WEAK CHEBYSHEV SUBSPACES AND CONTINUOUS
SELECTIONS FOR THE METRIC PROJECTION

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ABSTRACT. Let G be an n-dimensional subspace of C[a, b]. It is shown that there exists a continuous selection for the metric projection if for each f in C[a, b] there exists exactly one alternation element g, i.e., a best approximation for f such that for some a < x_0 < \ldots < x_n < b,

\[ \epsilon(-1)^i(f - g_i)(x_i) = \|f - g\|, \quad i = 0, \ldots, n, \epsilon = \pm 1. \]

Further it is shown that this condition is fulfilled if and only if G is a weak Chebyshev subspace with the property that each g in G, g \neq 0, has at most n distinct zeros. These results generalize in a certain sense results of Lazar, Morris and Wulbert for n = 1 and Brown for n = 5.

If G is a nonempty subset of a normed linear space E then for each f in E, we define P_G(f) = \{ g_0 \in G: \|f - g_0\| = \inf(\|f - g\|: g \in G) \}. P_G defines a set-valued mapping of E into 2^G which in the literature is called the metric projection onto G. A continuous mapping s of E into G is called a continuous selection for the metric projection P_G (or, more briefly, continuous selection) if s(f) is in P_G(f) for each f in E. In this paper we treat the problem of the existence of continuous selections for n-dimensional subspaces G of C[a, b], with C[a, b] as usual the Banach space of real-valued continuous functions on [a, b] under the uniform norm.

A. Lazar, P. Morris and D. Wulbert [4] have characterized the 1-dimensional subspaces of C(X) with X compact Hausdorff, which admit a continuous selection. They raised the problem of characterizing the corresponding n-dimensional subspaces. The only known result for higher dimensional subspaces has been given by A. Brown [1], who has shown the existence of continuous selections for certain 5-dimensional subspaces of C[-1, 1].

To obtain continuous selections, Lazar, Morris and Wulbert [4] and Brown [1] proceeded as follows: For each f in C[a, b] they considered all g in P_G(f) which can be written as g = a_1g_1 + \cdots + a_ng_n, where g_1, \ldots, g_n is a basis...
of \( G \), and chose the unique element \( g \) in \( P_G(f) \) with maximal coefficient \( a_n \). This works in the cases \( n = 1 \) and \( n = 5 \).

Using this kind of selection it does not seem possible to get a general theorem for \( n \)-dimensional subspaces in \( C[a,b] \). With new methods, however, and in the setting of weak Chebyshev subspaces we can give a sufficient condition for the existence of continuous selections.

R. Jones and L. Karlovitz [2, Theorem 4] have shown that an \( n \)-dimensional subspace \( G \) of \( C[a,b] \) is weak Chebyshev if and only if for each \( f \) in \( C[a,b] \) there exists at least one alternation element \( g_f \) (see Definition 1 below) in \( P_G(f) \). We show that if for each \( f \) in \( C[a,b] \) there exists exactly one alternation element \( g_f \) in \( P_G(f) \), then \( s(f) = g_f \) defines a continuous selection (Proposition 2). From Theorem 8 and Theorem 11, which together represent the main result of this paper, it follows that for an \( n \)-dimensional weak Chebyshev subspace \( G \) each \( f \) in \( C[a,b] \) has exactly one alternation element in \( P_G(f) \) if and only if each \( g \in G, g \neq 0 \), has at most \( n \) distinct zeroes. (In particular, \( g \) may not vanish on intervals.)

Using this result and Proposition 2, we immediately get an existence theorem for continuous selections for \( n \)-dimensional subspaces (Corollary 9). Brown [1] uses essentially stronger conditions to guarantee the existence of continuous selections for 5-dimensional subspaces of \( C[-1,1] \). Brown's result disproves a claim of Lazar, Morris and Wulbert [4], who tried to show that for \( n \)-dimensional subspaces \( G \) in \( C(X) \) (\( X \) a connected, compact, Hausdorff space) such that 1 is in \( G \) and each \( g \in G, g \neq 0 \), does not vanish on an open set in \( X \), there does not exist a nontrivial continuous selection.

Finally, Let us remark that from P. Schwartz [8] it follows that under the assumption of Corollary 9 the continuous selection is unique.

