BOUNDED POINT EVALUATIONS AND SMOOTHNESS PROPERTIES OF FUNCTIONS IN $R^p(X)$

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ABSTRACT. Let $X$ be a compact subset of the complex plane $\mathbb{C}$. We denote by $R_0(X)$ the algebra consisting of the (restrictions to $X$ of) rational functions with poles off $X$. Let $m$ denote 2-dimensional Lebesgue measure. For $p > 1$, let $L^p(X) = L^p(X, dm)$. The closure of $R_0(X)$ in $L^p(X)$ will be denoted by $R^p(X)$. Whenever $p$ and $q$ both appear, we assume that $1/p + 1/q = 1$.

If $x$ is a point in $X$ which admits a bounded point evaluation on $R^p(X)$, then the map which sends $f$ to $f(x)$ for all $f \in R_0(X)$ extends to a continuous linear functional on $R^p(X)$. The value of this linear functional at any $f \in R^p(X)$ is denoted by $f(x)$. We examine the smoothness properties of functions in $R^p(X)$ at those points which admit bounded point evaluations. For $p > 2$ we prove in Part I a theorem that generalizes the "approximate Taylor theorem" that James Wang proved for $R(X)$.

In Part II we generalize a theorem of Hedberg about the convergence of a certain capacity series at a point which admits a bounded point evaluation. Using this result, we study the density of the set $X$ at such a point.

PART I. SMOOTHNESS PROPERTIES OF FUNCTIONS IN $R^p(X)$

Let $X$ be a compact subset of the complex plane $\mathbb{C}$. We denote by $R_0(X)$ the algebra consisting of the (restrictions to $X$ of) rational functions with poles off $X$. Let $m$ denote 2-dimensional Lebesgue measure. For $p > 1$, let $L^p(X) = L^p(X, dm)$. The closure of $R_0(X)$ in $L^p(X)$ will be denoted by $R^p(X)$. Whenever $p$ and $q$ both appear, we will assume that $1/p + 1/q = 1$.

1. Bounded point derivations.

Definition (1.1). For $x \in X$ we say that $x$ admits a bounded point derivation of order $s$ on $R^p(X)$ if there exists a constant $C$ such that $|f^{(s)}(x)| \leq C\|f\|_p$ for all $f \in R_0(X)$.

When $x$ admits a bounded point derivation of order $s$ on $R^p(X)$, the map $f \mapsto f^{(s)}(x)/s!$ extends from $R_0(X)$ to a bounded linear functional on $R^p(X)$.
We denote this bounded linear functional by $D_x$.

**Definition (1.2).** When $x$ admits a bounded point derivation of order 0, we say that $x$ admits a *bounded point evaluation*. For $f \in R^p(X)$ we define $f(x) = D_x^0 f$.

**Definition (1.3).** For each $p > 2$ the *inner set* for $R^p(X)$ is the set of points in $X$ which admit bounded point evaluations, and we denote it by $S^p(X)$.

**Proposition (1.1).** For each $p > 2$, $S^p(X)$ is an $F_σ$ set.

**Proof.** Write $S^p(X) = \bigcup_{n=1}^{\infty} S^n_p(X)$ where

$$S^n_p(X) = \{x \in X | |f(x)| < n\|f\|_p \text{ for all } f \in R^p(X)\}.$$ 

We show that each set $S^n_p(X)$ is closed. Suppose that $\{x_k\} \subset S^n_p(X)$ and that $x_k \to x \in X$. Let $L_{x_k} f = f(x_k)$ and observe that the $L_{x_k}$ are a family of linear functionals bounded in norm by $n$. Since $L_{x_k} f \to f(x)$ for $f \in R_0(X)$, and $R_0(X)$ is dense in $R^p(X)$, it follows that $x \in S^n_p(X)$. Thus each $S^n_p(X)$ is closed.

2. **Potentials and representing functions.** In this paper $z$ will denote the identity function.

**Definition (2.1).** Let $\psi$ be a positive nondecreasing function on $(0, \infty)$. For each $g \in L^q(X)$, $q > 1$, we define the $\psi$-potential of $g$, $U^\psi_g$, by

$$U^\psi_g (y) = \int \frac{|g|}{\psi(|z - y|)} \, dm.$$ 

If $1/\psi(|z|)$ is locally summable with respect to $m$, Fubini’s theorem implies that $U^\psi_g$ is locally summable; hence $U^\psi_g < \infty$ a.e. ($m$).

**Definition (2.2).** When $\psi(r) = r$, we denote $U^\psi_g$ by $\hat{g}$.

**Definition (2.3).** When $\psi(r) = r^q$, $1 < q < 2$, we denote $U^\psi_g$ by $U^q_g$.

**Definition (2.4).** We define the Cauchy transform of $g$ to be

$$\hat{g}_0 (y) = \int (z - y)^{-1} g \, dm \quad \text{for all } y \text{ where } \hat{g}_0 (y) < \infty.$$ 

For the proof of the following lemma we refer the reader to Sinanjan [16] or Brennan [1, pp. 10-11]. Brennan’s proof uses the Cauchy transform.

**Lemma (2.1).** Let $X \subset C$ be compact and have no interior. Then $R^p(X) = L^p(X)$ for $1 < p < 2$.

It follows from the Riesz representation theorem that if $x \in S^p(X)$, then there is a $g \in L^q(X)$ such that $f(x) = \int fg \, dm$ for all $f \in R^p(X)$. We call such a $g$ a *representing function* for $x$. If $R^p(X) \neq L^p(X)$, there is a nonzero function $g \in L^q(X)$ such that $\int fg \, dm = 0$ for all $f \in R^p(X)$. We call such a $g$ an annihilating function.
The following lemma was proved by Bishop for the sup norm case: We assume that $1 < q < 2$.

**Lemma (2.2).** Let $g \in L^q(X)$ be an annihilating function. Suppose that $\hat{g}(y)$ is defined and $\neq 0$, and that $(z - y)^{-1}g \in L^q(X)$. Then $\hat{g}(y)^{-1}(z - y)^{-1}g$ is a representing function for $y$.

**Proof.** If $f \in R_0(X)$, then $f = f(y) + (z - y)h$ for some $h \in R_0(X)$. Hence
\[
\int (z - y)^{-1}fg\,dm = f(y)\hat{g}(y) + \int hg\,dm = f(y)\hat{g}(y).
\]

**Corollary (2.1).** Let $g \in L^q(X)$ be a representing function for $x$. Let
\[
c(y) = \int (z - x)(z - y)^{-1}g\,dm = 1 + (y - x)\hat{g}(y).
\]
Then $c(y)^{-1}(z - x)(z - y)^{-1}g$ is a representing function for $y$ whenever $c(y)$ is defined and $\neq 0$.

