we have $g \in N(T_1 \hat{U}_2 A_0)$. Checking centralizers we have $t^g \in \hat{U}_2 t$. So $g$ normalizes

$$C_{T_1 \hat{U}_2 A_0}(\langle t^{N_2(x)} \rangle) = T_1.$$

Now consider $N(T_1) / T_1$. We argue as in (3.14) that $N(T_1)$ does not involve $L_3(2)$. As $g$ normalizes $T_1 \hat{U}_2 \langle t \rangle / T_1$ we have $\langle E_1, g \rangle$ inducing a 2-transitive group on $T_1 \hat{U}_2 \langle t \rangle / T_1 \hat{U}_2 \langle \hat{U}_2 \rangle$. Using the arguments of (2.7) together with an application of the 3-subgroups theorem we conclude that there is a 2-group $\hat{U}_2 \langle t \rangle$ with $[\hat{U}_2, t] = \hat{U}_2$ and $\hat{U}_2 \leq C(T_1)$. However, $\hat{U}_2 \langle t \rangle \in \text{Syl}_2(C(T_1))$. This is a contradiction. We conclude that $T = T_1 \hat{U}_2 T_3 A_0$. We now have $S$ in all cases. If $R_2 = U_2$, let $\hat{U}_2 = \hat{U}_2$. Then $S = T_1 \hat{U}_2 T_3 A_0$.

If $T_3 A_0$ is cyclic, then we have a contradiction as follows. It is easy to check centralizers to get $t^g \cap T_1 \hat{U}_2 = \emptyset$. So by the Thompson transfer lemma and the action of $D_1$, $T_1 \hat{U}_2 \in \text{Syl}_2(\langle T_1 \hat{U}_2 \rangle^G)$, against (3.3). We may assume that $T_3 A_0 = T_3 \times A_0 > A_0$ and let $\langle v \rangle = \Omega_1(T_3)$.

Let $X = N_2(A_1)$ and let bars denote images in $X / A_1 = \bar{X}$. Then using (2.6) we have $T_1 \hat{U}_2$ strongly closed in $\bar{S}$ unless possibly $\hat{U}_2 \cong R_2$, in which case $T_1 \hat{U}_2 A_0$ is strongly closed in $\bar{S}$. By Goldschmidt [11] $(\bar{T}_3 \times A_0) \cap \langle 1 \rangle = 1$. From here we check centralizers of elements of $T_3 \times A_0$ acting on $T_1 \hat{U}_2$ and use the fact that $\bar{S} / T_1 \hat{U}_2$ is abelian to see that

$$t^x \cap T_1 \hat{U}_2 v = t^x \cap T_1 \hat{U}_2 t v = \emptyset.$$

In particular, $t \not\sim v$ and $t \not\sim t v$ in $X$. Using the action of $E_1$ on $T_1 \hat{U}_2$ and information on multipliers, we conclude that $T_1 \hat{U}_2 O(\bar{X}) \leq \bar{X}$.

If $t^g = v$ or $t v$, then for some Sylow 2-subgroups, $H$, of $M_8$, $A_1 = \Omega_1(H)$ and we may suppose $g \in N(A_1)$. But we have just seen this to be impossible. So $t \not\sim v$ and $t \not\sim t v$. Consequently, if we set $K = C_G(v) / \langle v \rangle$, then $M_2 = C_{M_2}(v) / \langle v \rangle$ is *-standard in $K$ and $N_K(M_2) \cap C(M_2)$ contains $\langle t, v \rangle / \langle v \rangle$ as a Sylow 2-subgroup. By induction we have the structure of $K / O(K)$. Similarly for $C_G(t v) / \langle t v \rangle$.