In the following let \( G \) be an \( n \)-dimensional subspace of \( C[a,b] \).

1. Definition. If \( f \) is in \( C[a,b] \), then \( g \) in \( P_G(f) \) is called an alternation element (A-element) of \( f \) if there exist \( n + 1 \) distinct points \( a < x_0 < \cdots < x_n < b \) such that

\[
\epsilon (-1)^i (f - g)(x_i) = \|f - g\|, \quad i = 0, \ldots, n, \quad \epsilon = \pm 1.
\]

The points \( a < x_0 < \cdots < x_n < b \) are called alternating extreme points of \( f - g \).

First, we want to show that when each \( f \) has a unique A-element then we can always define a continuous selection.

2. Proposition. Suppose for each \( f \) in \( C[a,b] \) there exists exactly one A-element \( g_f \) in \( P_G(f) \). Define \( s: C[a,b] \to G \) by \( s(f) = g_f \) for each \( f \in C[a,b] \). Then, \( s \) is a continuous selection for \( P_G: C[a,b] \to 2^G \).

Proof. We suppose \( s \) is not continuous.
Because of the finite dimensionality of $G$, there exist $f \in C[a,b]$, $g \in G$ and a sequence $(f_m) \subset C[a,b]$ so that $f_m \to f$, $s(f_m) \to g$, but $g \neq s(f)$.

We will show that $g$ is an $A$-element of $f$ and this will contradict the uniqueness of the $A$-element.

By definition, $s(f_m)$ is an $A$-element of $f_m$, $m \in N$. Therefore, there are extreme points $a < x_0^{(m)} < x_1^{(m)} < \cdots < x_n^{(m)} < b$ of $f_m - s(f_m)$.

We can assume that

$$(1) \quad (-1)^i(f_m - s(f_m))(x_i^{(m)}) = \|f_m - s(f_m)\|, \quad i = 0, \ldots, n, \ m \in N.$$  

Here it may be necessary to choose a subsequence of $(f_m)$ and perhaps work with $-f$ and $-f_m$ in place of $f$ and $f_m$. We can also assume (again choosing a subsequence if necessary) that $\lim_{m \to \infty} x_i^{(m)} = x_i$ exists, $i = 0,1,\ldots, n$. Now, since $\lim_{m \to \infty} f_m = f$ and $\lim_{m \to \infty} s(f_m) = g$, we have

$$\|f - g\| = \lim_{m \to \infty} \|f_m - s(f_m)\|$$

$$= (-1)^i \lim_{m \to \infty} (f_m - s(f_m))(x_i^{(m)})$$

$$= (-1)^i (f - g)(x_i)$$

where in the second equality we used (1) and the uniform convergence. This shows that $g$ is an $A$-element, which is the desired contradiction.

Jones and Karlovitz [2] have characterized those $n$-dimensional subspaces of $C[a,b]$ which have at least one $A$-element for each $f$ in $C[a,b]$. For this characterization, we need the following definition:

3. Definition. $G$ is called weak Chebyshev if each $g$ in $G$ has at most $n - 1$ changes of sign, i.e., there do not exist points $a < x_0 < \cdots < x_n < b$ such that $g(x_i) \cdot g(x_{i+1}) < 0$, $i = 0, \ldots, n - 1$.

Jones-Karlovitz [2] have proved the following theorem:

4. Theorem. $G$ is weak Chebyshev if and only if for each $f$ in $C[a,b]$ there exists at least one $A$-element in $P_G(f)$.

To get a continuous selection under application of Proposition 2, we examine what additional conditions a weak Chebyshev subspace has to fulfill in order that each $f$ in $C[a,b]$ has exactly one $A$-element.

We need the following standard definition:

5. Definition. A zero $x_0$ of $f$ in $C[a,b]$ is said to be a simple zero if $f$ changes sign at $x_0$ or if $x_0 = a$ or $x_0 = b$.

A zero $x_0$ of $f$ in $C[a,b]$ is said to be a double zero if $f$ does not change sign at $x_0$ and $x_0 \neq a, x_0 \neq b$.

In the following, we count simple zeroes as one zero and double zeroes as two zeroes. To prove the following results we need the lemma below.