**Proof.** $(z - x)g$ is an annihilating function.

We now present a lemma of Brennan in [2, p. 288] which will be very useful.

**Lemma (2.3).** If $p > 2$, then $R^p(X) \neq L^p(X)$ if and only if $S^p(X)$ has positive 2-dimensional measure.

**Proof.** Suppose that $S^p(X) \neq \emptyset$ and $x \in S^p(X)$ is represented by a nonzero function $g \in L^q(X)$. Then $R^p(X) \neq L^p(X)$ because $(z - x)g \in L^q(X)$, and $\int (z - x)gf\,dm = 0$ for all $f \in R^p(X)$.

Now suppose that $R^p(X) \neq L^p(X)$ and let $g \in L^q(X)$ be a nonzero annihilating function. Then $\hat{g}$ fails to vanish on a set of positive measure in $X$. Hence there is a set $S \subset X$ of positive measure such that for $y \in S$, $\hat{g}(y) \neq 0$ and $\hat{g}(y)^{-1}(z - y)^{-1}g \in L^q(X)$. It follows from Corollary (2.1) that $S \subset S^p(X)$, and the lemma is proved.

**Remark.** If we know that there is an $x \in S^2(X)$, the difficulty in showing that there are other points in $S^2(X)$ by the above method is that $z^{-1} \not\in L^2_{\text{loc}}$.

3. **Admissible functions.** Fix $x \in C$ and let $\Delta_n = \{y \in C: |y - x| < 1/n\}$. We say that a set $E \subset C$ has full area density at $x$ if $\lim_{n \to \infty}m(E \cap \Delta_n)/m(\Delta_n) = 1$. Let $F$ be a function defined on $X$, $x \in X$. We say that $a$ is the approximate limit of $F$ at $x$, and write $\text{app lim}_{y \to x}F(y) = a$ if there exists a subset $E$ of $X$ having full area density at $x$, such that $\lim_{y \to x}F(y) = a$. We say that $F$ is approximately continuous at $x$ if $\text{app lim}_{y \to x}F(y) = F(x)$.

If $\phi$ is a positive function on $(0, \infty)$ with $\lim_{r \to 0}\phi(r) = 0$, we say that $F$ admits $\phi$ as a modulus of approximate continuity at $x$ if $|F(y) - F(x)| < \phi(|y - x|)$. 
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\( \phi(|y - x|) \) for all \( y \) in a set having full area density at \( x \). We say that \( F \) satisfies an approximate Hölder condition of order \( \alpha \) at \( x \) if \( F \) admits \( C^{r^\alpha} \) as a modulus of approximate continuity at \( x \) for some constant \( C \).

**Definition (3.1).** We say that \( \phi \) is an admissible function if
(a) \( \phi \) is a positive, nondecreasing function defined on \((0, \infty)\), and
(b) the associated function \( \psi \), defined by \( \psi(r) = r/\phi(r) \), is nondecreasing, with \( \psi(0 + ) = 0 \).

**Example.** For any \( \alpha, 0 < \alpha < 1 \), \( \phi(r) = r^\alpha \) is admissible.

**Remarks.** 1. If \( \phi \) is admissible and \( 0 < \beta < 1 \), then \( \phi^\beta \) is also admissible because
\[ r/\phi^\beta(r) = (r/\phi(r)) \cdot \phi^{1-\beta}(r). \]

2. In using an admissible function \( \phi \) we will often refer to the triangle inequality: \( \phi(r) \leq \phi(r_1) + \phi(r_2) \) whenever \( r \leq r_1 + r_2 \). This follows from the definition of an admissible function since
\[ \phi(r) \leq \phi(r_1 + r_2) = (r_1 + r_2)/\psi(r_1 + r_2) \leq r_1/\psi(r_1) + r_2/\psi(r_2) = \phi(r_1) + \phi(r_2). \]

Wang introduced a special kind of admissible function in [17, p. 349].

**Definition (3.2).** We say that the admissible function \( \phi \) is nice if
\[ \int_0^{\infty} \phi(r)^{-q} dr < \infty. \]

For each \( q, 1 < q < 2 \), we will be interested in a subset of the set of nice admissible functions.

**Definition (3.3).** We say that the admissible function \( \phi \) is \( q \)-nice if
\[ \int_0^{\infty} (q/\phi(r))^{-q} dr < \infty. \]

Note that a nice admissible function is \( 1 \)-nice and that \( \phi(r) = r^\alpha \) is \( q \)-nice for \( \alpha < (2 - q)/q \). When \( p > 2 \), the \( q \)-nice admissible functions will be the most likely ones to be moduli of approximate continuity for functions in the unit ball of \( R^p(X) \) at points in \( S^p(X) \).

The following lemma is due to Wang [17]:

**Lemma (3.1).** Let \( g \in L^q(X), q \geq 1 \), and let \( x \in X \). Then there exists a nice admissible function \( \phi \) with \( \phi(0 + ) = 0 \) such that \( \phi(|z - x|)^{-1}g \in L^q(X) \).

**Proof.** See Wang [17].

Our proof of the next lemma is in the spirit of Browder’s result [3, p. 157]. It will be useful for studying the density of \( X \) at points in \( S^p(X) \). Let \( E \subseteq X \) be measurable. Define \( \rho_n \) by \( \pi \rho_n^2 = m(\Delta_n \setminus E) \). Denote \( m|\Delta_n \setminus E \) by \( m_n \).

**Lemma (3.2).** Let \( \psi \) be associated with an admissible \( \phi \). For \( q, 0 < q < 2 \), let \( \tau = \psi^q \). Then if \( g \in L^1(X) \),
\[ \lim_{n \to \infty} \frac{n^q}{\rho_n^{2-q}} \int \tau(|y - x|)U_g^*(y) \, dm_n(y) = 0. \]
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**Proof.** Define

$$F_n(\xi) = n^q \rho_n^{q-2} \int \psi(|y - x|) \cdot \psi(|\xi - y|)^{-q} \, dm_n(y).$$

Then $F_n(x) < \infty$ and if $\xi \neq x$, we have for large $n$

$$|F_n(\xi)| < n^q \rho_n^q \psi(n^{-1})^q \cdot \psi(|x - \xi| - n^{-1})^q \to 0 \quad \text{as } n \to \infty.$$ 

Next, we will show that the $F_n$ are bounded independently of $n$. Let $D_n = \Delta(\xi, \rho_n)$. Since $\psi^q$ is increasing,

$$|F_n(\xi)| \leq n^q \rho_n^{q-2} \psi(n^{-1})^q \int \psi(|y - \xi|)^{-q} \, dm_n(y)$$

$$\leq n^q \rho_n^{q-2} \psi(n^{-1})^q \int_{D_n} \psi(|y - \xi|)^{-q} \, dm(y)$$

$$= 2\pi n^q \rho_n^{q-2} \psi(n^{-1})^q \int_0^{\rho_n} \psi(r)^{-q} \, dr$$

$$< 2\pi n^q \rho_n^{q-2} \psi(n^{-1})^q \phi(\rho_n)^q \int_0^{\rho_n} r^{1-q} \, dr$$

$$= 2\pi n^q \rho_n^{q-2} \psi(n^{-1})^q \phi(\rho_n)^q \rho_n^{2-q}(2-q)^{-1}$$

$$< 2\pi(2-q)^{-1}.$$ 

Thus, the $F_n$ converge boundedly a.e. to zero. We apply the dominated convergence theorem and Fubini's theorem to obtain the lemma.