We claim that $t^G \cap T_1 \hat{U}_2 \langle v \rangle = \emptyset$. For suppose $t^g \in T_1 \hat{U}_2 \langle v \rangle$. Then $A_1 < C_t(t^g)$ and we first show that $A_1 < M_8^f$. If $a \in A_1 - M_8^f t^g$, then since all involutions in $M_8^f a$ are conjugate we have $a \not\sim v$ or $t v$. But then $C_G(v)$ or $C_G(t v)$ contains a conjugate of $T_1 \hat{U}_2 \langle t, v \rangle$, contradicting the above paragraph. Therefore $A_1 < M_8^f \langle t^g \rangle$. All involutions in $M_8^f t^g$ are fused to $t^g$, so $A_1 < M_8^f$. So we may assume that $g \in X = N_G(A_1)$, whereas we have already shown that $t^g \cap T_1 \hat{U}_2 \langle v \rangle = \emptyset$. So the claim holds and, similarly, $t^G \cap T_1 \hat{U}_2 \langle t v \rangle = \emptyset$.

Apply the Thompson transfer lemma to $S$, with $S_0 = T_1 \hat{U}_2 T_3$. By the above, $t \not\in G'$, although from the structure of $N(T_1 \hat{U}_2)$ we have $T_1 \hat{U}_2 < G$. So $S \cap G' = T_1 \hat{U}_2 \langle l \rangle$ where $\Omega_1(\langle l \rangle) = \langle v \rangle, \langle t v \rangle$, or 1. Any involution in
$T_1 \hat{U}_2$ is centralized by an abelian subgroup of order $q^3$. But from the known structure of $C_G(t)$ and $C_G(tv)$, we conclude that $v^G \cap T_1 \hat{U}_2 = (tv)^G \cap T_1 \hat{U}_2 = \emptyset$. Consequently, we may apply the Thompson transfer theorem once again and obtain a normal subgroup of $G$ with $T_1 \hat{U}_2$ as Sylow 2-subgroup. This contradicts (3.3), completing the proof of (6.4).

(6.5) $|\Omega| = q^2$, $Y^*$ is transitive on $\Omega$, and $Y^*$ contains a regular normal subgroup.

**Proof.** As $A_s^0 \cap T_1 \mathcal{R}_2 = \emptyset$ (check centralizers), $\Omega \subseteq iT_1 \mathcal{R}_2 / \mathcal{R}_2$ and $|\Omega| < q^2$. Using the action of $D_1$ we see that if $\Omega$ is not of order $q^2$, then $|\Omega| = 1 + \frac{1}{3}(q^2 - 1)$ or $1 + \frac{2}{3}(q^2 - 1)$. In the first case $Y^*$ satisfies the hypotheses, but not the conclusion, of Theorem 1.1 of Hering, Kantor and Seitz [16]. In the second case $|\Omega|$ is odd, so $Y^*$ has cyclic Sylow 2-subgroup and is solvable. By order considerations $Y^*$ is primitive. Also $D^*_1$ is semiregular on $\Omega - \{AR_2 / \mathcal{R}_2\}$. So if $N^*$ is a minimal normal subgroup of $Y^*$, $N^*$ is semiregular on $\Omega$ and $N^*D^*_1$ is a Frobenius group. But then $D^*_1$ cannot act fixed-point-freely on $T_1 \mathcal{R}_2 / \mathcal{R}_2$. So $|\Omega| = q^2$ and, by definition of $\Omega$, $Y^*$ is transitive on $\Omega$.

It remains to show that $Y^*$ contains a regular normal subgroup. In the action on $\Omega$, $\Omega \mathcal{R}_1(T_3)$ fixes all (if $T_3 = 1$) or exactly $q$ points of $\Omega$, so if $T_3 \neq 1$, then $C_y(\Omega \mathcal{R}_1(T_3))$ has Sylow 2-subgroups of order dividing $q|T_3|$. If $n$ is even then $3|q + 1$, $Y^*$ is 2-transitive and Theorem (1.1) of [16] gives the result. So suppose $n$ is odd. Then $|T_3| = 1$ or 2. Consider $Y_0 = C_y(A_i) \leq Y$. Then $Y_0^*$ has orbits of equal length on $\Omega$ and the stabilizer in $Y_0^*$ of $AR_2 / \mathcal{R}_2$ is $D^*\Omega_1(T_2)^*$. So $|Y_0^*| = (q + 1)2^a$ or $\frac{1}{3}(q + 1)2^a$ for some integer $a > 2$. Choose $N$ minimal normal in $Y^*$ with $N < Y_0$. If $D^* \cap N = 1$, then $N$ is a 2-group and is consequently a regular normal subgroup. So suppose $D^* \cap N \neq 1$. But $|N : C_N(D^* \cap N)|$ is a power of 2, so we obtain a contradiction from Burnside's Theorem.