6. Lemma. If $f$ is in $C[a,b]$ and if there exist $n + 1$ points $a < x_0 < \cdots <
$x_n < b$ such that
\[\epsilon(-1)^i f(x_i) > 0, \quad i = 0, \ldots, n, \epsilon = \pm 1,\]
then $f$ has at least $n$ zeroes $y_i$ such that
\[x_0 < y_0 < x_1 < y_1 < \cdots < x_{n-1} < y_{n-1} < x_n.\]

7. Lemma. If $G$ is an $n$-dimensional weak Chebyshev subspace of $C[a,b]$ such that there exists a $g$ in $G$, $g \neq 0$, with at least $n + 2$ zeroes, then there exists a $\tilde{g}$ in $G$, $\tilde{g} \neq 0$, with at least $n + 1$ distinct zeroes.

Proof. Let $g$ be in $G$, $g \neq 0$, with at least $n + 2$ zeroes in $[a,b]$, but only $r$, $r < n$, distinct zeroes. Suppose first that $g(a) = g(b) = 0$, and set $\bar{x} = \max\{x \in [a,b] | g(x) = 0\}$.

Let $a < x_1 < \cdots < x_s < \bar{x}$ be the simple zeroes of $g$.

(a) $s + n - 1$ is an even number.

We choose $n - 1 - s$ points $\bar{x} < x_{n+1} < \cdots < x_{n-1} < b$. Since $G$ is weak Chebyshev, by Jones and Karlovitz [2, p. 140] there exists a $\tilde{g} \in G$, $\tilde{g} \neq 0$, with
\[\epsilon(-1)^i \tilde{g}(x) > 0, \quad x_{i-1} < x < x_i, \quad i = 1, \ldots, n, \epsilon = \pm 1,\]
where $x_0 = a$, $x_n = b$. By Lemma 6, $\tilde{g}$ has at least $n - 1$ distinct zeroes. We choose $\epsilon$ such that $\text{sgn}(g(x) \cdot \tilde{g}(x)) > 0$ if $x \in [a,x_{n+1}]$. Let $a = y_1 < \cdots < y_n = b$ be the distinct zeroes of $g$ in $[a,b]$.

Then
\[M := \min_{i=1,2,\ldots,n} \| g \|_{[y_i, y_{i+1}]} > 0.\]
We define $\bar{g} := M\tilde{g} / (2\| \tilde{g} \|)$.

The function $\bar{g}$ has at least two further distinct zeroes in $[a,b]$, otherwise the function $g - \bar{g}$ would have at least $n$ changes of sign. This would be a contradiction.

(b) $s + n - 1$ is an odd number.

We choose $x_0 = a$ and $n - s - 2$ points
\[\bar{x} < x_{n+1} < \cdots < x_{n-2} < b.\]
Since $G$ has an $(n - 1)$-dimensional weak Chebyshev subspace (see Sommer and Strauss [11, Theorem 2.6]), by Jones and Karlovitz [2, p. 140] there exists a $\tilde{g} \in G$, $\tilde{g} \neq 0$ with $\epsilon(-1)^i \tilde{g}(x) > 0, x_{i-1} < x < x_i, i = 1, \ldots, n - 1, \epsilon = \pm 1$ where $x_{n-1} = b$.

As before let $\text{sgn}(g(x) \cdot \tilde{g}(x)) > 0$ if $x \in [a,x_{n+1}]$.

Following (a) we construct a function $\bar{g}$.

As before it follows that either the function $\bar{g}$ or the function $g - \bar{g}$ has $n + 1$ distinct zeroes in $[a,b]$.
If not $g(a) = g(b) = 0$, the assertion can be shown in an analogous way. This completes the proof.

8. THEOREM. If $G$ is an $n$-dimensional weak Chebyshev subspace of $C[a,b]$ such that each $g$ in $G, g \neq 0$, has at most $n$ distinct zeroes, then each $f$ in $C[a,b]$ has exactly one $A$-element $g_f$ in $P_G(f)$.

PROOF. Assumption. There exists a function $f$ in $C[a,b]$ which has two $A$-elements $g_1$ and $g_2$ in $P_G(f)$.

Let $a < x_0 < \cdots < x_n < b$ be $n + 1$ alternating extreme points of $f - g_1$ and let $a < y_0 < \cdots < y_n < b$ be $n + 1$ alternating extreme points of $f - g_2$.

We distinguish two cases:

First case.