**Lemma (3.3).** Let $\psi$ be associated with an admissible $\phi$. For $0 < q < 2$, let $\tau = \psi^q$. Then if $g \in L^q(X)$, and $\delta > 0$, the set $E = \{ y \in \mathbb{C} : \tau(|y - x|) U^\tau(g)(y) < \delta \}$ has full area density at $x$.

**Proof.** It is sufficient to prove that $\lim_{n \to \infty} m(\Delta_n \setminus E) / m(\Delta_n) = 0$ where $\Delta_n = \Delta(x, 1/n)$. We observe that since

$$m(\Delta_n \setminus E) \leq \delta^{-1} \int_{\Delta_n} \tau(|y - x|) U^\tau(g)(y) \, dm(y),$$

it is sufficient to prove that

$$\lim_{n \to \infty} n^2 \int_{\Delta_n} \tau(|y - x|) U^\tau(g)(y) \, dm(y) = 0.$$ 

This follows from Lemma (3.2) if we take $E$ in that lemma to be the empty set.

**4. The main theorem.** The following lemma in the sup norm case is due to Wilken [20]. For $x \in S^p(X)$, $p > 2$, it gives a condition for $x$ to admit a bounded point derivation of order $s$.

**Lemma (4.1).** Suppose there exist a representing function $g \in L^q(X)$ for
Let \( x \in S^p(X) \), \( p > 2 \), and a nonnegative integer \( s \) such that \( (z - x)^{-s}g \in L^q(X) \). Let \( c_j = \int (z - x)^{-j} g \ dm \) \((0 < j < s)\) and define \( G_0, \ldots, G_s \) by:

\[
G_0 = g, \quad G_j = (z - x)^{-j} g - \sum_{k < j} c_{j-k} G_k.
\]

Then \( D_x f \) exists, and \( D_x f = \int f G_j \ dm \) for all \( f \in R^p(X), 0 < j < s \).

An additional lemma will be needed in proving the theorem.

**Lemma (4.2).** Let \( s \) be a nonnegative integer, and \( g \in L^q(X), 1 < q < 2 \). Suppose that \( (z - x)^{-s}g \in L^q(X) \). Set \( H_j = (z - x)^{-j} g \) \((0 < j < s)\). For any \( f \in L^p(X) \) and \( y \in C \)

\[
\int (z - y)^{-1} f g \ dm = \sum_{j=1}^s (y - x)^{j-1} \int f H_j \ dm + (y - x)^{s} \int (z - y)^{-1} f H_s \ dm.
\]

**Proof.** Since \( H_j = (z - x) H_{j+1} \) for \( 0 < j < s \),

\[
\int (z - y)^{-1} f H_{j} \ dm = \int f H_{j+1} \ dm + (y - x) \int (z - y)^{-1} f H_{j+1} \ dm
\]

which implies the lemma.

Our main theorem generalizes the “approximate Taylor’s theorem” which Wang obtained for functions in \( R(X) \) \([17, p. 352]\).

**Theorem (4.1).** Let \( \phi \) be an admissible function and \( s \) a nonnegative integer. Suppose that \( p > 2 \) and that there is an \( x \in S^p(X) \) represented by a \( g \in L^q(X) \) such that \( (z - x)^{-s} \phi(|z - x|) \in L^q(X) \). Then for every \( \epsilon > 0 \) there is a set \( E \) in \( X \) having full area density at \( x \) such that for every \( f \in R^p(X) \)

(i) \( f = \sum_{j=0}^s D_{x}^j f(z - x)^j + R \) where \( R \in R^p(X) \) satisfies

(ii) \( |R(y)| < \epsilon |y - x|^s \phi(|y - x|) \| f \|_p \) for all \( y \in E \), and

(iii) \( \text{app lim}_{y \to x} \{ R(y)/|y - x|^s \phi(|y - x|) \} = 0 \).

**Proof.** Since \( (z - x)^{-s}g \in L^q(X) \), Lemma (4.1) implies that the \( D_x^j \) exist for \( 0 < j < s \). To each \( D_x^j \), \( 0 < j < s \), there corresponds a constant \( C_j \) such that \( |D_x^j f| < C_j \| f \|_p \) for all \( f \in R^p(X) \). By Minkowski's inequality there is another constant \( C \) such that if \( R \) is defined as in (i), \( \| R \|_p < C \| f \|_p \) for all \( f \in R^p(X) \).

Choose \( \delta > 0 \) so that \( 0 < C \delta (1 - \delta)^{-1} < \epsilon / 2 \). If \( y \in E_1 = \{ y \in C : |y - x| \hat{g}(y) < \delta \} \), then \( c(y) = 1 + (y - x) \hat{g}(y) \) is well defined, and \( |c(y)| > 1 - \delta \). By Corollary (2.1),
\[
R(y) = c(y)^{-1} \int \left[ R(z - x)/(z - y) \right] g \, dm \\
= c(y)^{-1} \int R[1 + (y - x)/(z - y)] g \, dm \\
= c(y)^{-1}(y - x) \int [R/(z - y)] g \, dm.
\]

Next, we claim that \( R(y) = c(y)^{-1}(y - x)^{t+1}(z - x)^{-s}(z - y)^{-1}Rg \, dm \). This claim depends on Lemma (4.2). Each of the functions \( (z - x)^{-j}g \), \( 0 < j < s \), is a linear combination of functions representing \( D_x^k \), \( 0 < k < j \), which implies that \( f(z - x)^{-j}Rg \, dm = 0 \) for \( 0 < j < s \), and the claim is proved.