**Notation.** Let $Y_1$ be the kernel of $Y$ on $\Omega$ and let $U$ be a Sylow 2-subgroup of the preimage of the regular normal subgroup of $Y^*$. We may assume $R_2 \lhd U \cap Y_1$, that $T_1 \mathcal{R}_2 \triangleleft U$, and that there is a subgroup $E_1$ of odd order such that $T_3$ normalizes $E_1$, $E_1$ normalizes $U_1$ and $E_1$ induces $D_1$ on $T_1 \mathcal{R}_2$. Then $U = U_1 \langle t \rangle$, where $U_1 = [U, E_1]$. Let $\hat{R}_2 = U_1 \cap Y_1$. Standard arguments imply $R_2 = \hat{R}_2$ or $\hat{R}_2 / \mathcal{R}_2 \cong A_1$ as $E_1$-modules.

(6.6) Let $U_0 = [U_1, E]$, where $E = E_1^{-1}$.

(i) $U_0$ covers $U_1 / \hat{R}_2$.
(ii) $U_1 / \hat{R}_2$ is elementary abelian.
(iii) $[U_0, \hat{R}_2] = 1$.

**Proof.** (i) is clear as $U_1 / T_1 \hat{R}_2$ and $T_1 \hat{R}_2 / \hat{R}_2$ are equivalent $E_1$-modules.
Also, by (2.2), \(U_1/R_2\) is abelian. As the normal closure of \(E\) in \(Y\) centralizes \(R_2\), \(U_0\) is in \(C(R_2)\), proving (iii).

So \(U_0\) has class 2 and for \(x, y \in U_0\), \([x^2, y^2] = [x^4, y] \in [R_2, y] = 1\). If \(U_1/R_2\) is not elementary, this would imply \(T_1\) is abelian, which is absurd. This proves (ii).

(6.7) (i) \(R_2\) is elementary abelian, \(R_2 = \hat{R}_2\), and \(U'_1 = R_2\).
(ii) \(R_2\) and \(U_1\) are characteristic in \(U_1T_3A_0 = \hat{U}\).

**Proof.** Suppose that \(R_2\) is homocyclic of rank \(n\). As \(U_0\) has class 2 and \(\hat{R}_2 \cap U_0 < Z(U_0), x, y \in U_0\) implies that \(1 = [x^2, y] = [x, y]^2\). So \([U_0, U_0] < A_1\) and for \(x \in U_0\), \(|C_U(x)| > q^5\). Choose \(x \in T_1 - A_1\). Then \(t\) normalizes \(C_U(x)\) and \(|C_T(x)| = q^2\). Consequently, \(U_1 = T_1C_{U_0}(x)\). As \([U_1, t]R_2 = T_1R_2\), we obtain a contradiction by looking at \([t, C_U(x)]\). Namely, \(C_U(x)\) covers \(U_1/T_1R_2\), so

\[T_1R_2 = [t, U_1]R_2 = [t, C_U(x)]R_2.\]

On the other hand, \([t, C_U(x)] < C_{T_1R_2}(x)\), which does not cover \(T_1R_2/R_2\). So \(R_2\) is elementary and \(R_2 = \hat{R}_2\). Also the above argument shows that \(U'_1 = R_2\).