- For $i = 0, \ldots, n$,
  \[ (-1)^i(f - g_1)(x_i) = \|f - g_1\|, \]
  \[ (-1)^i(f - g_2)(y_i) = \|f - g_2\|, \]

Then

- For $i = 0, \ldots, n$,
  \[ (-1)^i(g_2 - g_1)(x_i) > 0, \]
  \[ (-1)^i(g_2 - g_1)(y_i) < 0, \]

We treat only the case

(ii) $x_{i-2} < y_i < x_{i+2}, \quad i = 0, \ldots, n,$

where the points $x_i$ for $i = -2, -1, n + 1, n + 2$ are omitted. In the other case, if $y_i < x_{i-2}$ for some $i$, we choose the points $y_0, \ldots, y_i, x_{i-2}, \ldots, x_n$ fulfilling

- For $j = 0, \ldots, i$,
  \[ (-1)^j(g_2 - g_1)(y_j) < 0, \]
  \[ (-1)^j(g_2 - g_1)(x_{j-3}) < 0, \quad j = i + 1, \ldots, n + 3. \]

By Lemma 6, $g_2 - g_1$ has at least $n + 3$ zeroes. Applying Lemma 7 we get a contradiction of the hypothesis that elements of $G$ have at most $n$ distinct zeroes.

A similar argument works for $x_{i+2} < y_i$.

Now we prove by induction that $g_1 - g_2$ has at least $n + 1$ distinct zeroes. This is a contradiction of the hypothesis on $G$. If $x_i = y_j, i = 0, \ldots, n$, then $g_1 - g_2(x_i) = 0, \quad i = 0, \ldots, n$. We may assume $x_i < y_i$ for some $i = 0, \ldots, n$.

We show: $x_j \leq y_j, j = 0, \ldots, n$.

If $y_{j_0} < x_{j_0}$ for any $j_0 \in \{0, \ldots, n\}$ we choose

- $y_0, \ldots, y_{j_0}, x_{j_0}, \ldots, x_i, y_i, \ldots, y_n$ if $j_0 < i$

and
Because of (i) in both cases \( g_1 - g_2 \) has at least \( n + 2 \) zeroes by Lemma 6. Applying Lemma 7 we get a contradiction of the hypothesis on \( G \).

Now we show by induction that \( g_1 - g_2 \) has at least \( n + 1 \) distinct zeroes in \([x_0, y_n] \): \( n = 1 \).

If \( x_0 < y_0 < x_1 < y_1 \) (respectively \( x_0 < y_0 = x_1 < y_1 \) or \( x_0 < x_1 < y_0 < y_1 \)), then \( g_1 - g_2 \) has one zero in each interval \([x_0, y_0] \), \([x_1, y_1] \) (respectively \([x_0, y_0], (x_1, y_1) \) or \([x_0, x_1], [y_0, y_1] \)).

Let the statement be true for \( n - 1 \).

If \( y_{n-1} < x_n < y_n \), then by assumption \( g_1 - g_2 \) has \( n \) distinct zeroes in \([x_0, y_{n-1}] \) and a further zero in \([x_n, y_n] \).

If \( y_{n-1} = x_n < y_n \), then by assumption \( g_1 - g_2 \) has \( n \) distinct zeroes in \([x_0, y_{n-1}] \) and a further zero in \((x_n, y_n) \).

Finally we consider the case \( x_n < y_{n-1} < y_n \):

Since \( (-1)^n(g_2 - g_1)(x_n) > 0 \), \( (-1)^n(g_2 - g_1)(y_{n-1}) > 0 \), and \( y_{n-2} < x_n \) we conclude as in the case \( y_{n-1} < x_n < y_n \).

Second case.

\[
(-1)^i(f - g_1)(x_i) = \|f - g_1\|, \quad i = 0, \ldots, n,
\]

\[
-(-1)^i(f - g_2)(y_i) = \|f - g_2\|, \quad i = 0, \ldots, n.
\]

We treat only the case that \( f - g_1 \) and (iii) \( f - g_2 \) have exactly \( n + 1 \) alternating extreme points.

Otherwise we can apply the first case.