Factoring \( g = \phi(|z - x|)h \) where \( h \in L^q(X) \), we obtain by the “triangle inequality” that
\[
|g| \leq \phi(|z - y|)|h| + \phi(|y - x|)|h|.
\]

Consequently,
\[
|R(y)| \leq c(y)^{-1}|y - x|^{t+1} \left[ \int |z - y|^{-1}|z - x|^{-s} \phi(|z - y|) |Rh| \, dm \\
+ \int |z - y|^{-1}|z - x|^{-s} \phi(|y - x|) |Rh| \, dm \right].
\]

Denote the first integral by \( I_1 \) and the second by \( I_2 \). We have
\[
I_1 = |c(y)|^{-1}|y - x|^{t+1} \int |z - y|^{-1}|z - x|^{-s} \phi(|y - x|) \psi(|y - x|) \int \psi(|z - y|)^{-1}|z - x|^{-s} |Rh| \, dm.
\]

Let \( \tau = \psi^q \), \( k = (z - x)^{-q}h^q \), and
\[
E_2 = \{ y \in \mathbb{C}: \tau(|y - x|) U_k^i(y) < \delta^q \}.
\]

For \( y \in E_2 \) we apply Hölder’s inequality to obtain
\[
I_1 \leq (1 - \delta)^{-1}|y - x|^{t+1} \phi(|y - x|) \psi(|y - x|) \int |R|^{p} \, dm \right)^{1/p} \{ U_k^i(y) \}^{1/q}
\leq (1 - \delta)^{-1}|y - x|^t \phi(|y - x|) C \| f \|_p \delta
\leq (\varepsilon/2)|y - x|^t \phi(|y - x|) \| f \|_p.
\]

To estimate \( I_2 \) we define
\[
E_3 = \{ y \in \mathbb{C}: |y - x|^q \leq \delta^q \} \quad \text{and let } y \in E_2 \cap E_3.
\]

By Hölder’s inequality,
By Lemma (3.3) the set \( E = E_2 \cap E_3 \) has full area density at \( x \), and we have proved that for \( y \in E \)

\[
|R(y)| \leq I_1 + I_2 \leq \varepsilon |y - x| \phi(|y - x|) \|f\|_p
\]

for any \( f \in R^p(X) \). To prove (iii) let \( L_y f = R(y)/|y - x| \phi(|y - x|) \). The above result implies that \( \|L_y\| \leq \varepsilon \) for \( y \in E \). Let \( y \to x \) in such a way that \( y \) stays in \( E \). Then \( L_y f \to 0 \) as \( y \to x \) for \( f \in R_0(X) \), and since \( R_0(X) \) is dense in \( R^p(X) \), (iii) follows.

An interesting consequence of the above theorem is that we can take the limit of Newton quotients in the set \( E \) to evaluate \( D_y f \). For \( f \) a function defined on a subset of \( X \), \( h \in C \), we set

\[
\Delta_h f = f(z + h) - f
\]

so \( \Delta_h f \) is a function defined on a subset of \( X \). We define inductively \( \Delta_h^0 = \text{id}, \Delta_h^j = \Delta_h \circ \Delta_h^{-1} \) for \( j > 1 \). The sup norm version of the following corollary is proved in [17].

**Corollary (4.1).** If \( x \) admits a bounded point derivation of order \( s \) on \( R^p(X), p > 2 \), then for all \( f \in R^p(X) \)

\[
\Delta_h^s f = \text{app lim}_{h \to 0} \frac{\Delta_h^s f(x)}{s! h^s}.
\]

**Lemma (4.3).** Let \( \phi \) be a \( q \)-nice admissible function. If \( x \in S^p(X), p > 2 \), then \( \{y \in X: \exists \text{ a function } g_y \text{ that represents } y \text{ for } R^p(X) \text{ and satisfies } \phi(|z - y|)^{-1} g_y \in L^q(X)\} \) has full area density at \( x \).

**Proof.** Let \( g \in L^q(X) \) represent \( x \).

Let

\[
F = \left\{ y \in C: \int |z - y|^{-q} \phi(|z - y|)^{-q} |g|^q \, dm < \infty \right\}.
\]

Since \( |z|^{-q} \phi(|z|)^{-q} \) is locally summable with respect to \( m, m(C \setminus F) = 0 \). Fix \( \delta, 0 < \delta < 1 \), and put \( E = F \cap E_1 \) where \( E_1 = \{ y \in C: |y - x| \bar{g}(y) < \delta \} \).

By Lemma (3.3) the set \( E \) has full area density at \( x \). For each \( y \in E \) the function \( g_y = c(y)^{-1}[(z - x)/(z - y)] g \) represents \( y \). Moreover,
\[
\int \phi(|z - y|)^{-q} |g_y|^q \, dm = |c(y)|^{-q} \int |z - y|^{-q} \phi(|z - y|)^{-q} |z - x|^q |g|^q \, dm \\
\leq C \int |z - y|^{-q} \phi(|z - y|)^{-q} |g|^q \, dm < \infty.
\]

This proves the lemma.

**Corollary (4.2).** Suppose that \( \phi \) is \( q \)-nice. Then at almost every point of \( S^p(X) \), \( p > 2 \), the functions in the unit ball of \( R^p(X) \) admit \( \phi \) as a modulus of approximate continuity.

**Proof.** Combine Theorem (4.1) with Lemma (4.3).

In particular, it follows that at a.e. \( x \in S^p(X) \), \( p > 2 \), the unit ball of \( R^p(X) \) satisfies an approximate uniform Hölder condition of order \( \alpha \) for every \( \alpha < (2 - q)/q \).

**Lemma (4.4).** Let \( \phi \) be admissible and \( g \in L^q(X) \), \( 1 < q < 2 \). Then if \( \phi(|z - x|)^{-1} g \in L^q(X) \), \( \delta > 0 \), and

\[
E = \left\{ y \in C : |y - x|^q \int |y - z|^{-q} |g|^q \, dm < \delta \right\},
\]

it follows that \( m(\Delta_n \setminus E) = o(n^{-1})^2 / n^2 \).