By (6.6) \(R_2 < Z(U_1)\). Also \([U_1, t]R_2 = T_1R_2\) and \(\hat{U}' < U_1\), so \(C_{\hat{U}'}(\hat{U}'') = R_2\), is \(t\)-invariant and intersects \(T_1\) in \(A_1\). This forces \(C_{\hat{U}''}(\hat{U}'') = R_2\), so \(R_2\) is characteristic in \(\hat{U}\). In \(\hat{U}/R_2\), \(U/R_2\) is unique of its isomorphism type, proving (ii).

(6.8) \(t^G \cap U_1\langle t \rangle = t^{U_1} = U_1t\).
(ii) \(T_3A_0 = T_3 \times A_0 > A_0\).

**Proof.** Both \(U_1/R_2\) and \(R_2\) are free \(F_2\langle t \rangle\)-modules, proving (i) (no conjugate of \(t\) centralizes \(R_2\), so \(t^G \cap U_1 = 0\)). If \(T_3A_0\) is cyclic, then from (6.7)(ii) and (i) we have \(S = U_1T_3A_0\). However, we can then transfer out \(T_3A_0\), contradicting (3.3). So \(T_3A_0\) is not cyclic and (ii) holds.

(6.9) \(S > U_1T_3A_0\).

**Proof.** Suppose that \(S = U_1(T_3 \times A_0)\). First assume that \(C_S(R_2) > U_1\). Then \(C_S(R_2) = U_1\langle k \rangle\), where \(\langle k \rangle\) or \(\langle kt \rangle\) is \(\Omega_1(T_3)\). By (3.5)(a) and transfer, \(G = O_2^2(G)A_0\) with \(O_2^2(G) \cap A_0 = 1\). Clearly \(U_1 < O_2^2(G)\), so \(U_1\langle l \rangle \in Syl_2(O_2^2(G))\) with \(\Omega_1(\langle l \rangle) = \langle k \rangle\) and \(k' = k\) or \(kt\). If \((k')^{O_2^2(G)} \cap U_1 = 0\), then we have a contradiction by transfer. So assume \((k')^{G} \cap U_1 \neq 0\). Involutions in \(U_1\) have centralizers of order at least \(q^4\) (as \(U'_1 = R_2\)). Since \(U_1/R_2\) is a free \(F_2\langle k' \rangle\)-module, each involution in \(U_1k'\) is conjugate to one in \(R_2k'\). Now \(C_{U_1}(k')\) does not cover \(C(k') \cap U_1/R_2\), since \(C_{T_1}(k') = A_1\). Consequently, elements of \((k')^{G} \cap U_1\langle k' \rangle\) that are extremal in \(U_1\langle l \rangle\) are all in \(U_1\). Choosing a Sylow 2-subgroup of \(C(k')\) and extending to a Sylow 2-subgroup of \(G\), we see that there is a conjugate \(U_f^g\) of \(U_1\) such that \(k' \in U_f^g\).
and \( t \in N(U_1) \). Since \( t \not\in U_1 \), both \( U_1/R_2 \) and \( R_2 \) are free \( F_2\langle \langle t \rangle \rangle \)-modules. It follows that \( |C(t) \cap U_1| = q^3 \). Modulo \( O(M) \), \( C(t) \cap U_1 = T_t \langle t, v \rangle \), where \( T_t \subset U_1 \) and has index 2 in \( T_t \). But \( [k, T_t] \) is homocyclic of exponent 4, whereas \( [k, U_1] \not\subset R_2 \). This is impossible.