Then

\[
(-1)^i(g_2 - g_1)(x_i) > 0, \quad i = 0, \ldots, n,
\]

\[
(-1)^i(g_2 - g_1)(y_i) > 0, \quad i = 0, \ldots, n.
\]

It is now enough to treat the case (v) \( x_{i-1} < y_i < x_{i+1} \), \( i = 0, \ldots, n \), where the points \( x_{-1} \) and \( x_{n+1} \) are omitted. Otherwise we can conclude as in the first case. Applying the first case to the points \( x_0, \ldots, x_{n-1}, y_1, \ldots, y_n \) because of (v) \( g_1 - g_2 \) has \( n \) distinct zeroes \( z_1, \ldots, z_n \) in \([x_0, y_n] \).

We first prove: \( z_1, \ldots, z_n \in (a, b) \). If \( z_1 = x_0 \), then \( y_0 < x_0 \). Otherwise \( f - g_2 \) has \( n + 2 \) alternating extreme points \( x_0, y_0, \ldots, y_n \). This is a contradiction to (iii). Therefore \( z_1 > a \).

If \( z_n = y_n \), then \( x_n > y_n \). Otherwise \( f - g_1 \) has \( n + 2 \) alternating extreme points \( x_0, \ldots, x_n, y_n \). This is a contradiction to (iii). Therefore \( z_n < b \).

If \( g_1 - g_2 \) has \( n + 1 \) distinct zeroes, then we would get a contradiction of the hypothesis on \( G \).
Therefore we know $g_1 - g_2$ has no further zero in $[a, b]$. Because of $a < z_1 < \cdots < z_n < b$ and $G$ weak Chebyshev $g_1 - g_2$ has at most $n - 1$ changes of sign under the points $z_i$. We show $g_1 - g_2$ has at most $n - 2$ changes of sign:

If $g_1 - g_2$ has $n - 1$ changes of sign under the points $z_i$, then there exists exactly one zero $z_j \in (a, b)$ such that $g_1 - g_2$ does not change sign at $z_j$.

Then it holds: If $z_1 > x_0$, then because of (iv)

$$(-1)^0(g_2 - g_1)(x) > 0 \quad \text{if } a < x < z_1 \quad \text{and finally}$$

$$(-1)^n(g_2 - g_1)(x) < 0 \quad \text{if } z_n < x < b.$$  

If $z_1 = x_0$, then $y_0 < x_0$ and (vi) is also valid.

Now we get a contradiction to (iv):

(1) $x_n > y_n$.

Then $z_n < x_n$ and because of (iv) $(-1)^n(g_2 - g_1)(x_n) > 0$. This is a contradiction.

(2) $x_n < y_n$.

If $z_n < y_n$ we also get a contradiction because of (iv). But if $z_n = y_n$, then $x_n > y_n$ is always valid because of (iii).

We have shown:

If $g_1 - g_2$ has exactly $n$ distinct zeroes, then $g_1 - g_2$ has at most $n - 2$ changes of sign. But in this case there exist $n + 2$ zeroes of $g_1 - g_2$ because of $a < z_1, z_n < b$.

Applying Lemma 7 we get a contradiction to the assumption.

Schwartz [8] has shown that for an $n$-dimensional subspace $G$ of $C(X)$ with the property that no $g$ in $G$, $g \neq 0$, vanishes identically on an open subset of $X$, the set of functions in $C(X)$ having unique best approximation in $G$ is dense in $C(X)$. Therefore there exists at most one continuous selection. By this result, Proposition 2 and Theorem 8 the next corollary follows immediately.

**9. Corollary.** If $G$ is an $n$-dimensional weak Chebyshev subspace such that each $g$ in $G$, $g \neq 0$, has at most $n$ distinct zeroes, then there exists a unique continuous selection $s: C[a, b] \to G$ for $P_G: C[a, b] \to 2^G$.

Now we will give some nontrivial examples of subspaces $G$ in $C[a, b]$ fulfilling the assumption of Corollary 9.

**10. Examples.** (a) $G: = \langle x, x^2, \ldots, x^n \rangle \subset C[0, 1]$. $G$ is Chebyshev in $(0, 1]$ and therefore the assumption of Corollary 9 is fulfilled, but $G$ is not Chebyshev in $[0, 1]$.