**Proof.** We observe that

\[
m(\Delta_n \setminus E) \leq \delta^{-1} \int |y - x|^q \int |z - y|^{-q} |g|^q \, dm \, dm_n(y).
\]

Factor \( g = \phi(|z - x|) h \) where \( h \in L^q(X) \). Then

\[
|g|^q \leq C \left[ \phi(|z - y|)^q |h|^q + \phi(|y - x|)^q |h|^q \right]
\]

where \( C \) is some constant. We have

\[
m(\Delta_n \setminus E) \leq \delta^{-1} C \left[ \int |y - x|^q \int |z - y|^{-q} \phi(|z - y|)^q |h|^q \, dm \, dm_n(y) \\
+ \int |y - x|^q \int |z - y|^{-q} \phi(|y - x|)^q |h|^q \, dm \, dm_n(y) \right].
\]

By substituting \( |y - x|^q = \phi(|y - x|)^q \psi(|y - x|) \) in the first integral, and using the fact that \( \phi(|y - x|)^q \leq \phi(n^{-1})^q \) for \( y \in \Delta_n \), we obtain

\[
m(\Delta_n \setminus E) \leq \delta^{-1} C \phi(n^{-1})^q \psi(|y - x|)^q \int \psi(|z - y|)^{-q} |h|^q \, dm \, dm_n(y) \\
+ \int |y - x|^q \int |z - y|^{-q} |h| \, dm \, dm_n(y).
\]

Let \( A_n \) denote the sum of the two integrals on the right. Replacing \( m(\Delta_n \setminus E) \) by \( \pi^2 \rho_n^2 \), we obtain

\[
\pi \rho_n^2 \leq \delta^{-1} C \phi(n^{-1})^q \rho_n^2 - q \rho_n^{-q} (A_n).
\]
where \( \lim_{n \to \infty} A_n = 0 \) by Lemma (3.2). Divide both sides by \( p_n^{2-q} \) to get

\[
\pi p_n^{2-q} \leq \delta^{-1} C \phi(n^{-1})^q n^{-q} (A_n).
\]

Now raise both sides to the power \( 2/q \), and the conclusion of the lemma follows.

In the next corollary we consider functions \( f \in R^p(X) \) to be defined on \( C \) by setting \( f(x) = 0 \) for \( x \not\in X \).

**Corollary (4.3).** Let \( \epsilon > 0 \). If \( x \in S^p(X) \), \( p > 2 \), is represented by \( g \in L^q(X) \), and \( (z - x)^{-\alpha} g \in L^q(X) \) for some \( \alpha > q - 1 \), then there is an integer \( N_x \) depending on \( x \) such that for \( n > N_x \)

\[
m(\Delta_n)^{-1} \int_{\Delta_n} |f - f(x)| \, dm \leq \epsilon \|f\|_p \quad \text{for all } f \in R^p(X).
\]

**Proof.** Let \( E \) be the set in the conclusion of Theorem (4.1) when \( \epsilon/2 \) and \( x \in S^p(X) \) are given and \( \phi(r) \equiv 1 \).

\[
m(\Delta_n)^{-1} \int_{\Delta_n} |f - f(x)| \, dm
\]

\[
\leq m(\Delta_n)^{-1} \left[ \int_{\Delta_n \cap E} |f - f(x)| \, dm + \int_{\Delta_n \setminus E} |f - f(x)| \, dm \right]
\]

\[
\leq (\epsilon/2) \|f\|_p m(\Delta_n)^{-1} m(\Delta_n \cap E) + \pi^{-1} n^2 \int_{\Delta_n \setminus E} |f - f(x)| \, dm
\]

\[
\leq (\epsilon/2) \|f\|_p + \pi^{-1} n^2 \int_{\Delta_n \setminus E} |f - f(x)| \, dm.
\]

Let \( \chi_{\Delta_n \setminus E} \) be the characteristic function of \( \Delta_n \setminus E \). Then by Hölder's inequality,

\[
\pi^{-1} n^2 \int_{\Delta_n \setminus E} |f - f(x)| \, dm = \pi^{-1} n^2 \int \chi_{\Delta_n \setminus E} |f - f(x)| \, dm
\]

\[
\leq C n^2 \left[ m(\Delta_n \setminus E) \right]^{1/q} \|f\|_{L^q(\Delta_n \setminus E)}
\]

where \( C \) is a constant. By Lemma (4.4)

\[
\left[ m(\Delta_n \setminus E) \right]^{1/q} = o(n^{-(2/q) - (2\alpha/q)}).
\]

Thus if \( \alpha > q - 1 \), we can choose an integer \( N_x \) so that \( n > N_x \) implies that \( C n^2 [m(\Delta_n \setminus E)]^{1/q} < \epsilon/2 \). Hence,

\[
m(\Delta_n)^{-1} \int_{\Delta_n} |f - f(x)| \, dm \leq (\epsilon/2) \|f\|_p + (\epsilon/2) \|f\|_{L^p(\Delta_n \setminus E)}
\]

\[
\leq \epsilon \|f\|_p.
\]

This completes the proof.
FUNCTIONS IN $R^p(X)$

**Corollary (4.4).** If $p > 2 + \sqrt{2}$, then for a.e. $x \in S^p(X)$,

$$
\lim_{n \to \infty} m(\Delta_n)^{-1} \int_{\Delta_n} |f - f(x)| \, dm = 0 \quad \text{for any } f \in R^p(X).
$$

**Proof.** This follows from Lemma (4.3) and Corollary (4.3).

Given $f \in L^1(dm)$, the set of points $x \in C$ such that

$$
\lim m(\Delta_n)^{-1} \int_{\Delta_n} |f - f(x)| \, dm = 0
$$

is called the Lebesgue set of $f$. For an arbitrary $f \in L^1(dm)$, a.e. $(m)$ point $x \in C$ belongs to the Lebesgue set of $f$ (see [5, p. 156]). The above corollary identifies points belonging to the Lebesgue sets of all $f \in R^p(X)$. It would be interesting to know whether the corollary holds for $p > 2$.

**Part II. Capacity and bounded point evaluations**

1. **Capacity theorems.** Before proving a capacity result about bounded point evaluations, we will need two lemmas of Hedberg [9]. Let $\Omega$ denote the complex plane when $p > 2$ and the unit disk when $p = 2$.

**Definition (1.1).** Let $A' \subset \Omega$ be a compact set. Then

$$
\Gamma_q(A') = \inf \int |\text{grad } \omega|^q \, dm
$$

where the inf is taken over Lipschitz functions $\omega$ with compact support contained in $\Omega$ such that $\omega(z) > 1$ on $A'$.

For noncompact sets $F$, $q$-capacity is defined by $\Gamma_q(F) = \sup_{K \subset F} \Gamma_q(K)$, $K$ compact.

Let $U$ be an open set (bounded if $p = 2$) in the complex plane and denote by $L_p^U$ the space of analytic functions in $L_p^U$. If $f$ is analytic in $\Omega \setminus X$ where $X \subset \Omega$ is compact, we write $\alpha(f) = (2\pi i)^{-1} \int_C f(z) \, dz$ where $C$ is any Jordan curve in $\Omega$ enclosing $X$.

**Lemma (1.1).** Let $X \subset \Omega$ be compact. Then there are positive constants $C_1$ and $C_2$, depending only on $p$, such that

$$
C_1 \Gamma_q(X)^{1/q} \leq \sup_f |\alpha(f)| \leq C_2 \Gamma_q(X)^{1/q}
$$

where the sup is taken over functions $f$ in $L_p^U$, $2 < p < \infty$, with $\int_{\Omega \setminus X} |f(z)|^p \, dm \leq 1$.