We now have \( C_5(R_2) = U_1 \). Then both \( U_1/R_2 \) and \( R_2 \) are free modules for \( \langle t \rangle \), \( \langle v \rangle \), and \( \langle tv \rangle \), so we conclude that each involution in \( S - U_1 \) is conjugate in \( U_1 \) to one of \( t, v, tv \). The arguments of the previous paragraph show that \( t^G \cap U_1 = (tv)^G \cap U_1 = \emptyset \), and we know that \( t^G \cap U_1 = \emptyset \), as \( t \) cannot centralize a conjugate of \( R_2 \). Suppose \( t^G \cap U_1 = \emptyset \). Then apply the Thompson transfer theorem twice to conclude that \( U_1 \in \text{Syl}_2(U_1^G) \), which contradicts (3.3). So assume that \( t^g \in U_1 \langle v \rangle \) for some \( g \in G \). By the above we may assume that \( t^g = v \). Also we may assume \( g \) normalizes \( C_{M_1}(v)^{\langle \infty \rangle} \) and \( A_1 \). So \( (A_1 \langle t \rangle)^g = A_1 \langle v \rangle \). A Sylow 2-subgroup of \( N_G(A_1 \langle v \rangle) \) is a conjugate of \( T_t R_2 T_3 \langle t \rangle \). Note that \( R_2 \leq N(A_1 \langle v \rangle) \). We claim that \( R_2 \) is strongly closed in \( T_t R_2 T_3 \langle t \rangle \) with respect to \( G \). Suppose \( r \in R_2 \) and \( r' \in T_t R_2 T_3 A_0 - R_2 \). Then \( r' \notin T_t R_2 \), as \( \Omega_4(T_t R_2) = R_2 \). So \( [t, r'] \in T_t - A_1 \). However, \( \langle t, r' \rangle \) is dihedral, so \( t \) inverts \( [t, r'] \). This is impossible. By the above we have now established the claim. So we may take \( g \in N(R_2) \). Now use (2.6) and Goldschmidt [11] to conclude that \( U_1 O(N_G(R_2)) \leq N_G(R_2) \). Consequently, we may assume \( g \in N(U_1) \).

It now follows that \( g \) can be chosen as a 3-element in \( N(S) \) and \( t \sim v \sim tv \). In particular, \( T_3 = \langle v \rangle \). Consider the group \( N = N_G(U_1) \) and let bars denote images in \( N/C_N(U_1/R_2) \). The group \( C_N(R_2) \) is 2-closed and \( \overline{E} = C_N(R_2) \). Also, since \( g \) normalizes \( A_1 \) and \( C_{M_1}(v)^{\langle \infty \rangle} \), we may assume \( g \) normalizes \( E_{1+1} \). Say \( \overline{H} \) is minimal normal in \( N \) with \( H \preceq C_N(R_2) \). \( \overline{H} \) is an elementary \( l \)-group for some prime \( l \). Then

\[
\overline{H} = C_{\overline{H}}(t)C_{\overline{H}}(v)C_{\overline{H}}(tv).
\]

But \( C_{\overline{H}}(t) \not\leq \overline{E} \) and \( C_{\overline{H}}(t) \cap C_{\overline{H}}(tv) = 1 \). So \( \overline{H} \) has order \( l^2 \). As \( E_{1+1} \) centralizes \( C_{\overline{H}}(t) \), we use the action of \( g \) to see that \( E_{1+1} \overline{H} = E_{1+1} \times H = B \). However, \( U_1/R_2 \) has at most 2 \( B \)-composition factors, as \( E_{1+1} \times C_{\overline{H}}(t) \) is irreducible on \( T_t R_2/R_2 \) and on \( U_1/T_t R_2 \). This implies \( B \) has rank at most 2, and we have a contradiction. This completes the proof of (6.9).

Let \( N_1 = N_G(U_1) \) and let bars denote images in \( N_1 \) modulo \( C_{N_1}(U_1/R_2) \). Write \( \langle v \rangle = \Omega_4(T_3) \). There is an element \( s \in N_5(U_1 T_3 A_0) \) such that \( t^s \not\in U_1 t \). Now \( s \) acts on \( T_3 A_0 U_1/U_1 = T_3 \times A_0 \). So if \( |T_3| > 2 \) we have \( t^s \in U_1 tv \). If \( |T_3| = 2 \) we rechoose \( T_3 \), if necessary, so that in all cases \( t^s \in U_1 tv \). Since each of \( R_2 \) and \( U_1/R_2 \) is a free \( F_2\langle \langle t^s \rangle \rangle \)-module, we may assume \( t^s = tv \). Then \( s \) normalizes \( \overline{N} = C_{\overline{M}}(t, t^s) \). So \( s \) normalizes \( \overline{E}_{00,0} \), where \( E_{00} = E_{1+1} \). Also \( s \) normalizes \( O(N_1) \geq \overline{E} \).