(b) For $n \geq 2$ and $n$ even, we define $G: = \langle 1, x(1 - x^2), x^2, x^2(1 - x^2), x^4, \ldots, x^{n-1}(1 - x^2), x^n \rangle \subset C[-1, 1]$. The dimension of $G$ is $n + 1$. Each
function $g$ in $G$ is a polynomial of degree $\leq n + 1$ and has therefore at most $n + 1$ zeroes in $[-1, 1]$. Such a function $g$ can be written as $g = g_1 - g_2$ where

$$g_1(x) = \sum_{i=0}^{n/2} a_{2i} x^{2i} \quad \text{and} \quad g_2(x) = x(1 - x^2) \sum_{i=1}^{n/2} a_{2i-1} x^{2i-2}.$$ 

Because of the behaviour of $g_1(x)$ and $g_2(x)$ for $x \to \pm \infty$ it can be shown that $g_1 - g_2$ has a zero in $(-\infty, -1] \cup (1, \infty)$. Therefore $G$ is Chebyshev in $(-1, 1]$.

$G$ is not Chebyshev in $[-1, 1]$ because there exists a function

$$g_0(x) = x(1 - x^2) \sum_{i=1}^{n/2} a_{2i-1} x^{2i-2} \quad \text{in } G, \ g \neq 0,$$

having exactly $n + 1$ zeroes in $[-1, 1]$.

A similar example has been given by Brown [1] in the case $n = 5$.

(c) $G$: $= \langle |x|, x^3 \rangle \subset C[-1, 1]$. $G$ is weak Chebyshev and each $g \in G, g \neq 0$, has at most 2 distinct zeroes, but $G$ is not Chebyshev in $[-1, 1)$ or $(-1, 1]$.

Finally we ask how strong the assumption of Theorem 7 is for the uniqueness of $A$-elements and we show that this is the weakest condition because the converse of Theorem 7 is true.

11. Theorem. If $G$ is an $n$-dimensional weak Chebyshev subspace of $C[a, b]$ such that for each $f$ in $C[a, b]$ there exists exactly one $A$-element in $P_G(f)$ then each $g$ in $G, g \neq 0$, has at most $n$ distinct zeroes.

Proof. Assumption. There exists a $\tilde{g}_0$ in $G$, $\tilde{g}_0 \equiv 0$, with at least $n + 1$ distinct zeroes.

We define: $g_0^* = \tilde{g}_0/\|g_0\|$. Then $\|g_0\| = 1$.

Since $G$ is weak Chebyshev, $g_0^*$ has at most $n - 1$ changes of sign. Therefore $n + 1$ distinct zeroes $x_0, \ldots, x_n$ of $g_0$ exist such that $e_i g_0(x) > 0$, $x \in [x_i, x_{i+1}]$, $i = -1, 0, \ldots, n$, $e_i = \pm 1$, $x_{-1} = a$, $x_{n+1} = b$.

We construct a function $f$ in $C[a, b]$, having two $A$-elements in $P_G(f)$. We define $f$ in the following way:

(1) $\varepsilon_{-1}(1)^j f(x_i) = 1, \quad i = 0, \ldots, n$,

(2) $\|f\| = 1$,

(3) $0, g_0$ in $P_G(f)$.

Then $g_0$ and $0$ are $A$-elements of $f$.

Construction of $f$:

(a) We may assume $g \succ 0$ for $x \in [a, x_0]$.

We define: $f(x) = 1$ if $x \in [a, x_0], (-1)^i f(x_i) = 1, i = 0, \ldots, n$. 

(b) Definition of $f$ in $[x_0, x_1]$:

**First case.** $g_0(x) > 0$ if $x \in [x_0, x_1]$.  
Let $\tilde{x} := (x_0 + x_1)/2$ and $f(\tilde{x}) = 0$.  
Let $f$ be linear in $[x_0, \tilde{x}]$:  
\[
 f(x) := g_0(x) - g_0(\tilde{x}) + 2(g_0(\tilde{x}) - 1) \frac{x - \tilde{x}}{x_1 - x_0} \quad \text{if } x \in [\tilde{x}, x_1].
\]

**Second case.** $g_0(x) \leq 0$ for $x \in [x_0, x_1]$.  
\[
 f(x) := g_0(x) - g_0(\tilde{x}) + (1 + g_0(\tilde{x})) \frac{x - \tilde{x}}{\tilde{x} - x_0} \quad \text{if } x \in [x_0, \tilde{x}],
\]
\[
 f(\tilde{x}) = 0.
\]