We denote the annulus $\{z: 2^{-n-1} \leq |z - x| \leq 2^{-n}\}$ by $A_n(X)$. We write $A_n = A_n(0)$.

**Lemma (1.2).** Let $X \subset \Omega$ be compact. There is a constant $C$, depending only on $p$, such that for $z \notin A_{n-1} \cup A_n \cup A_{n+1}$
for $f$ analytic outside $A_n \setminus X$, $f(\infty) = 0$ and $\int_{\Omega \setminus X} |f(z)|^p \, dm < \infty$.

The following theorem was proved in the sup norm case by Wang [18, p. 223]. Wang essentially followed O'Farrell [13], who elaborated on a method of Gamelin [7, p. 206]. We assume that $x = 0$ and that $0 \in \partial X$.

**Theorem (1.1).** Let $\phi$ be an admissible function and $s$ a nonnegative integer. Suppose that there is a function $v \in L^q(X)$ which represents 0 for $R^p(X)$ such that $|z|^{-s} \phi(|z|)^{-1} v \in L^q(X)$. Then

$$\sum_{n=1}^{\infty} 2^{q(s+1)n} \phi(2^{-n})^{-q} \Gamma_q(A_n \setminus X) < \infty.$$ 

**Proof.** Suppose that

$$\sum_{n=1}^{\infty} 2^{q(s+1)n} \phi(2^{-n})^{-q} \Gamma_q(A_n \setminus X) = \infty.$$ 

We will show that this leads to a contradiction. We may assume that for each $n$

$$2^{q(s+1)n} \phi(2^{-n})^{-q} \Gamma_q(A_n \setminus X) \leq 1.$$ 

If not, choose $Y_n$ compact, $Y_n \subset A_n$ such that

$$\frac{1}{2} \leq 2^{q(s+1)n} \phi(2^{-n})^{-q} \Gamma_q(A_n \setminus X \cup Y_n) < 1,$$

and set $Y = \bigcup Y_n \cup X$. Then define $v^*(z) = v(z)$ for $z \in X$ and $v^*(z) = 0$ for $z \in Y \setminus X$. Clearly, $|z|^{-s} \phi(|z|)^{-1} v^* \in L^q(Y)$ and $v^*$ represents 0 for $R^p(Y)$.

Now choose integers $M_1 < N_1 < M_2 < N_2 < \cdots$ so that

$$1 \leq \sum_{n=M_j}^{N_j} 2^{q(s+1)n} \phi(2^{-n})^{-q} \Gamma_q(A_n \setminus X) < 2.$$ 

For each $n$ we choose by Lemma (1.1) compact sets $K_n \subset A_n \setminus X$ and functions $f_n \in L^p_\Omega (\Omega \setminus K_n)$ so that:

(i) $|\alpha(f_n)| \geq C_1 \int_{A_n \setminus X} \left( \int_{\partial K_n} |f_n(z)|^p \, dm \right)^{1/p}$

$$= C_1 \int_{A_n \setminus X} \|f_n\|_{A_n \setminus K_n}^p,$$

where $f_n = 0$ on $K_n$ and

(ii) $\|f_n\|_{A_n \setminus X} = 2^{q(s+1)n} \phi(2^{-n})^{-q} \Gamma_q(A_n \setminus X)^{1/p}$. 

Let \( g_j(z) = \phi(|z|)z^{s-j+1}\sum_{n=0}^{N_j} f_n(z) \). We will show that \( \|g_j\|_{X,p} \leq C \) for all \( j \).

In the following discussion \( C \) will denote any constant that is independent of \( n \) and \( j \). Lemma (II.1.2) implies that for \( z \in A_k, k < n - 1 \),
\[
|f_n(z)| < C 2^{q(s+1)n+k}\phi(2^{-n})^{-q} \Gamma_q(A_n \setminus X),
\]
and for \( z \in A_k, k > n + 1 \),
\[
|f_n(z)| < C 2^{q(s+1)n+n+\phi(2^{-n})^{-q} \Gamma_q(A_n \setminus X)}.
\]

We may assume that \( X \subset \{|z| < 1\} \). Then for \( z \in A_k \cap X, k < n - 1 \),
\[
\phi(|z|)|z|^{s+1}|f_n(z)| < C 2^{q(s+1)n+k}\phi(2^{-n})^{-q} \Gamma_q(A_n \setminus X)
\]

For \( z \in A_k, k > n + 1 \),
\[
\phi(|z|)|z|^{s+1}|f_n(z)| < C 2^{q(s+1)n+n-s+1}\phi(2^{-n})^{-q} \Gamma_q(A_n \setminus X).
\]

Now
\[
\int_X |g_j(z)|^p \, dm = \sum_{k=0}^{\infty} \int_{A_k \cap X} \left| \sum_{n=M_j}^{N_j} \phi(|z|)z^{s-j+1}f_n(z) \right|^p \, dm
\]

\[
< C \sum_{k=0}^{\infty} \int_{A_k \cap X} \left\{ \sum_{n=M_j; n \neq k-1,k,k+1}^{N_j} \phi(|z|)|z|^{s+1}|f_n(z)| + \sum_{n=k-1}^{k+1} \phi(|z|)|z|^{s+1}|f_n(z)| \right\}^p \, dm.
\]

By the above estimates and the choice of \( M_j, N_j \), we have for \( z \in A_k \)
\[
\sum_{n=\max(k+2,M_j)}^{N_j} \phi(|z|)|z|^{s+1}|f_n(z)| < C \sum_{n=M_j}^{N_j} 2^{q(s+1)n+\phi(2^{-n})^{-q} \Gamma_q(A_n \setminus X)} < C.
\]

Similarly,
\[
\sum_{n=M_j}^{N_j} \phi(|z|)|z|^{s+1}|f_n(z)| < C \sum_{n=M_j}^{N_j} 2^{q(s+1)n+\phi(2^{-n})^{-q} \Gamma_q(A_n \setminus X)} < C.
\]

Thus
\[
\sum_{k=0}^{\infty} \int_{A_k \cap X} \left\{ \sum_{n=M_j; n \neq k-1,k,k+1}^{N_j} \phi(|z|)|z|^{s+1}|f_n(z)| \right\}^p \, dm < C.
\]