Now \( [\overline{E}_{00}, \overline{E}] = [\overline{E}_{00}, \overline{E}^2] = 1 \), so that \( W = \langle \overline{E}_{00}, \overline{E}, \overline{E}^2 \rangle \) centralizes \( \overline{E}_{00} \). It
follows that \( V = U_1/R_2 \) may be regarded as an \( F_2 \)-module for \( W \) and that \( V \) is either irreducible under the action of \( W \) or the sum of two irreducibles. That is, \( V \) is the sum of irreducible submodules of dimension 1 or 2. Passing to splitting fields and noting that \( W \) has odd order, we see that each of the submodules splits into linear factors. Consequently, \( W \) is abelian.

Since \( t \) centralizes \( E \) and \( t^2 \) inverts \( E \), we know that \( EE^t = E \times E^t \). From here we see that \( V \) splits into a sum \( V = V_1 \oplus V_2 \) of inequivalent irreducible modules for \( E_1E^t = E_0 \times E \times E^t = W \).

This decomposition of \( V \) gives further information about \( U_1 \) as follows. As \( E_1 \) is fixed-point-free on \( V \), \( t \) cannot stabilize \( V_1 \) and \( V_2 \). For otherwise \( t \) would centralize \( W/C_w(V_i), i = 1, 2 \), which implies that \( [W, t] = C_w(V_i) \cap C_w(V_j) = 1 \), a contradiction. So \( t \) interchanges \( V_1, V_2 \). Also \( t \) acts on \( \{V_1, V_2\} \) as \( s \in N(W) \). Consequently, \( t \in \langle s, t \rangle \) must fix each of \( V_1 \) and \( V_2 \).

For \( i = 1, 2 \) let \( F_i = C_{E_i}(V_i) \). Then \( F = F_1F_2 = EE^t \). Write \( J_i/R_2 = V_i \). Then for \( j \) in \( J_1, j \in J_2, [J_1, J_2] = [J_1, J_2] \) for each \( g \in F_i \). So \( J_2, F \) are \( C(J_1) \), and, as \( J_2, F_1 \) covers \( J_2/R_2 \), we conclude that \( [J_1, J_2] = 1 \).

Let \( J_1 \) in \( J_1 \). As \( |J_1/R_2| = |R_2| = q^2, [J_1, J_1] < R_2 \). Suppose \( q \neq 8 \). Then \( F_2 \) is irreducible on \( J_1/R_2 \), and \( g \in F_2 \) implies \([J_1, J_1] = [J_2, J_1] \). So here \( [J_1, J_1] < R_2 \), and using the fact that \( [J_1, J_2] = 1 \) and the action of \( E_0 \), we have \([J_1] = q \). If \( q = 8 \) an easy Lie ring argument shows that \([J_1] = 8 \). Similarly, \([J_2] = q \). Setting \( Y_t = [J_1, E] \) and using (6.7) we now have \( U_1 = Y_1 \times Y_2 \) with \( t \) interchanging \( Y_1 \) and \( Y_2 \).

Since \( t \) is an involution, \( Y_1 \simeq Y_2 \simeq C_{U_1}(t) = T_1 \). Hence \( R_2 = \Omega_{1}(U_1) \). The Krull-Schmidt theorem implies that \( \{Y_1Z(U_1), Y_2Z(U_1)\} \) is invariant under \( N_G(U_1) = N_1 \). Also

\[
N_1 = N_{11}(Y_1Z(U_1)) \langle t \rangle, \quad N_0 = N_{11}(Y_1Z(U_1)) < N_1.
\]