Let $f$ be linear in $[\tilde{x}, x_1]$.  
This construction of $f$ is continued in an analogous way for the intervals $[x_1, x_2], \ldots, [x_{n-1}, x_n], [x_n, b]$. Obviously $f$ is continuous in $[a, b]$.  
We show: $|f(x)| \leq 1$ if $x \in [x_0, x_1]$.  
In the first case:
\[
 -1 \leq g_0(x) - g_0(\tilde{x}) + g_0(\tilde{x}) - 1 \\
 \leq g_0(x) - g_0(\tilde{x}) + 2(g_0(\tilde{x}) - 1) \frac{x - \tilde{x}}{x_1 - x_0} = f(x) \leq g_0(x) - g_0(\tilde{x}) \leq g_0(x) \leq 1 \quad \text{if } \tilde{x} \in [x, x_1].
\]

In the second case:
\[
 -1 \leq g_0(x) \leq g_0(x) - g_0(\tilde{x}) \\
 \leq g_0(x) - g_0(\tilde{x}) + (1 + g_0(\tilde{x})) \frac{x - \tilde{x}}{\tilde{x} - x_0} = f(x) \leq g_0(x) - g_0(\tilde{x}) + (1 + g_0(\tilde{x})) \\
 \leq g_0(x) + 1 \leq 1 \quad \text{if } x \in [x_0, \tilde{x}].
\]

Therefore $|f(x)| \leq 1$ if $x \in [x_0, x_1]$.  
We can show in an analogous way: $|f(x) - g_0(x)| \leq 1$ if $x \in [x_0, x_1]$.  
These estimations hold in each interval because of the construction of $f$.  
Therefore $f - 0$ and $f - g_0$ have $x_0, \ldots, x_n$ as alternating extreme points.  
If $0$ and $g_0$ are not in $P_G(f)$, then there would exist a function $g$ in $G$ such that $\|f - g\| < \|f\| = \|f - g_0\| = 1$. Since $(-1)^i f(x_i) = 1 = \|f\| > (-1)^i (f(x_i) - g(x_i))$ it follows $(-1)^i g(x_i) > 0, i = 0, \ldots, n$.  
Hence $g$ has at least $n$ changes of sign in $[a, b]$. This is a contradiction to the assumption that $G$ is weak Chebyshev. Therefore $0$ and $g_0$ are two $A$-elements of $f$ in $P_G(f)$.  
This completes the proof.  
Finally we show in Proposition 14 that a large class of weak Chebyshev
subspaces in \( C[a, b] \) whose nonzero functions have only finitely many zeroes fulfill the assumption of Corollary 9 and therefore admit a unique continuous selection.

We need the following definition (cf. Singer [10, p. 126]):

12. **Definition.** A linear subspace \( G \) of a normed linear space \( E \) is called \( k \)-Chebyshev (where \( k \) is an integer with \( 0 < k < \infty \)), if for each \( f \) in \( E \) we have \( 0 < \dim P_G(f) < k \).

Finite-dimensional \( k \)-Chebyshev subspaces in \( C(X) \) (\( X \) compact) are characterized in Singer [10, p. 240]:

13. **Theorem.** If \( G \) is an \( n \)-dimensional subspace of \( C(X) \) (\( X \) compact) and \( k \) an integer with \( 0 < k < n - 1 \). Then \( G \) is a \( k \)-Chebyshev subspace if and only if there do not exist \( n - k \) distinct points \( x_1, \ldots, x_{n-k} \) in \( X \) and \( k + 1 \) linearly independent functions \( g_0, g_1, \ldots, g_k \) in \( G \), such that
\[
  g_i(x_j) = 0, \quad j = 1, \ldots, n - k, \quad i = 0,1, \ldots, k.
\]

Using the methods in the proof of Lemma 7 and Theorem 13 we can show in a straightforward manner that the following Proposition holds:

14. **Proposition.** If \( G \) is an \( n \)-dimensional, weak Chebyshev subspace which is \((n - 1)\)-Chebyshev and if each \( g \) in \( G \), \( g \neq 0 \), has only finitely many zeroes, then each \( g \) in \( G \), \( g \neq 0 \), has at most \( n \) distinct zeroes.

**References**


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