Next, we estimate
For each $k$, \[
\int_{A_k \cap X} \left( \phi(|z|)|z|^{s+1}|f_{k-1}(z)| \right)^p \, dm
\]
\[
\leq C \left( \phi(2^{-k+1})^p 2^{-p(k-1)} \|f_{k-1}\|_{X,p}^p \right)
\]
\[
\leq C \phi(2^{-k+1})^{p-\rho q} 2^{(k-1) [1-\rho + \rho q (s+1)]} \Gamma_q(A_{k-1} \setminus X)
\]
\[
\leq C 2^{q(s+1)(k-1)\phi(2^{-k+1})^{-q} \Gamma_q(A_{k-1} \setminus X)}
\]
and similarly for $f_k$ and $f_{k+1}$. Thus
\[
\sum_{k=0}^{\infty} \int_{A_k \cap X} \left( \phi(|z|)|z|^{s+1}|f_n(z)| \right)^p \, dm
\]
\[
\leq C \sum_{k=M_j}^{N_j} 2^{q(s+1)k\phi(2^{-k})^{-q} \Gamma_q(A_k \setminus X)}
\]
\[
\leq C \text{ by choice of } M_j \text{ and } N_j.
\]
Combining the above estimates, we obtain
\[
\int_X |g_j|^p \, dm \leq C \text{ for all } j.
\]
Next we pass to a subsequence of the $\{g_j\}$ that converges weakly to $g \in L^p(X)$. Denote the subsequence also by $\{g_j\}$. We form $h_j(z) = z\phi(|z|)^{-1} g_j(z)$ and $F_j(z) = z^{-s-1} h_j(z)$, which are analytic in $\mathbb{C} \setminus \Delta(0, 2^{-M_j})$. By the above estimates the functions $h_j$ and $F_j$ are uniformly bounded on compact subsets of $\mathbb{C} \setminus \{0\}$. Hence, there are subsequences that converge uniformly on compact subsets of $\mathbb{C} \setminus \{0\}$ to $h(z) = z\phi(|z|)^{-1} g(z)$ and $F(z) = z^{-s-1} h(z)$ respectively.

We claim that $h$ is a polynomial of degree $s + 1$ with $h(0) = 0$. The above estimates show that there is a number $M > 0$ that bounds the $h_j$ in the following sense: to any $z \in \Delta(0, 1) \setminus \{0\}$ there corresponds an integer $J$ such that for $j > J$ and $|z| > |\xi|, |h_j(\xi)| < M$. This implies that $h$ is bounded near 0, so $h$ is entire and $\lim_{z \to 0} h(z) = 0$. To show that $h$ is a polynomial we consider
\[
\lim_{z \to \infty} z^{-s-1} h(z) = F(\infty) = \lim_{j \to \infty} F_j(\infty).
\]
For all $j$, $F_j(\infty) = \sum_{n=M_j}^{N_j} f_n(\infty)$ lies in $[C_1/2, 3C_2]$ where $C_1$ and $C_2$ are the constants of Lemma (1.1). Therefore, we have that $\lim_{j \to \infty} F_j(\infty) = \beta \in [C_1, 2C_2]$, and
$h(z) = \beta z^{s+1} + \sum_{i=1}^{s} \beta_i z^i$ where $\beta_i$ is a constant for each $i$.

Thus

$$g_f = \phi(|z|)z^{-1}h_j \to \phi(|z|)z^{-1}h = \beta \phi(|z|)z^s + \sum_{i=1}^{s} \beta_i \phi(|z|)z^{i-1}$$

weakly and pointwise on each bounded subset of $C \setminus \{0\}$.

This means that if $u \in L^q(X)$, then

$$\int g_j u \, dm \to \int \beta \phi(|z|)z^s u \, dm + \sum_{i=1}^{s} \beta_i \int \phi(|z|)z^{i-1}u \, dm.$$ 

Wilkin's lemma (Lemma (1.4.1)) and the original hypothesis imply that there is a function $v_j \in L^q(X)$ which is a linear combination of the functions $z^{-j}v$, $0 < j < s$, such that

$$\int f v_j \, dm = \frac{f^{(s)}(0)}{s!}$$

for all $f \in R_q(X)$. Taking $u = \phi(|z|)^{-1}v_j$, we get a contradiction.

The next theorem may be proved in a similar way, and we omit many of the details.

**Theorem (1.2).** Let $\phi$ be an admissible function and $s$ a nonnegative integer. Suppose that there is a function $v \in L^q(X)$ representing $0$ for $R^p(X)$ such that $|z|^{-s} \phi(|z|)^{-1}v \in L^q(X)$. Then

$$\lim_{r \to 0} r^{-qs-q} \phi(r)^{-q} \Gamma_q(\Delta(0, r) \setminus X) = 0.$$ 

**Proof.** Suppose that there is a sequence $r_n \to 0$ and a $b > 0$ such that

$$r_n^{-qs-q} \phi(r_n)^{-q} \Gamma_q(\Delta(0, r_n) \setminus X) > b \quad \text{for all } r_n.$$ 

We may assume as before that

$$2^{q(s+1)n} \phi(2^{-n})^{-q} \Gamma_q(A_n \setminus X) \leq 1 \quad \text{for all } n.$$ 

Note that if $2^{-k} > r_n$, and $|2^{-k} - r_n| < 2^{-k-1}$,

$$2^{q(s+1)} \sum_{n=k}^{\infty} 2^{q(s+1)n} \phi(2^{-n})^{-q} \Gamma_q(A_n \setminus X) > b.$$ 

Thus there is a sequence of integers $M_1 < N_1 < M_2 < N_2 < \cdots$ such that

$$2 > \sum_{n=M_j}^{N_j} 2^{q(s+1)n} \phi(2^{-n})^{-q} \Gamma_q(A_n \setminus X) > 2^{-q(s+1)b}$$

for all $j$. The proof then proceeds as before.

2. **Density at bounded point evaluations.** We will get an estimate for $\Gamma_q$ capacity in terms of the measure $m$. The following lemma is in [4].
**Lemma (2.1).** Let μ be a measure of total mass 1 (i.e. \( \int d\mu = 1 \)). If \( 1 < q < 2 \) and \( p = q/(q - 1) \), then
\[
\int_C \left\{ \int |z|^{-1} d\mu(z) \right\}^p dm \leq C \left\{ \sup_{z \in C} \int |z|^{q-2} d\mu(z) \right\}^{q-1}
\]
where \( C \) is some constant depending only on \( p \).

**Lemma (2.2).** For each \( q, 1 < q < 2 \), there is a positive constant \( C \) such that
\[
\Gamma_q(X) > C m(X)^{(2-q)/2}
\]
for all compact sets \( X \subset \mathbb{C} \).