We choose \( T_3 \) and \( s \) so that \( \langle T_3, s \rangle < N_0 \). Note that \( s \) and \( v \) normalize \( E_0 \), \( Y_1 \), and \( Y_2 \). Also \( F_1 = F_1 \times F_2 = E \times E^t \). Let \( \tilde{P}_1 \) be a Sylow \( p \)-subgroup of \( \tilde{F}_1 \), where \( p = 3 \) if \( q = 8 \) and \( p \) a primitive divisor of \( q + 1 \) if \( q \neq 8 \). Then \( \tilde{s} \) must centralize one of \( \tilde{P}_1 \) and \( \tilde{P}_2 \) and invert the other. Say \( \tilde{s} \) centralizes \( \tilde{P}_1 \). Then \( \tilde{P}_1 \) normalizes \( \langle \tilde{s}, V_2 \rangle \). If \( \tilde{s} \notin C(E_0) \), then considering the Frobenius group \( [E_0, \tilde{s}] \langle \tilde{s} \rangle \), we conclude that \( V_2 \) is a free \( F_2 \langle \langle \tilde{s} \rangle \rangle \)-module. But then \( \tilde{P}_1 \) cannot act on \( \langle V_2, \tilde{s} \rangle \). Therefore \( \tilde{s} \) centralizes \( E_0 \) and, as \( \tilde{E}_0 \tilde{P}_1 \) is irreducible on \( V_2, \tilde{s} \in C(V_2) \). This forces \( \tilde{s} \in C(\tilde{F}_1) \).

Now consider the action of \( \langle E_0, s, v \rangle \) on \( R_2 \) to get \( \langle s, v \rangle \leq C(R_2) \). Further, an application of the 3-subgroups lemma to \( \langle s \rangle, Y_2 \) and \( F_1 \) shows that \( [s, Y_2] = 1 \).

We know that \( \Omega_{1}(T_3) \langle t \rangle \) is a Sylow 2-subgroup of \( C_{\tilde{C}}(t) \), so the Sylow 2-subgroups of \( C(t) \)(and \( U_1 \)) are dihedral or quasidihedral, where \( C = N(U_1) \cap C(R_2) \). Now \( C \cap C(Y_i) \leq C \) for \( i = 1, 2 \), and from these facts we see that \( \langle s, v \rangle U_1 \in \text{Syl}_2(C) \), \( \langle s, v \rangle U_1/U_1 \) is klein, and \( C/U_1 \) has 2-complement
Let \( C_i = C_N(Y_i) \) so that \( F_i \subset C_i \). Then \([C_1, C_2] \subset C(U_1)\) and \([C_1, C_2] = 1\). As \( C_2 \supseteq C_1, C_1 C_2 \supseteq C_1 C(t)\). This shows that \( F_1 F_2 = O(C(C_0)) \). Let \( C_i \), \( C_j \). Let \( C_i \), \( C_j \).

As \( U_1 < C < C(R_2) \) implies that \( U_1 \cap U_1 < C \), and the involutions in \( U_1 < C \), \( v^t \) are in \( U_1 \setminus U_1 \). As \( U_1 \) is irreducible on \( Y_2 \). As \( N_0 < C(F_1), N_0 \) induces a cyclic group on \( V_2 \). Similarly for \( V_1 \). So \( N_0 \) is abelian of odd order and of rank at most 2. Since \( C(t) \) covers \( C(t) \), \( N_0 \), \( N_0 \) has Sylow 2-subgroups of rank 2. It is now easy to check that \( U_1 \) is characteristic in \( S \), so \( S = S \cap N_1 \subset Syl_2(G) \).

As \( \Omega_1(S \cap N_0) < U_1 < C(R_2), t^G \cap (S \cap N_0) = \emptyset \), and we apply transfer to obtain a subgroup \( G_0 \) of index 2 in \( G \) with \( U_1 < G_0 \). Considering the action of \( t \) on a Sylow 2-subgroup of \( C_G(v) \), we see that \( v^G \cap U_1 < S > = G_0 \). Again we can apply transfer to get \( v \not\in G' \). But then \( s^t \not\in s V_1 \) implies that \( U_1 \subset Syl_2(G') \), contradicting (3.3).

We have now completed the proof of the main theorem.

**Bibliography**


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