**Proof.** Define \( f = m(X)^{-1} \int_X (z - \xi)^{-1} dm(\xi) \). Then \( f \) is analytic in \( \mathbb{C} \setminus X \) and \( f'(\infty) = 1 \). To estimate \( ||f||_{\mathbb{C} \setminus X} \) we apply Lemma (II.2.1) with \( \mu = m(X)^{-1} \chi_X \) where \( \chi_X \) is the characteristic function of \( X \). We get
\[
||f||_{\mathbb{C} \setminus X} < C \left\{ \sup_{z \in C} m(X)^{-1} \int_X |z - \xi|^{q-2} dm(\xi) \right\}^{1/q}
\]
We will use \( C \) to denote any constant depending only on \( p \). Choose \( R > 0 \) so that \( R^2 = m(X) \), and let \( D = \Delta(\xi, R) \). Then since \( r^{q-2} \) is a decreasing function of \( r \),
\[
m(X)^{-1} \int_X |z - \xi|^{q-2} dm(\xi) \leq \pi^{-1} R^{-2} \int_0^{2\pi} \int_0^R r^{q-2} dr d\theta
\]
\[
= \pi^{-1} R^{-2} 2\pi \int_0^R r^{q-1} dr
\]
\[
= 2(q - 1)^{-1} R^{-2} R^q = 2(q - 1)^{-1} R^{q-2}.
\]
Applying the above inequality for \( ||f||_{\mathbb{C} \setminus X} \), we have
\[
||f||_{\mathbb{C} \setminus X} < CR^{(q-2)/q}.
\]
Define \( g = f/||f||_{\mathbb{C} \setminus X} \). Then \( g \) is analytic in \( \mathbb{C} \setminus X \) and \( ||g||_{\mathbb{C} \setminus X} = 1 \). Moreover,
\[
g'(\infty) = f'(\infty)/||f||_{\mathbb{C} \setminus X} > CR^{(2-q)/q} > C m(X)^{(2-q)/2q}.
\]
By Lemma (II.1.1) we conclude that
\[
\Gamma_q(X) > C m(X)^{(2-q)/2},
\]
and the proof is complete.

**Corollary (2.1).** Let \( \phi \) be an admissible function and \( s \) a nonnegative integer. Suppose that there is a function \( v \in L^q(X) \) representing 0 for \( R^p(X) \),
functions in $R^p(X)$

$p > 2$, such that $|z|^{-2} \phi(|z|)^{-1} \in L^q(X)$. Then

$$m(\Delta(0, n^{-1}) \setminus X) = o\left(\phi(n^{-1})^{2t}(n^{-1})^{2t(z+1)}\right),$$

where $t = q/(2 - q)$.

**Proof.** This follows from Theorem (II.1.2) and Lemma (II.2.2).

3. An example. In this section we use Hedberg's capacity theorems to construct a Swiss cheese $Y$ such that $\cap_{p > 2} S^p(Y) = \{0\}$. Let $X_0$ be the closure of a set having positive measure whose boundary consists of finitely many analytic curves. The first step is to show that for a given $\varepsilon > 0$ and $p > 2$ one can construct a Swiss cheese $X = X_0 \setminus \bigcup_{i=1}^{\infty} D_i$ such that:

1. $\sum_{i=1}^{\infty} r_i^{2-\eta} < \varepsilon$ where $r_i$ is the radius of $D_i$; and
2. for some $p', p > p' > 2$, $S^{p'}(X) = \emptyset$. For $n = 1, 2, \ldots$ we define $X_n$ inductively by letting $X_n = X_{n-1} \setminus G_n$ where $G_n = \bigcup \{\Delta(i2^{-n}, (e2^{-n})^{3/(2-q)}), \text{ where the summation is taken over all Gaussian integers } t \text{ such that } |t2^{-n}| < 1\}$. Then set $X = \cap_{n=0}^{\infty} X_n$. Since each $G_n$ consists of $< 2^{2n}$ disks

$$\sum_{i=1}^{\infty} r_i^{2-\eta} < \sum_{i=1}^{\infty} 2^{2i}[(e2^{-i})^{3/(2-q)}]^{2-\eta} = \varepsilon.$$

Now choose $q', q < q' < 2$, so that $3(2 - q')/(2 - q) < q'$. Let $x \in X$. We claim that $x \notin S^{p'}(X)$ where $1/p' + 1/q' = 1$. Within any disk centered at $x$ and having radius $2^{-n}$, there is a disk in $C \setminus X$ having radius at least $4^{-1}(e2^{-r})^{3/(2-q)}$. Hence

$$\lim_{n \to \infty} 2^{nq'} \Delta(x, 2^{-n}) \setminus X$$

$$> 4^{q'-2} \lim_{n \to \infty} 2^{nq'}(e2^{-n})^{3(2-q')/(2-q)} > 0.$$

Thus by Theorem (II.1.2), $x \notin S^{p'}(X)$, and $X$ is the desired set.

Given $\varepsilon > 0$ and $p > 2$, it is possible by the above construction to remove open disks $D_{j,k}$ of radius $r_{j,k}$ from $A_j(0)$ to obtain a Swiss cheese $Y_j$ such that $\Sigma_{k=1}^{\infty} r_{j,k}^{2-\eta} < \varepsilon$ (where $r_j + 1/q_j = 1$), and $S^p(Y_j) = \emptyset$ for some $p_j, p_j > p' > 2$. Choose the $\varepsilon_j$ so that $\Sigma_{j=1}^{\infty} 2^{j\varepsilon_j} < \infty$, and define $Y = \cup_{j=0}^{\infty} Y_j$.

We will use Hedberg's theorem [9] to prove that for any $p > 2, 0 \in S^p(Y)$. Let $p > 2$. There is an integer $J$ such that $p > p_j > 2$ for $j > J$. Hence,

$$\sum_{j=J}^{\infty} 2^{j\varepsilon_j} \Gamma_q(A_j(0) \setminus X) < C \sum_{j=J}^{\infty} 2^{j\varepsilon_j} \sum_{k=1}^{\infty} r_{j,k}^{2-\eta} < C \sum_{j=J}^{\infty} 2^{j\varepsilon_j} < \infty.$$

By Hedberg's theorem $0 \in S^p(Y)$, and since $p > 2$ was arbitrary, $0 \in \cap_{p > 2} S^p(Y)$. That $0$ is the only point in $\cap_{p > 2} S^p(Y)$ follows from the construction of $Y$ and the fact that $x \in S^p(Y)$ if and only if $x \in S^p(Y \cap \overline{\Delta(x,r)})$ for any $r > 0$.

Given any compact set $X$ it would be interesting to find necessary and sufficient conditions for $\cap_{p > 2} S^p(X)$ to have positive measure. Lemma (I.2.3)
implies that a sufficient condition is that there exist a single $g$ which represents 0 for $R^p(X)$ for all $p > 2$.

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